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Global Asymptotic Stability of High-Order Hopfield Type Neural Networks with Time Delays

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Abstract—This paper studies the problem of global asymptotic stability of a class of high-order Hopfield type neural networks with time delays. By utilizing Lyapunov functionals, we obtain some sufficient conditions for the global asymptotic stability of the equilibrium point of such neural networks in terms of linear matrix inequality (LMI). Numerical examples are given to illustrate the advantages of our approach.

Keywords—Time delay, High-order Hopfield type neural networks, Global asymptotic stability.

1. INTRODUCTION

Higher-order neural networks have attracted considerable attention in recent years (see, e.g., [1–7]). This is due to the fact that high-order neural networks have stronger approximation property, faster convergence rate, greater storage capacity, and higher fault tolerance than lower-order neural networks. Hopfield neural networks with time delays have been extensively investigated over the years, and various sufficient conditions for the stability of the equilibrium point of this class of neural networks have been presented in [8–12]. However, there are very few results on the stability of the equilibrium point for high-order Hopfield type neural networks with time delays. In this paper, we shall consider a class of such neural networks. By utilizing Lyapunov functionals, we obtain some sufficient conditions on global asymptotic stability of the equilibrium point. Our conditions are expressed in terms of linear matrix inequality (LMI) and are less conservative. Even in the special case, our results improve the existing theorems. Some numerical examples are worked out to illustrate the advantages of our approach.

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2. PRELIMINARIES

Consider the following second-order Hopfield type neural networks with time delays:

$$\begin{aligned} C_i\frac{du_i(t)}{dt} &= -\frac{u_i(t)}{R_i} + \sum_{j=1}^{n} T_{ij} g_j(u_j(t-\tau_j)) \\ &+ \sum_{j=1}^{n}\sum_{k=1}^{n} T_{ijk} g_j(u_j(t-\tau_j)) g_k(u_k(t-\tau_k)) + I_i, \qquad i = 1,2,\dots,n, \end{aligned} \tag{1}$$

where $C_i > 0$, $R_i > 0$, and I_i are, respectively, the capacitance, resistance, and external input of the i^{th} neuron; T_{ij} and T_{ijk} the first- and second-order synoptic weights of the neural networks, which are not necessary symmetric, $u_i(t)$ the output of the i^{th} neuron; g_i the neuron input-output activation function; τ_i the time delay of the i^{th} neuron, which satisfies $0 \le \tau_i \le \tau$ $(i = 1,2,\dots,n)$, τ is a positive constant.

Let $u_i(s) = \varphi_i(s)$, $s \in [-\tau, 0]$ $(i = 1,2,\dots,n)$ be the initial condition, where $\varphi_i : [-\tau, 0] \to R$ are continuous functions.

We assume that the activation functions $g_i : R \to R$ $(i = 1,2,\dots,n)$ satisfy $|g_i(u)| \le M_i$ for all $u \in R$, and $0 \le (g_i(u))/u \le K_i$ for any $0 \ne u \in R$ $(i = 1,2,\dots,n)$, where $L_i > 0$, $M_i > 0$ $(i = 1,2,\dots,n)$ are constants.

Let $u^* = (u_1^*, u_2^*, \dots, u_n^*)^\top$ be an equilibrium point of system (1), and set $x = u - u^* = (x_1, x_2, \dots, x_n)^\top$, $f_i(x_i) = g_i(x_i + u_i^*) - g_i(u_i^*)$. Then, we see that

$$|f_i(z)| \le K_i|z| \quad \text{and} \quad z f_i(z) \ge 0, \qquad \text{for all } z \in R, \tag{2}$$

and system (1) is equivalent to

$$\begin{aligned} C_i\frac{dx_i(t)}{dt} &= -\frac{x_i(t)}{R_i} + \sum_{j=1}^{n} T_{ij} f_j(x_j(t-\tau_j)) + \sum_{j=1}^{n}\sum_{k=1}^{n} T_{ijk}\left[f_j(x_j(t-\tau_j)) f_k(x_k(t-\tau_k))\right. \\ &\left. + f_k(x_k(t-\tau_k)) g_j\left(u_j^*\right) + f_j(x_j(t-\tau_j)) g_k\left(u_k^*\right)\right], \qquad i = 1,2,\dots,n. \end{aligned} \tag{3}$$

The initial condition becomes $x_i(t) = \phi_i(t)$, $t \in [-\tau, 0]$, where $\phi_i(t) = \varphi_i(t) - u_i^*$, $t \in [-\tau, 0]$, $i = 1,2,\dots,n$.

