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# Independence free graphs and vertex connectivity augmentation☆

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**Abstract**

Given an undirected graph $G$ and a positive integer $k$, the $k$-vertex-connectivity augmentation problem is to find a smallest set $F$ of new edges for which $G+F$ is $k$-vertex-connected. Polynomial algorithms for this problem have been found only for $k \leqslant 4$ and a major open question in graph connectivity is whether this problem is solvable in polynomial time in general.

In this paper, we develop an algorithm which delivers an optimal solution in polynomial time for every fixed $k$. In the case when the size of an optimal solution is large compared to $k$, our algorithm is polynomial for all $k$. We also derive a min–max formula for the size of a smallest augmenting set in this case. A key step in our proofs is a complete solution of the augmentation problem for a new family of graphs which we call $k$-independence free graphs. We also prove new splitting off theorems for vertex connectivity.



*Keywords:* Connectivity of graphs; Vertex-connectivity augmentation; Algorithms

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## 1. Introduction

An undirected graph $G = (V, E)$ is *k-vertex-connected*, or more simply *k-connected*, if $|V| \geqslant k+1$ and the deletion of any $k-1$ or fewer vertices leaves a connected graph. Given a graph $G = (V, E)$ and a positive integer $k$, the $k$-vertex-connectivity augmentation problem is to find a smallest set $F$ of new edges for which $G' = (V, E \cup F)$ is $k$-connected. This problem (and a number of versions with different connectivity requirements and/or edge weights) is an important and well-studied optimization problem in network design. The complexity of the vertex-connectivity augmentation problem is one of the most challenging open questions of this area. It is open even if the graph $G$ to be augmented is $(k-1)$-vertex-connected. Polynomial algorithms have been developed only for $k = 2, 3, 4$ by Eswaran and Tarjan [5], Watanabe and Nakamura [22] and Hsu [11], respectively. Values of $k$ close to $|V| = n$ are also of interest. The case $k = n-1$ is easy, $k = n-2$ is equivalent to finding a maximum matching, and $k = n-3$ is open. Near-optimal solutions can be found in polynomial time for every $k$, see [13,12].

In this paper, we give an algorithm which delivers an optimal solution in polynomial time for any fixed $k \geqslant 2$. We also obtain a min–max formula which determines the size of an optimal solution when it is large compared to $k$. In this case, the running time of the algorithm is $O(n^6)$, where $n$ is the size of the input graph. When the size of an optimal solution is small compared to $k$, the running time is bounded by $O(c_k n^3)$, where $c_k$ is a constant if $k$ is fixed. A key step in our proofs is a complete solution of the augmentation problem for a new family of graphs which we call $k$-independence free graphs. We follow some of the ideas of the approach of [15], which obtained a near-optimal solution in the special case when the graph to be augmented is $(k-1)$-connected. We also develop new 'splitting off' theorems for $k$-vertex-connectivity.

We remark that the other three basic augmentation problems (where one wants to make $G$ $k$-edge-connected or wants to make a digraph $k$-edge- or $k$-vertex-connected) have been shown to be polynomially solvable. These results are due to Watanabe and Nakamura [21], Frank [6], and Frank and Jordán [8], respectively. For more results on connectivity augmentation and its algorithmic aspects, see the survey papers by Frank [7] and Nagamochi [20], respectively. In the remainder of this section, we introduce some definitions and our new lower bounds for the size of an augmenting set which makes $G$ $k$-vertex-connected. We also state our main min–max results.

In what follows we deal with finite undirected graphs. We shall reserve the term 'graph' for graphs without loops or multiple edges and use 'multigraph' if loops and multiple edges are allowed. Let $G = (V, E)$ be a multigraph, $v \in V$ and $X \subseteq V - v$. We use $d_G(v)$ to denote the *degree* of $v$ in $G$ and $d_G(v, X)$ for the number of edges of $G$ from $v$ to $X$. Let $N_G(X)$ denote the set of *neighbours* of $X$, that is, $N_G(X) = \{v \in V - X : uv \in E$ for some $u \in X\}$, and $n_G(X)$ denote $|N_G(X)|$. (We will suppress the subscript $G$ in the above functions when it is obvious to which graph we are referring.) We use $X^*$ to denote $V - X - N_G(X)$. We say that $X$ is a *fragment* of $G$ if $X \neq \emptyset \neq X^*$. A *k-deficient fragment* is a fragment $X$ for which $n(X) < k$, for some integer $k$. For two vertices $x, y$ of $G$ we shall use $\kappa(x, y, G)$ to denote the maximum number of openly disjoint paths from $x$ to $y$ in $G$. We use $\kappa(G)$ to denote the minimum of $\kappa(x, y, G)$ over all pairs of vertices of $G$. By Menger's theorem $\kappa(G)$ equals the minimum size of a vertex cut in $G$, unless $G$ is complete.