Using Taylor's theorem, we can write (3) as

$$C_i\frac{dx_i(t)}{dt} = -\frac{x_i(t)}{R_i} + \sum_{j=1}^{n}\left(T_{ij} + \sum_{k=1}^{n}(T_{ijk} + T_{ikj})\zeta_k\right) f_j(x_j(t-\tau_j)), \qquad i = 1,2,\dots,n, \tag{4}$$

where ζ_k lies between $g_k(u_k(t-\tau_k))$ and $g_k(u_k^*)$.

Let $C = \operatorname{diag}(C_1, C_2, \dots, C_n)$, $R = \operatorname{diag}(R_1, R_2, \dots, R_n)$, $I = (I_1, I_2, \dots, I_n)^\top$, $T = (T_{ij})_{n\times n}$, $T_i = (T_{ijk})_{n\times n}$ $(i = 1,2,\dots,n)$, $\Pi = (T_1 + T_1^\top, T_2 + T_2^\top, \dots, T_n + T_n^\top)^\top$, $g(u(t-\bar\tau)) = (g_1(u_1(t-\tau_1)), g_2(u_2(t-\tau_2)), \dots, g_n(u_n(t-\tau_n)))^\top$, $\phi = (\phi_1, \phi_2, \dots, \phi_n)^\top$, $G(u(t-\bar\tau)) = \operatorname{diag}(g(u(t-\bar\tau)), \dots, g(u(t-\bar\tau)))$, $u(t) = (u_1(t), u_2(t), \dots, u_n(t))^\top$, $K = \operatorname{diag}(K_1, K_2, \dots, K_n)$, $M = (M_1, M_2, \dots, M_n)^\top$, $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)^\top$, $\Gamma = \operatorname{diag}(\zeta, \zeta, \dots, \zeta)$, $f(x(t-\bar\tau)) = (f_1(x_1(t-\tau_1)), f_2(x_2(t-\tau_2)), \dots, f_n(x_n(t-\tau_n)))^\top$. Then, system (1) can be rewritten in the following vector-matrix form:

$$C\frac{du(t)}{dt} = -R^{-1}u(t) + Tg(u(t-\bar\tau)) + \frac{1}{2}G^\top(u(t-\bar\tau))\Pi g(u(t-\bar\tau)) + I, \tag{5}$$

and system (4) becomes

$$C\frac{dx(t)}{dt} = -R^{-1}x(t) + Tf(x(t-\bar\tau)) + \Gamma^\top \Pi f(x(t-\bar\tau)). \tag{6}$$

We denote the vector norm of y on R^n by $|y| = \sqrt{y^\top y}$, and the matrix norm of A by $|A|$ induced by the vector norm $|\cdot|$; i.e., $|A| = \sqrt{\lambda_{\max}(A^\top A)}$. $A^\top$ is the transpose of A. If A is symmetric, then $A > 0$ means that A is positive definite. Similarly, $A < 0$ ($A \le 0$) means that A is negative definite (negative semidefinite).

Consider the following autonomous time delay equation:

$$\dot{x}(t) = f(x_t), \tag{7}$$

where $x_t \in C([-\tau, 0]), R^n)$ is defined by $x_t(\theta) = x(t+\theta)$, $-\tau \le \theta \le 0$, $f : C([-\tau, 0]), R^n) \to R^n$ is completely continuous. Assume that solutions of (7) depend continuously on the initial data. We denote by $x(\phi)$ the solution through $(0, \phi)$, $\phi \in C([-\tau, 0]), R^n)$, and $C([-\tau, 0]), R^n)$ the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^n with the topology of uniform convergence. We denote the norm of ϕ in $C([-\tau, 0]), R^n)$ by $\|\phi\| = \sup_{-\tau \le \theta \le 0} |\phi(\theta)|$.

If $V : C([-\tau, 0]), R^n) \to R$ is a continuous functional, we define the generalized derivative of V along a solution of (7) by

$$\dot{V}(\phi)|_{(7)} = \limsup_{h \to 0^+} \frac{1}{h} [V(x_h(\phi)) - V(\phi)].$$

LEMMA 1. *(See [13].) Suppose $V : C([-\tau, 0]), R^n) \to R$ is continuous and there exist nonnegative functions $a_1(r)$, $a_2(r)$, and $b(r)$ such that $a_1(r) \to \infty$ as $r \to \infty$*

$$a_1(|\phi(0)|) \le V(\phi) \le a_2(\|\phi\|), \qquad \dot{V}(\phi) \le -b(|\phi(0)|).$$

Then, the solution $x = 0$ of equation (7) is stable and every solution is bounded. If, in addition, $b(r)$ is positive definite, then every solution approaches zero as $t \to \infty$.

3. GLOBAL ASYMPTOTIC STABILITY

In this section, we shall show that system (1) has a unique equilibrium point which is globally asymptotically stable.

THEOREM 1. *Assume that the activation functions $g_i(u)$ ($i = 1, 2, \dots, n$) are bounded on R, and there are some constants $K_i > 0$ such that $0 \le (g_i(u))/u \le K_i$ for any $0 \ne u \in R$ ($i = 1, 2, \dots, n$). Then system (1) has an equilibrium point.*

PROOF. If u^* is an equilibrium point of system (1), then u^* satisfies the vector-matrix equation

$$u^* = RTg(u^*) + \frac{1}{2} RG^\top(u^*)\Pi g(u^*) + RI.$$

Define a map $F : R^n \to R^n$ by $F(u) = RTg(u) + (1/2)RG^\top(u)\Pi g(u) + RI$. Obviously, F is continuous. Let

$$\Omega = \left\{ u \in R^n \mid |u - RI| \le |R| \left(|T| + \frac{1}{2}|\Pi|L \right) L \right\},$$

and

$$L = \sup_{s \in R} \left\{ \sqrt{\sum_{i=1}^{n} g_i^2(s)} \right\}.$$

Then Ω is a bounded and closed set on R^n.

Since for any $u \in \Omega$,

$$\begin{aligned} |F(u) - RI| &= \left| R \left(Tg(u) + \frac{1}{2} G^\top(u)\Pi g(u) \right) \right| \\ &\le |R| \left(|T| + \frac{1}{2} \left| G^\top(u) \right| |\Pi| \right) |g(u)| \le |R| \left(|T| + \frac{1}{2}|\Pi|L \right) L, \end{aligned}$$

it follows that F maps Ω into itself. By the Brouwer's fixed-point theorem, the map F has at least one fixed point u^*. This means that there exists at least one equilibrium point for system (1). The proof is complete. ∎

We shall need the following lemma.

LEMMA 2. *(See [14].) For any constant* $\varepsilon > 0$, $2u^\top \nu \le \varepsilon u^\top u + (1/\varepsilon)\nu^\top \nu$, *where* $u \in R^n$, $\nu \in R^m$.

THEOREM 2. *The equilibrium point* u^* *of system (1) is unique and globally asymptotically stable, if there exists a symmetric matrix* $P > 0$, *diagonal matrices* $Q = \mathrm{diag}(q_1, q_2, \dots, q_n) > 0$, $H = \mathrm{diag}(h_1, h_2, \dots, h_n) > 0$, *and constants* $\varepsilon_i > 0$ $(i = 1, 2, 3, 4)$ *such that*

$$\begin{bmatrix} A & PC^{-1}T & PC^{-1} & KHC^{-1} & KHC^{-1} \\ T^\top C^{-1}P & -\varepsilon_1 I_{n\times n} & & & \\ C^{-1}P & & -\dfrac{\varepsilon_2 I_{n\times n}}{|M|^2} & & \\ C^{-1}HK & & & -\varepsilon_3 I_{n\times n} & \\ C^{-1}HK & & & & -\dfrac{\varepsilon_4 I_{n\times n}}{|M|^2} \end{bmatrix} < 0, \tag{8}$$

and

$$\varepsilon_1 I_{n\times n} + \varepsilon_3 T^\top T + (\varepsilon_2 + \varepsilon_4)\Pi^\top \Pi - 2Q \le 0, \tag{9}$$

where $A = 2KQK - 2KR^{-1}C^{-1}H - PC^{-1}R^{-1} - R^{-1}C^{-1}P$.