Let $G$ be a graph with at least $k+1$ vertices. A *$k$-augmenting set* for $G$ is a set of edges $F$ such that $G+F$ is $k$-connected. (When the value of $k$ is obvious we shall refer to $F$ simply as an *augmenting set* for $G$.) Let $a_k(G)$ denote the size of a smallest $k$-augmenting set for $G$. It is easy to see that every $k$-augmenting set for $G$ must contain at least $k-n(X)$ edges from $X$ to $X^*$ for every fragment $X$. By summing up these 'deficiencies' over pairwise disjoint $k$-deficient fragments, we may obtain a useful lower bound on $a_k(G)$, similar to the one used in the corresponding edge-connectivity augmentation problem. Let

$$t_k(G) = \max\{\sum_{i=1}^{r} k - n(X_i) : X_1, \ldots, X_r \text{ are pairwise disjoint fragments in } V\}.$$

Then

$$a_k(G) \geqslant \lceil t_k(G)/2 \rceil. \tag{1}$$

Another lower bound for $a_k(G)$ comes from 'shredders'. For $K \subset V$ let $b_G(K)$, or simply $b(K)$ when it is clear to which graph we are referring to, denote the number of components in $G-K$. We say that $K$ is an *$s$-shredder* (or simply a *shredder*) in $G$ if $|K| = s$ and $b(K) \geqslant 3$. Let $b_k(G) = \max\{b(K) : K \text{ is a } (k-1)-\text{shredder in } G\}$. Since $(G+F)-K$ has to be connected for every $k$-augmenting set $F$ and every $(k-1)$-shredder $K$, we have $|F| \geqslant b(K)-1$. This gives the second lower bound:

$$a_k(G) \geqslant b_k(G) - 1. \tag{2}$$

These lower bounds extend the two natural lower bounds used for example in [5,11,15]. Although these bounds suffice to characterize $a_k(G)$ for $k \leqslant 3$, there are examples showing that $a_k(G)$ can be strictly larger than the maximum of these lower bounds. For example, if we take $G$ to be the complete bipartite graph $K_{3,3}$ with target connectivity $k=4$, we have $t_4(G)=6$, $b_4(G)=3$ and $a_4(G)=4$. We shall show in Section 3 that $a_k(G) = \max\{b_k(G)-1, \lceil t_k(G)/2 \rceil\}$ when $G$ is a '$k$-independence free graph'. We use this result in Section 4 to show that if $G$ is $(k-1)$-connected and $a_k(G)$ is large compared to $k$, then again we have $a_k(G) = \max\{b_k(G)-1, \lceil t_k(G)/2 \rceil\}$. Our proof technique is to find a set of edges $F_1$ such that $a_k(G+F_1) = a_k(G) - |F_1|$ and $G+F_1$ is $k$-independence free. The same result is not valid if we remove the hypothesis that $G$ is $(k-1)$-connected. To see this consider the graph $G$ obtained from $K_{m,k-2}$ by adding a new vertex $x$ and joining $x$ to $j$ vertices in the $m$ set of the $K_{m,k-2}$, where $j < k < m$. Then $b_k(G) = m$, $t_k(G) = 2m+k-2j$ and $a_k(G) = m-1+k-j$. We shall see in Section 7, however, that if we modify the definition of $b_k(G)$ slightly, then we may obtain an analogous min–max theorem for augmenting graphs of arbitrary connectivity. For a $(k-1)$-shredder $K$ of $G$ we define $\delta(K) = \max\{0, \max\{k-d(x) : x \in K\}\}$ and $\hat{b}(K) = b(K) + \delta(K)$. We let $\hat{b}_k(G) = \max\{\hat{b}(K) : K \text{ is a } (k-1)-\text{shredder in } G\}$. It is easy to see that

$$a_k(G) \geqslant \hat{b}_k(G) - 1.$$

We shall prove in Section 7 that if $G$ is a graph of arbitrary connectivity and $a_k(G)$ is large compared to $k$, then

$$a_k(G) = \max\{\hat{b}_k(G) - 1, \lceil t_k(G)/2 \rceil\}.$$

Our proof technique is to find a set of edges $F_1$ such that $a_k(G + F_1) = a_k(G) - |F_1|$ and either $G + F_1$ is $(k-1)$-connected or $G + F_1$ is $(k-2)$-connected and has a special structure. In the former case, we apply the result of Section 4 to $G + F_1$. In the latter case, we find an optimal $k$-augmenting set for $G + F_1$ using a result on 'detachments' of 2-connected graphs.

Our proofs are algorithmic and give rise to polynomial algorithms for finding an optimal $k$-augmenting set in each of the cases mentioned above. In the remaining case, when $a_k(G)$ is small compared to $k$, we simply check all possible $k$-augmenting sets (spanned by a small set of vertices) to find an optimal solution. This is the only part where our algorithm is polynomial only if $k$ is fixed.

In what follows, we shall suppress the subscript $k$ in the parameters $t_k(G)$, $b_k(G)$, $\hat{b}_k(G)$ when the value of $k$ is obvious.

## 2. Preliminaries

In this section, we first introduce some submodular inequalities for the function $n$ and then describe the 'splitting off' method. We also prove some preliminary results on edge splittings and shredders.