PROOF. Define the Lyapunov functional $V(\phi)$ by

$$V(\phi) = \phi^\top(0)P\phi(0) + 2\sum_{i=1}^{n} q_i \int_{-\tau_i}^{0} f_i^2(\phi_i(s))\,ds + 2\sum_{i=1}^{n} h_i \int_0^{\phi_i(0)} f_i(s)\,ds,$$

and let $a_1(r) = \lambda_{\min}(P)r^2$, $a_2(r) = (\lambda_{\max}(P) + 2\max_{1\le i\le n}\{h_i K_i\} + 2\tau \max_{1\le i\le n}\{q_i K_i^2\})r^2$. Then $a_1(r) \to \infty$ as $r \to \infty$, and $a_1(|\phi(0)|) \le V(\phi) \le a_2(\|\phi\|)$.

The derivative of $V(\phi)$ along the trajectories of system (6) is

$$\begin{aligned} \dot V\Big|_{(6)} &= -x^\top(t)\left(PC^{-1}R^{-1} + R^{-1}C^{-1}P\right)x(t) + 2x^\top(t)PC^{-1}Tf(x(t-\bar\tau)) \\ &\quad + 2x^\top(t)PC^{-1}\Gamma^\top \Pi f(x(t-\bar\tau)) - 2x^\top(t)R^{-1}C^{-1}Hf(x(t)) \\ &\quad + 2f^\top(x(t))HC^{-1}Tf(x(t-\bar\tau)) + 2f^\top(x(t))HC^{-1}\Gamma^\top\Pi f(x(t-\bar\tau)) \\ &\quad + 2f^\top(x(t))Qf(x(t)) - 2f^\top(x(t-\bar\tau))Qf(x(t-\bar\tau)). \end{aligned} \tag{10}$$

By Lemma 2, the terms on the right-hand side of (10) satisfy the following inequalities:

$$\begin{aligned} 2x^\top(t)PC^{-1}Tf(x(t-\bar\tau)) \le{}& \frac{1}{\varepsilon_1}x^\top(t)PC^{-1}TT^\top C^{-1}Px(t) \\ &+ \varepsilon_1 f^\top(x(t-\bar\tau))f(x(t-\bar\tau)), \end{aligned} \tag{11}$$

$$\begin{aligned} 2x^\top(t)PC^{-1}\Gamma^\top\Pi f(x(t-\bar\tau)) \le{}& \frac{1}{\varepsilon_2}x^\top(t)PC^{-1}\Gamma^\top\Gamma C^{-1}Px(t) \\ &+ \varepsilon_2 f^\top(x(t-\bar\tau))\Pi^\top\Pi f(x(t-\bar\tau)), \end{aligned} \tag{12}$$

$$\begin{aligned} 2f^\top(x(t))HC^{-1}Tf(x(t-\bar\tau)) \le{}& \frac{1}{\varepsilon_3}f^\top(x(t))HC^{-1}C^{-1}Hf(x(t)) \\ &+ \varepsilon_3 f^\top(x(t-\bar\tau))T^\top Tf(x(t-\bar\tau)), \end{aligned} \tag{13}$$

$$\begin{aligned} 2f^\top(x(t))HC^{-1}\Gamma^\top\Pi f(x(t-\bar\tau)) \le{}& \frac{1}{\varepsilon_4}f^\top(x(t))HC^{-1}\Gamma^\top\Gamma C^{-1}Hf(x(t)) \\ &+ \varepsilon_4 f^\top(x(t-\bar\tau))\Pi^\top\Pi f(x(t-\bar\tau)). \end{aligned} \tag{14}$$

Since $\Gamma^\top\Gamma = |\zeta|^2 I_{n\times n}$ and $|\zeta| \le |M|$, it follows that

$$x^\top(t)PC^{-1}\Gamma^\top\Gamma C^{-1}Px(t) \le |M|^2 x^\top(t)PC^{-2}Px(t), \tag{15}$$

$$f^\top(x(t))HC^{-1}\Gamma^\top\Gamma C^{-1}Hf(x(t)) \le |M|^2 f^\top(x(t))HC^{-2}Hf(x(t)). \tag{16}$$

By (2), we have

$$x_i(t)f_i(x_i(t)) \ge \frac{f_i^2(x_i(t))}{K_i}, \qquad i = 1,2,\ldots,n,$$

and hence

$$\begin{aligned} x^\top(t)R^{-1}C^{-1}Hf(x(t)) &= \sum_{i=1}^n \frac{h_i}{R_iC_i}x_i(t)f_i(x_i(t)) \ge \sum_{i=1}^n \frac{h_i}{R_iC_iK_i}f_i^2(x_i(t)) \\ &= f^\top(x(t))R^{-1}C^{-1}K^{-1}Hf(x(t)). \end{aligned} \tag{17}$$

Substituting (11)–(17) into (10), we have

$$\begin{aligned} \dot V\Big|_{(6)} &\le x^\top(t)\left(\frac{1}{\varepsilon_1}PC^{-1}TT^\top C^{-1}P + \frac{|M|^2}{\varepsilon_2}PC^{-2}P - PC^{-1}R^{-1} - R^{-1}C^{-1}P\right)x(t) \\ &\quad + f^\top(x(t))\left[\left(\frac{1}{\varepsilon_3} + \frac{|M|^2}{\varepsilon_4}\right)HC^{-2}H - 2R^{-1}C^{-1}K^{-1}H + 2Q\right]f(x(t)) \\ &\quad + f^\top(x(t-\bar\tau))\left[\varepsilon_1 I_{n\times n} + \varepsilon_3 T^\top T + (\varepsilon_2+\varepsilon_4)\Pi^\top\Pi - 2Q\right]f(x(t-\bar\tau)) \\ &\le x^\top(t)\left(\frac{1}{\varepsilon_1}PC^{-1}TT^\top C^{-1}P + \frac{|M|^2}{\varepsilon_2}PC^{-2}P - PC^{-1}R^{-1} - R^{-1}C^{-1}P\right)x(t) \\ &\quad + x^\top(t)K\left[\left(\frac{1}{\varepsilon_3} + \frac{|M|^2}{\varepsilon_4}\right)HC^{-2}H - 2R^{-1}C^{-1}K^{-1}H + 2Q\right]Kx(t) \\ &\quad + f^\top(x(t-\bar\tau))\left[\varepsilon_1 I_{n\times n} + \varepsilon_3 T^\top T + (\varepsilon_2+\varepsilon_4)\Pi^\top\Pi - 2Q\right]f(x(t-\bar\tau)) \\ &= x^\top(t)\Psi x(t) + f^\top(x(t-\bar\tau))\left[\varepsilon_1 I_{n\times n} + \varepsilon_3 T^\top T + (\varepsilon_2+\varepsilon_4)\Pi^\top\Pi - 2Q\right]f(x(t-\bar\tau)), \end{aligned}$$

where

$$\begin{aligned} \Psi &= \frac{1}{\varepsilon_1}PC^{-1}TT^\top C^{-1}P + \frac{|M|^2}{\varepsilon_2}PC^{-2}P - PC^{-1}R^{-1} - R^{-1}C^{-1}P \\ &\quad + K\left[\left(\frac{1}{\varepsilon_3} + \frac{|M|^2}{\varepsilon_4}\right)HC^{-2}H - 2R^{-1}C^{-1}K^{-1}H + 2Q\right]K. \end{aligned}$$

By (9) we get $\dot V|_{(6)} \le \lambda_{\max}(\Psi)|\phi(0)|^2$. On the other hand, by the Schur complement [15], LMI (8) is equivalent to $\Psi < 0$. Hence, if we let $b(r) = -\lambda_{\max}(\Psi)r^2$, then $b(r)$ is positive definite. Therefore, it follows from Lemma 2 that the equilibrium point $x = 0$ of system (6) or, equivalently, the equilibrium point u^* of system (1) is globally asymptotically stable. Consequently, the equilibrium point u^* is unique. Thus, the proof is complete. ∎