### 2.1. Submodular inequalities

The following inequalities are fundamental to our proof technique. Inequality (4) is well-known, see, for example [15].

**Proposition 2.1.** *In a graph $H = (V, E)$ every pair $X, Y \subseteq V$ satisfies*

$$n(X) + n(Y) = n(X \cap Y) + n(X \cup Y) + |\,(N(X) \cap N(Y)) - N(X \cap Y)| + |\,(N(X) \cap Y) - N(X \cap Y)| + |\,(N(Y) \cap X) - N(X \cap Y)|. \quad (3)$$

**Proof.** Readers may find it helpful to follow the proof given below if they imagine $V(G)$ represented by a $3 \times 3$ grid, in which the two pairs of opposite sides represent $(X, X^*)$ and $(Y, Y^*)$, respectively, and the 9 subsquares represent the corresponding partition of $V(G)$ into 9 subsets. Then (3) follows from the following equalities:

$$n(X) = |N(X) \cap Y| + |N(X) \cap N(Y)| + |N(X) \cap Y^*|,$$

$$n(Y) = |X \cap N(Y)| + |N(X) \cap N(Y)| + |X^* \cap N(Y)|,$$

$$n(X \cup Y) = |N(X) \cap Y^*| + |N(X) \cap N(Y)| + |X^* \cap N(Y)|$$

and

$$n(X \cap Y) = |N(X \cap Y) \cap X| + |N(X \cap Y) \cap Y| + |N(X \cap Y) \cap (N(X) \cap N(Y))\,|. \qquad \square$$

**Proposition 2.2.** *In a graph $H = (V, E)$ every pair $X, Y \subseteq V$ satisfies*

$$n(X) + n(Y) \geqslant n(X \cap Y) + n(X \cup Y), \tag{4}$$

$$n(X) + n(Y) \geqslant n(X \cap Y^*) + n(Y \cap X^*). \tag{5}$$

**Proof.** Inequality (4) follows immediately from (3). Inequality (5) can be proved in a similar way to Proposition 2.1 □

The following inequality is new and may be applicable in other vertex-connectivity problems as well.

**Proposition 2.3.** *In a graph $H = (V, E)$ every triple $X, Y, Z \subseteq V$ satisfies*

$$\begin{aligned} n(X) + n(Y) + n(Z) \geqslant\ & n(X \cap Y \cap Z) + n(X \cap Y^* \cap Z^*) + n(X^* \cap Y^* \cap Z) \\ & + n(X^* \cap Y \cap Z^*) - |N(X) \cap N(Y) \cap N(Z)|. \end{aligned} \tag{6}$$

**Proof.** Readers may find it helpful to follow the proof given below if they imagine $V(G)$ represented by a $3 \times 3 \times 3$ grid, in which the three pairs of opposite faces represent $(X, X^*)$, $(Y, Y^*)$, and $(Z, Z^*)$, respectively, and the 27 subcubes represent the corresponding partition of $V(G)$ into 27 subsets. We have

$$\begin{aligned} n(X) = & |N(X) \cap Y \cap Z| + |N(X) \cap N(Y) \cap Z| + |N(X) \cap Y^* \cap Z| \\ & + |N(X) \cap Y \cap N(Z)| + |N(X) \cap N(Y) \cap N(Z)| + |N(X) \cap Y^* \cap N(Z)| \\ & + |N(X) \cap Y \cap Z^*| + |N(X) \cap N(Y) \cap Z^*| + |N(X) \cap Y^* \cap Z^*| \end{aligned}$$

and

$$\begin{aligned} n(X \cap Y \cap Z) \leqslant\ & |X \cap Y \cap N(Z)| + |X \cap N(Y) \cap Z| + |X \cap N(Y) \cap N(Z)| \\ & + |N(X) \cap Y \cap Z| + |N(X) \cap Y \cap N(Z)| + |N(X) \cap N(Y) \cap Z| \\ & + |N(X) \cap N(Y) \cap N(Z)|. \end{aligned}$$

The lemma follows from the above (in)-equalities and similar (in)-equalities for $n(Y)$, $n(Z)$, $n(X \cap Y^* \cap Z^*)$, $n(X^* \cap Y^* \cap Z)$ and $n(X^* \cap Y \cap Z^*)$. □

### *2.2. Extensions and splittings*

In the so-called 'splitting off method' one extends the input graph $G$ by a new vertex $s$ and a set of appropriately chosen edges incident to $s$ and then obtains an optimal augmenting set by splitting off pairs of edges incident to $s$. This approach was initiated by Cai and Sun [2] for the $k$-edge-connectivity augmentation problem and further developed and generalized by Frank [6]. Here, we adapt the method to vertex-connectivity and prove several basic properties of the extended graph as well as the splittable pairs.

Given the input graph $G = (V, E)$, an *extension* $G + s = (V + s, E + F)$ of $G$ is obtained by adding a new vertex $s$ and a set $F$ of new edges from $s$ to $V$. Note that $F$ may contain multiple edges even though $G$ does not, and hence $G + s$ may be a multigraph. In $G + s$, we define $X^* = V - X - N_G(X)$ and $\bar{d}(X) = n_G(X) + d(s, X)$ for every $X \subseteq V$.

We say that $G+s$ is $(k,s)$-*connected* if $|V|\geqslant k+1$ and

$$\bar{d}(X)\geqslant k \quad \text{for every fragment } X \text{ of } G. \tag{7}$$

If, in addition, $F$ is an inclusionwise minimal set with respect to (7), then we say that $G+s$ is a *k-critical extension* of $G$. In this case, the minimality of $F$ implies that every edge $su$ is *k-critical*, that is, deleting $su$ from $G+s$ destroys (7). (Thus, an edge $su$ is $k$-critical if and only if there exists a fragment $X$ of $G$ with $u\in X$ and $\bar{d}(X)=k$.) A fragment $X$ with $d(s,X)\geqslant 1$ and $\bar{d}(X)=k$ is called *tight*. A fragment $X$ with $d(s,X)\geqslant 2$ and $\bar{d}(X)\leqslant k+1$ is called *dangerous*. Observe that if $G$ is $l$-connected then for every $v\in V$ we have $d(s,v)\leqslant k-l$ in any $k$-critical extension of $G$. The following lemma characterizes when we can have $d(s,v)\geqslant 2$.

**Lemma 2.4.** *Let $G+s$ be a $k$-critical extension of $G$. Suppose $d(s,v)\geqslant 2$ for some $v\in V$. Let $X$ be a fragment of $G$ with $v\in X$ and $|X|\geqslant 2$. Then $\bar{d}(X)>k$. Furthermore, $d_{G+s}(v)=k$.*

**Proof.** If $\bar{d}(X)=k$ then $\bar{d}(X-v)\leqslant k-d(s,v)+1<k$ which contradicts (7). Thus $\bar{d}(X)>k$. Since $G+s$ is $k$-critical we may choose a tight set $Y$ in $G+s$ with $v\in Y$. The first part of the lemma implies that $Y=\{v\}$. Hence $d_{G+s}(v)=\bar{d}(v)=k$. □

Since the function $d(s,X)$ is modular on the subsets of $V$ in $G+s$, Propositions 2.1–2.3 yield the following (in)equalities.

**Proposition 2.5.** *In a graph $G+s$ every pair $X,Y\subseteq V$ satisfies*

$$\begin{aligned}\bar{d}(X)+\bar{d}(Y)=\bar{d}(X\cap Y)+\bar{d}(X\cup Y)+|\,(N(X)\cap N(Y))-N(X\cap Y)|\\ +|\,(N(X)\cap Y)-N(X\cap Y)|+|\,(N(Y)\cap X)-N(X\cap Y)|,\end{aligned} \tag{8}$$

$$\bar{d}(X)+\bar{d}(Y)\geqslant \bar{d}(X\cap Y)+\bar{d}(X\cup Y), \tag{9}$$

$$\bar{d}(X)+\bar{d}(Y)\geqslant \bar{d}(X\cap Y^*)+\bar{d}(Y\cap X^*)+d(s,X-Y^*)+d(s,Y-X^*). \tag{10}$$

**Proposition 2.6.** *In a graph $G+s$ every triple $X,Y,Z\subseteq V$ satisfies*

$$\begin{aligned}&\bar{d}(X)+\bar{d}(Y)+\bar{d}(Z)\\ &\quad\geqslant \bar{d}(X\cap Y\cap Z)+\bar{d}(X\cap Y^*\cap Z^*)+\bar{d}(X^*\cap Y^*\cap Z)+\bar{d}(X^*\cap Y\cap Z^*)\\ &\qquad -|N_G(X)\cap N_G(Y)\cap N_G(Z)|+2d(s,X\cap Y\cap Z).\end{aligned} \tag{11}$$

**Lemma 2.7.** *Let $G+s$ be a $(k,s)$-connected extension of $G$. Then there exists a $k$-augmenting set $F$ of $G$ with $V(F)\subseteq N(s)$.*

**Proof.** Let $F$ be a set of edges such that $A=N(s)$ induces a complete graph in $H=G+F$. Suppose $H$ is not $k$-connected. Then there exists a $k$-deficient fragment $X$ in $H$. Since $A$ induces a clique in $H$, we have either $A\cap X=\emptyset$ or $A\cap X^*=\emptyset$. Assuming, without loss of generality, that $A\cap X=\emptyset$, we have $\bar{d}_{G+s}(X)=n_H(X)<k$. This contradicts the hypothesis that $G+s$ is $(k,s)$-connected. □

We can use Lemma 2.7 to obtain upper and lower bounds of $a_k(G)$ in terms of $d_{G+s}(s)$. The following result is an easy consequence of a theorem of Mader [18, Satz 1]. It was used in [15, p. 16] in the special case when $G$ is $(k-1)$-connected.