COROLLARY 1. *If there exists a symmetric matrix $P > 0$, a diagonal matrix*

$$Q = \operatorname{diag}(q_1, q_2, \ldots, q_n) > 0,$$

and constants $\varepsilon_i > 0$ $(i = 1,2,3,4)$ such that

$$\begin{bmatrix} \left(\frac{1}{\varepsilon_3} + \frac{|M|^2}{\varepsilon_4}\right)R^2 + 2KQK - 2I_{n\times n} - PC^{-1}R^{-1} - R^{-1}C^{-1}P & PC^{-1}T & PC^{-1} \\ T^\top C^{-1}P & -\varepsilon_1 I_{n\times n} & \\ C^{-1}P & & -\frac{\varepsilon_2}{|M|^2}I_{n\times n} \end{bmatrix} < 0,$$

and $\varepsilon_1 I_{n\times n} + \varepsilon_3 T^\top T + (\varepsilon_2+\varepsilon_4)\Pi^\top\Pi - 2Q \le 0$, then system (1) has a unique equilibrium point u^* which is globally asymptotically stable.

Corollary 1 follows from the proof of Theorem 2 in a straightforward manner by letting $H = CRK^{-1}$. If $H = 0$, then we get the following result.

COROLLARY 2. *If there exists a symmetric matrix $P > 0$, a diagonal matrix*

$$Q = \text{diag}(q_1, q_2, \dots, q_n) > 0,$$

and constants $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that

$$\begin{bmatrix} 2KQK - PC^{-1}R^{-1} - R^{-1}C^{-1}P & PC^{-1}T & PC^{-1} \\ T^{\mathrm{T}}C^{-1}P & -\varepsilon_1 I_{n\times n} & \\ C^{-1}P & & -\dfrac{\varepsilon_2}{|M|^2} I_{n\times n} \end{bmatrix} < 0,$$

and $\varepsilon_1 I_{n\times n} + \varepsilon_2 \Pi^{\mathrm{T}}\Pi - 2Q \le 0$, then system (1) has a unique equilibrium point u^* which is globally asymptotically stable.

4. EXAMPLES

To demonstrate the applicability of our results, we now consider some examples.

EXAMPLE 1. Consider the neural network

$$\begin{aligned} C_i \frac{du_i(t)}{dt} &= -\frac{u_i(t)}{R_i} + \sum_{j=1}^{n} T_{ij} g_j(u_j(t-\tau_j)) \\ &+ \sum_{j=1}^{n}\sum_{k=1}^{n} T_{ijk} g_j(u_j(t-\tau_j)) g_k(u_k(t-\tau_k)) + I_i, \qquad i = 1, 2, \dots, n, \end{aligned} \tag{1}$$

where $K = \text{diag}(0.7, 0.6, 0.8)$, $R^{-1} = \text{diag}(1.5, 1.8, 1.2)$,

$$C = I_{3\times 3}, \qquad M = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad T = \begin{bmatrix} 1.63 & 0.03 & -0.13 \\ -0.02 & 0.98 & 0.12 \\ 0.01 & -0.08 & 0.79 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 0.09 & 0.01 & -0.01 \\ 0.02 & 0.04 & 0.03 \\ 0.01 & 0.02 & 0.04 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 0.05 & 0.02 & -0.02 \\ 0.01 & 0.07 & 0.01 \\ -0.01 & 0.02 & 0.08 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} 0.06 & -0.01 & 0.02 \\ -0.01 & 0.03 & 0.01 \\ 0.02 & 0.01 & 0.09 \end{bmatrix}.$$