**Theorem 2.8.** *Let $F$ be a minimal $k$-augmenting set for a graph $G$ and let $B$ be the set of those vertices of $G$ which have degree at least $k+1$ in $G+F$. Then $F$ induces a forest on $B$.*

**Lemma 2.9.** *Let $G+s$ be a $(k,s)$-connected extension of $G$ and let $A$ be a minimal $k$-augmenting set for $G$ in which every edge in $A$ connects two vertices of $N(s)$. Then $|A| \leqslant d(s)-1$.*

**Proof.** Let $B = \{v \in N(s) : d_{G+A}(v) \geqslant k+1\}$ and let $C = N(s) - B$. Since $d_{G+A}(x) = k$ and $d_{G+s}(x) \geqslant k$, we have $d_A(x) \leqslant d(s,x)$ for each $x \in C$. By Theorem 2.8, $B$ induces a forest in $A$. Let $e_A(B)$ and $e_A(C)$ denote the number of those edges of $A$ which connect two vertices of $B$ and of $C$, respectively. The previous observations imply the following inequality.

$$\begin{aligned} |A| &= e_A(C) + d_A(B,C) + e_A(B) \leqslant \sum_{x\in C} d_A(x) + |B| - 1 \\ &\leqslant (d(s) - |B|) + |B| - 1 = d(s) - 1. \end{aligned}$$

This proves the lemma. □

To obtain a lower bound on $a_k(G)$ in terms of $d(s)$, we introduce a new parameter. Let $G = (V, E)$ be a graph. We say that a fragment $X$ of $G$ *separates* a pair of vertices $u, v \in V$ if $\{u, v\} \cap X \neq \emptyset \neq \{u, v\} \cap X^*$. A family $\mathcal{F}$ of fragments of $G$ is *half-disjoint* if every pair of vertices of $G$ is separated by at most two fragments in $\mathcal{F}$. Let $t'(G) = \max\{\sum_{X\in\mathcal{F}} k - n(X)\}$ where the maximum is taken over all half-disjoint families $\mathcal{F}$ of $k$-deficient fragments in $G$. Note that every family of pairwise disjoint fragments is half-disjoint and hence $t'(G) \geqslant t(G)$. Since every $k$-augmenting set for $G$ must contain at least $k - n(X)$ edges from $X$ to $X^*$ for every fragment $X$ of $G$, we obtain the lower bound:

$$a_k(G) \geqslant \lceil t'(G)/2 \rceil. \tag{12}$$

**Lemma 2.10.** *Let $G+s$ be a $k$-critical extension of a graph $G$. Then*

$$\lceil d(s)/2 \rceil \leqslant a_k(G) \leqslant d(s) - 1.$$

**Proof.** The last inequality follows immediately from Lemma 2.9. To verify the first inequality we choose a family $\mathcal{X} = \{X_1, \ldots, X_m\}$ of tight fragments of $G$ such that $N(s) \subseteq \cup_{i=1}^m X_i$ and such that $m$ is minimum and $\sum_{i=1}^m |X_i|$ is minimum. Such a family exists since the edges incident to $s$ in $G+s$ are $k$-critical. We claim that for every $1 \leqslant i < j \leqslant m$ either $X_i \cap X_j = \emptyset$ or at least one of $X_i^* \subseteq N(X_j)$ or $X_j^* \subseteq N(X_i)$ holds. Note that in the latter case no pair of vertices can simultaneously be separated by $X_i$ and $X_j$.

To verify the claim, suppose that $X_i \cap X_j \neq \emptyset$. Then by the minimality of $m$ the set $X_i \cup X_j$ cannot be tight. Thus (9) implies that $X_i^* \cap X_j^* = \emptyset$. Hence, either one of $X_i^* \subseteq N(X_j)$ or

$X_j^* \subseteq N(X_i)$ holds or $X_i \cap X_j^*$ and $X_j \cap X_i^*$ are both non-empty. In the former case we are done. In the latter case, we apply (10) to $X_i$ and $X_j$ and conclude that $X_i \cap X_j^*$ and $X_j \cap X_i^*$ are both tight and all the edges from $s$ to $X_i \cup X_j$ enter $(X_i \cap X_j^*) \cup (X_j \cap X_i^*)$. Thus we could replace $X_i$ and $X_j$ in $\mathcal{X}$ by two strictly smaller sets $X_i \cap X_j^*$ and $X_j \cap X_i^*$, contradicting the choice of $\mathcal{X}$. This proves the claim.