By MATLAB LMIToolbox, we know that there exist $\varepsilon_1 = 5.5485$, $\varepsilon_2 = 26.4433$, $\varepsilon_3 = 3.3531$, $\varepsilon_4 = 27.0682$, $Q = \text{diag}(9.2313, 9.8214, 6.8918) > 0$,

$$H = \text{diag}(5.1703, 3.4088, 3.3885) > 0, \qquad P = \begin{bmatrix} 2.5057 & 0.0098 & 0.0572 \\ 0.0098 & 2.6769 & -0.0133 \\ 0.0572 & -0.0133 & 4.6298 \end{bmatrix} > 0,$$

such that LMI (8) and (9) hold, and therefore from Theorem 2, the equilibrium point u^* of system (1) is globally asymptotically stable. Also, we know that there exist

$$\varepsilon_1 = 4.0589, \qquad \varepsilon_2 = 17.4413, \qquad P = \begin{bmatrix} 1.7769 & 0.0046 & 0.0543 \\ 0.0046 & 2.4417 & -0.0054 \\ 0.0543 & -0.0054 & 3.4101 \end{bmatrix} > 0,$$

$Q = \text{diag}(2.6621, 4.9189, 3.0831) > 0$, such that

$$\begin{bmatrix} 2KQK - PC^{-1}R^{-1} - R^{-1}C^{-1}P & PC^{-1}T & PC^{-1} \\ T^{\mathrm{T}}C^{-1}P & -\varepsilon_1 I_{n\times n} & \\ C^{-1}P & & -\dfrac{\varepsilon_2}{\|M\|^2} I_{n\times n} \end{bmatrix} < 0,$$

and $\varepsilon_1 I_{n\times n} + \varepsilon_2 \Pi^\top \Pi - 2Q \le 0$; therefore from Corollary 1, the equilibrium point u^* of system (1) is globally asymptotically stable.

The next example is the special case when $T_{ijk} \equiv 0$, which is the Hopfield neural network with time delays.

EXAMPLE 2. Consider the neural network

$$\frac{du_i(t)}{dt} = -\frac{u_i(t)}{R_i} + \sum_{j=1}^{n} T_{ij} g_j(u_j(t-\tau)) + I_i, \qquad i = 1, 2, \dots, n. \tag{18}$$

From Corollary 2, we obtain the following sufficient condition for the global asymptotic stability of the equilibrium point of this neural network.

There exists a symmetric matrix $P > 0$, a diagonal matrix $Q = \operatorname{diag}(q_1, q_2, \dots, q_n) > 0$, and a constant $\varepsilon > 0$ such that

$$\begin{bmatrix} 2KQK - PR^{-1} - R^{-1}P & PT \\ T^\top P & -\varepsilon I_{n\times n} \end{bmatrix} < 0 \qquad \text{and} \qquad \varepsilon I_{n\times n} - 2Q \le 0. \tag{19}$$

Let $R^{-1} = \operatorname{diag}(3.5, 1.8, 3.6, 3.6, 1.49, 1.95, 1.74, 1.55, 2.89, 3.62)$,

$$T = \frac{1}{2}\begin{bmatrix}
0.1 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & -0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & -3.6 & 0.5 & -0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.8 & 0.5 & 0.5 & -0.5 & 0.5 & 0.5 & 0.5 & -0.5 \\
0.5 & 0.5 & 0.5 & 0.2 & 0.5 & 0.5 & 0.5 & 0.5 & -0.5 & 0.5 \\
-0.5 & 0.5 & -0.5 & 0.5 & -2.65 & 0.5 & 0.5 & -0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & -0.5 & -2.45 & 0.5 & 0.5 & 0.5 & -0.5 \\
0.5 & 0.5 & -0.5 & 0.5 & 0.5 & 0.5 & -2.35 & 0.5 & 0.5 & -0.5 \\
0.5 & 0.5 & 0.5 & -0.5 & 0.5 & 0.5 & 0.5 & -2.9 & 0.5 & 0.5 \\
-0.5 & 0.5 & 0.5 & 0.5 & 0.5 & -0.5 & 0.5 & 0.5 & -3.55 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & -0.5 & 0.5 & 0.5 & -2.5
\end{bmatrix},$$

and $K = (3/\pi) \operatorname{diag}(1, 0.5, 0.8, 0.9, 0.5, 0.6, 0.7, 0.4, 0.8, 0.8)$ is system (18).