To finish the proof of the lemma, observe that $\sum_{i=1}^m k - n(X_i) = \sum_{i=1}^m d(s, X_i) \geqslant d(s)$. In other words, the sum of '$k$-deficiencies' of the fragments in $\mathcal{X}$ is at least $d(s)$. We shall show that $\mathcal{X}$ is half-disjoint. Suppose on the contrary that some pair $u, v \in V$ is simultaneously separated by three sets in $\mathcal{X}$, say $X_1, X_2, X_3$. By the above claim, $X_1, X_2, X_3$ are pairwise disjoint. This contradicts the fact that they each separate $u, v$ and hence $\{u, v\} \cap X_i \neq \emptyset$ for all $1 \leqslant i \leqslant 3$. Hence $\mathcal{X}$ is half-disjoint and $d(s) \leqslant t'(G)$, as required. □

Let $G + s$ be a $(k, s)$-connected extension of $G$. *Splitting off* two edges $su, sv$ in $G + s$ means deleting $su, sv$ and adding a new edge $uv$. Note that if we perform a sequence of splittings at $s$ starting with graph $G + s$, and denote the resulting graph by $G' + s$, then $G'$ is the graph obtained from $G$ by adding the split edges. A split is *$k$-admissible* if the graph obtained by the splitting also satisfies (7). We will also say that the pair of edges $su, sv$ is *$k$-admissible*, or simply *admissible* when the value $k$ is obvious. Notice that if $G + s$ has no edges incident to $s$ then (7) is equivalent to the $k$-connectivity of $G$. Hence, it would be desirable to know, when $G + s$ is a $k$-critical extension and $d(s)$ is even, that there is a sequence of admissible splittings such that $s$ is an isolated vertex in the resulting graph $G' + s$. In this case we would have $|E(G') - E(G)| = d_G(s)/2$, and, using the fact that $a_k(G) \geqslant d(s)/2$ by Lemma 2.10, the graph $G'$ would be an *optimal* $k$-augmentation of $G$. This approach works for the $k$-edge-connectivity augmentation problem [6] but does not always work in the vertex connectivity case. The reason is that such 'complete splittings' do not necessarily exist. On the other hand, we shall prove results which are 'close enough' to yield an optimal algorithm for $k$-connectivity augmentation using the splitting off method, which is polynomial for $k$ fixed.

Pairs of edges $sx, sy$ which do not give $k$-admissible splits can be characterized by tight and dangerous 'certificates' as follows. The proof of the following simple lemma is omitted.

**Lemma 2.11.** *Let $G + s$ be a $(k, s)$-connected extension of $G$ and $x, y \in N(s)$. Splitting off the pair $sx, sy$ is not $k$-admissible in $G + s$ if and only if one of the following holds*:

(i) *there exists a tight set $T$ with $x \in T$, $y \in N(T)$,*
(ii) *there exists a tight set $U$ with $y \in U$, $x \in N(U)$,*
(iii) *there exists a dangerous set $W$ with $x, y \in W$.*

### *2.3. Local separators and shredders*

For two vertices $u, v \in V$ a *$uv$-cut* is a set $K \subseteq V - \{u, v\}$ for which there is no $uv$-path in $G - K$. A set $S \subset V$ is a *local separator* if there exist $u, v \in V - S$ such that $S$ is an inclusionwise minimal $uv$-cut. We also say $S$ is a *local $uv$-separator* and we call the components of $G - S$ containing $u$ and $v$ *essential components* of $S$ (with respect to the

pair $u, v$). Note that $S$ may be a local separator with respect to several pairs of vertices and hence it may have more than two essential components. Clearly, $N(C) = S$ for every essential component $C$ of $S$. If $S$ is a local $uv$-separator and $T$ is a local $xy$-separator then we say *$T$ meshes $S$* if $T$ intersects the two essential components of $S$ containing $u$ and $v$, respectively.

**Lemma 2.12.** *If $T$ meshes $S$ then $S$ intersects every essential component of $T$ (and hence $S$ meshes $T$).*

**Proof.** Suppose $S$ is a $uv$-separator and let $C_u$, $C_v$ be the two essential components of $S$ containing $u$ and $v$ respectively. Let $C$ be an essential component of $T$. We need to show $S$ intersects $C$. Choose $w \in V(C)$. Without loss of generality, $w \notin S$ and $w \notin V(C_v)$. Choose $t \in T \cap C_v$. Then $t \notin S$. Let $P$ be a path in the subgraph of $C \cup T$ from $w$ to $t$ such that $P \cap T = \{t\}$. Then $P$ contains a vertex of $S$ since $S$ separates $w$ from $t$. Hence $C \cap S \neq \emptyset$. □

Lemma 2.12 extends a result of Cheriyan and Thurimella [4, Lemma 4.3(1)]. The next lemma extends a key observation from the same paper [4, Proposition 3.1] and will be used when we discuss algorithms in Section 8.

**Lemma 2.13.** *Let $K$ be a local $uv$-separator of size $k-1$ and suppose that there exist $k-1$ openly-disjoint paths $P_1, \ldots, P_{k-1}$ from $u$ to $v$ in $G$. Let $Q = \cup_{i=1}^{k-1} V(P_i)$.*

(a) *For each component $C$ of $G - K$ either $C \cap \{u, v\} \neq \emptyset$ or $C$ is a component of $G - Q$.*
(b) *If $K$ has at least three essential components then $K = N(C)$ for some component $C$ of $G - Q$.*

**Proof.** (a) Since $K$ is a local $uv$-separator of size $k-1$, $K$ contains exactly one vertex from each path $P_1, \ldots, P_{k-1}$. Let $C_u, C_v, C$ be distinct components of $K$ with $u \in C_u$ and $v \in C_v$. Then $Q - K \subseteq C_u \cup C_v$. Thus $C \cap Q = \emptyset$. Hence $C$ is a component of $G - Q$.

(b) Suppose $K$ has at least three essential components. Then we choose $C$ to be an essential component of $K$ distinct from $C_u, C_v$. Then $K = N(C)$ holds by (a). □

Let $K$ be a $(k-1)$-shredder of $G$ and $G + s$ be a $(k, s)$-connected extension of $G$. A component $C$ of $G - K$ is called a *leaf component of $K$ in $G + s$* if $d(s, C) = 1$ holds. Note that $d(s, C') \geqslant 1$ for each component $C'$ of $G - K$ by (7). The next lemma is easy to verify by (7).

**Lemma 2.14.** *Let $G + s$ be a $(k, s)$-connected extension of a graph $G$ and $K$ be a $(k-1)$-shredder in $G$.*

(a) *Let $C_1, C_2$ be leaf components of $K$ in $G + s$. Then there exist $k-1$ openly-disjoint paths in the subgraph of $G$ induced by $C_1 \cup C_2 \cup K$ from every vertex of $C_1$ to every vertex of $C_2$.*
(b) *If $d(s) \leqslant 2b(K) - 2$ then $K$ has at least two leaf components, $K$ is a local separator and every leaf component of $K$ is an essential component of $K$ in $G$.*

**Proof.** Assertion (a) follows from (7). Assertion (b) follows from the fact that $d(s, C) \geqslant 1$ for every component $C$ of $G - K$, and from (a). □

We shall use the following lemma to find $(k-1)$-shredders with many components in a graph $G$ when some edge incident to $s$ in $G + s$ belongs to many non-admissible pairs.

**Lemma 2.15.** *Let $G+s$ be a $(k, s)$-connected extension of a graph $G$. Suppose there exist $r$ dangerous sets $W_1, W_2, \ldots, W_r$ and a tight set $X_0$ in $G+s$ such that $r \geqslant 3$, $W_i \cap W_j = X_0$, and $W_i \cap W_j^* \cap W_h^* \neq \emptyset$ for all distinct $i, j, h \in \{1, 2, \ldots, r\}$. Then $K = N_G(X_0)$ is a $(k-1)$-shredder in $G$ with leaf components $C_0, C_1, \ldots, C_r$, where $V(C_0) = X_0$ and $V(C_i) = W_i - X_0$ for all $1 \leqslant i \leqslant r$.*

**Proof.** Applying (11) and using the facts that: $d(s, W_i \cap W_j \cap W_h) \geqslant 1$, since $W_i \cap W_j \cap W_h = X_0$, and $X_0$ is tight; and $n_G(W_i) = \bar{d}(W_i) - d(s, W_i) \leqslant k-1$ since $W_i$ is dangerous; we obtain

$$\begin{aligned} 3k+3 &\geqslant \bar{d}(W_i) + \bar{d}(W_j) + \bar{d}(W_h) \geqslant \bar{d}(W_i \cap W_j \cap W_h) + \bar{d}(W_i \cap W_j^* \cap W_h^*) \\ &\quad + \bar{d}(W_j \cap W_i^* \cap W_h^*) + \bar{d}(W_h \cap W_i^* \cap W_j^*) \\ &\quad - |N_G(W_i) \cap N_G(W_j) \cap N_G(W_h)| + 2d(s, W_i \cap W_j \cap W_h) \\ &\geqslant 4k - |N_G(W_i) \cap N_G(W_j) \cap N_G(W_h)| + 2 \geqslant 3k+3. \end{aligned} \tag{13}$$

Thus equality must hold throughout. Hence $d(s, X_0) = 1$, and $|N_G(W_i) \cap N_G(W_j) \cap N_G(W_h)| = n_G(W_i) = k-1$. Thus, $N_G(W_i) = N_G(W_j)$ for all $i, j \in \{1, 2, \ldots, r\}$. This implies that $N_G(W_i) \cap W_j = \emptyset$ for all $i, j \in \{1, 2, \ldots, r\}$ and hence that $N_G(X_0) \subseteq N_G(W_i)$. Since $\bar{d}(X_0) = k$, $d(s, X_0) = 1$ and $n_G(W_i) = k-1$; we have $N_G(X_0) = N_G(W_i) = K$, say, for all $i \in \{1, 2, \ldots, r\}$.