Using MATLAB LMIToolbox, we obtain a feasible solution of LMIs (19) as follows: $\varepsilon = 33.0793$, $Q = 16.5397 I_{10\times 10}$, and

$$P = \begin{bmatrix}
4.6159 & 0.0568 & -0.0407 & -0.0507 & 0.0784 & 0.0600 & -0.2019 & 0.0314 & 0.0417 & -0.0215 \\
0.0568 & 5.8910 & 0.0565 & -0.1429 & 0.0052 & 0.4260 & 0.5182 & 0.3111 & 0.3263 & 0.1385 \\
-0.0407 & 0.0565 & 4.4379 & -0.0428 & 0.1697 & -0.3497 & 0.0258 & 0.0284 & 0.0257 & -0.0959 \\
-0.0507 & -0.1429 & -0.0428 & 4.5054 & 0.1096 & 0.0535 & -0.0644 & -0.0070 & -0.1406 & 0.0053 \\
0.0784 & 0.0052 & 0.1697 & 0.1096 & 7.4133 & -0.0146 & 0.5118 & -0.3606 & 0.3388 & 0.2256 \\
0.0600 & 0.4260 & -0.3497 & 0.0535 & -0.0146 & 6.7007 & 0.3481 & 0.4740 & 0.0356 & -0.0769 \\
-0.2019 & 0.5182 & 0.0258 & -0.0644 & 0.5118 & 0.3481 & 7.3905 & 0.5371 & 0.3774 & -0.3063 \\
0.0314 & 0.3111 & 0.0284 & -0.0070 & -0.3606 & 0.4740 & 0.5371 & 6.9664 & 0.3361 & 0.1296 \\
0.0417 & 0.3263 & 0.0257 & -0.1406 & 0.3388 & 0.0356 & 0.3774 & 0.3361 & 4.6884 & 0.1209 \\
-0.0215 & 0.1385 & -0.0959 & 0.0053 & 0.2256 & -0.0769 & -0.3063 & 0.1296 & 0.1209 & 4.2062
\end{bmatrix} > 0.$$

It implies that the equilibrium point of system (18) is globally asymptotically stable. However, it can be shown that the sufficient condition $|RT| < 1/(\max_{1\le i\le n}\{K_i\})$ that guarantees global asymptotic stability of the equilibrium point of system (18) given in [10] does not hold.

As well, the sufficient condition $\max_{1\le i\le n}\{R_i \sum_{j=1}^{n} |T_{ji}|\} < 1$ that guarantees global asymptotic stability of the equilibrium point of system (18) given in [16] does not hold.

EXAMPLE 3. We consider the neural network

$$\frac{du_i(t)}{dt} = -u_i(t) + \sum_{j=1}^{n} T_{ij} g_j(u_j(t-\tau_j)) + I_i, \qquad i = 1, 2, \dots, n. \tag{20}$$

Using Corollary 2, a sufficient condition of the global asymptotic stability of the equilibrium point of this neural network is that there exists a symmetric matrix $P > 0$, a diagonal matrix $Q = \mathrm{diag}(q_1, q_2, \dots, q_n) > 0$, and a constant $\varepsilon > 0$ such that

$$\begin{bmatrix} 2KQK - 2P & PT \\ T^\top P & -\varepsilon I_{n\times n} \end{bmatrix} < 0, \qquad \varepsilon I_{n\times n} - 2Q \leq 0. \tag{21}$$

Let

$$T = \frac{1}{2} \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix},$$

$K = I_{3\times 3}$ in system (20). By simple calculation, we obtain a feasible solution of LMIs (21) is $\varepsilon = 431.2842$, $Q = 215.6421 I_{3\times 3}$, and

$$P = \begin{bmatrix} 427.2451 & 20.1084 & -16.0693 \\ 20.1084 & 437.8221 & -26.6464 \\ -16.0693 & -26.6464 & 473.9998 \end{bmatrix} > 0.$$

It shows that the equilibrium point of system (20) is globally asymptotically stable. But it can be verified that all the sufficient conditions for the global asymptotic stability of the equilibrium point of system (20) given in [12] do not hold.

5. CONCLUSION

In the present paper, we have derived several sufficient conditions for global asymptotic stability of the equilibrium point for a class of high-order Hopfield type neural networks with time delays. These conditions are expressed in terms of LMI and are less conservative. Even in the special case our results improve the existing results found in the literature, which has been illustrated by two numerical examples.

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