The fact that $W_i \cap N_G(W_j) = \emptyset$ for all $i, j \in \{1, 2, \ldots, r\}$ also implies that $W_i$ is the disjoint union of $W_i \cap W_j \cap W_h$ and $W_i \cap W_j^* \cap W_h^*$. Thus $W_i \cap W_j^* \cap W_h^* = W_i - X_0$ for all $i, j, h \in \{1, 2, \ldots, r\}$. Equality in (13) implies that $\bar{d}(W_i) = k+1$. Since $n_G(W_i) = k-1$, we have $d(s, W_i) = 2$. The fact that $d(s, X_0) = 1$ now implies that $d(s, W_i - X_0) = 1$. Since $N_G(W_i) = K$ we have $N_G(W_i^*) \subseteq K$ for all $i \in \{1, 2, \ldots, r\}$. Thus $N_G(W_i - X_0) = N_G(W_i \cap W_j^* \cap W_h^*) \subseteq K$. Since $d(s, W_i - X_0) = 1$ and $|K| = k-1$ we have $N_G(W_i - X_0) = K$. It follows that $K$ is the required $(k-1)$-shredder in $G$. □

Note that the existence of a $(k-1)$-shredder $K$ as described in Lemma 2.15 certifies that no pair of edges from $s$ to $\cup_{i=0}^{r} C_i$ is $k$-admissible since each of the sets $V(C_i) \cup V(C_j)$ is dangerous.

## 3. Independence free graphs

In this section, we give a complete solution of the $k$-connectivity augmentation problem for a special family of graphs which we call $k$-independence free graphs. This result is a key step in our proofs concerning arbitrary graphs. However, we shall only need a special case of the main result of this section: when we augment the connectivity of a $(k-1)$-connected

$k$-independence free graph by one. This is important from an algorithmic point of view, since, as we shall see in Section 8.1, we are able to check whether a $(k-1)$-connected graph is $k$-independence free. Thus, the reader may decide to focus on this special case at first reading.

Let $G=(V,E)$ be a graph and $k$ be an integer. Let $X_1, X_2$ be disjoint non-empty subsets of $V$. We say $(X_1, X_2)$ is a *$k$-deficient pair* if $d(X_1, X_2)=0$ and $|V-(X_1\cup X_2)|\leqslant k-1$. We say two $k$-deficient pairs $(X_1, X_2)$ and $(Y_1, Y_2)$ are *independent* if for some $i\in\{1,2\}$ we have either $X_i\subseteq V-(Y_1\cup Y_2)$ or $Y_i\subseteq V-(X_1\cup X_2)$. In this case no edge can simultaneously connect $X_1$ to $X_2$ and $Y_1$ to $Y_2$ and hence the two pairs give 'independent constraints' in the $k$-augmentation problem for $G$. We say $G$ is *$k$-independence free* if $G$ does not have two independent $k$-deficient pairs. The following observations follow from these definitions.

1. If $(X_1, X_2)$ is a $k$-deficient pair in $G$ then $X_1$ is a $k$-deficient fragment.
2. If $X$ is a $k$-deficient fragment in $G$ then $(X, X^*)$ is a $k$-deficient pair.
3. $(k-1)$-connected chordal graphs are $k$-independence free.
4. Graphs with minimum degree at least $2k-2$ are $k$-independence free.
5. All graphs are 1-independence free and all connected graphs are 2-independence free.
6. A graph with no edges and at least $k+1$ vertices is not $k$-independence free for any $k\geqslant 2$.
7. If $G$ is $k$-independence free and $H$ is obtained by adding edges to $G$ then $H$ is also $k$-independence free.
8. A $k$-independence free graph is $l$-independence free for all $l\leqslant k$.

In general, a main difficulty in vertex-connectivity problems is that vertex cuts (and hence tight and dangerous sets) can cross each other in many different ways. In the case of an independence free graph $G$, we can overcome these difficulties and provide both a complete characterization of the case when there is no admissible split containing a specified edge in an extension of $G$, and a min/max formula which determines the number of edges in an optimal $k$-augmentation for $G$.

**Lemma 3.1.** *Let $G+s$ be a $(k,s)$-connected extension of a $k$-independence free graph $G$ and $X, Y$ be fragments of $G$.*

(a) *If $X$ and $Y$ are tight then either: $X\cup Y$ is tight, $X\cap Y\neq\emptyset$ and $\bar{d}(X\cap Y)=k$; or $X\cap Y^*$ and $Y\cap X^*$ are both tight and $d(s, X-Y^*)=0=d(s, Y-X^*)$.*
(b) *If $X$ is a minimal tight set and $Y$ is tight then either $X\cup Y$ is right, $d(s, X\cap Y)=0$ and $n_G(X\cap Y)=k$; or $X\subseteq Y$; or $X\subseteq Y^*$.*
(c) *If $X$ is a tight set and $Y$ is a maximal dangerous set then either $X\subseteq Y$ or $d(s, X\cap Y)=0$.*
(d) *If $X$ is a tight set, $Y$ is a dangerous set and $d(s, Y-X^*)+d(s, X-Y^*)\geqslant 2$ then $X\cap Y\neq\emptyset$ and $\bar{d}(X\cap Y)\leqslant k+1$.*

**Proof.** (a) Suppose $X\cap Y^*, Y\cap X^*\neq\emptyset$. Then (10) implies that $\bar{d}(X\cap Y^*)=k=\bar{d}(Y\cap X^*)$ and $d(s, X-Y^*)=0=d(s, Y-X^*)$. Thus $X\cap Y^*$ and $Y\cap X^*$ are both tight. Hence, we may assume that either $X\cap Y^*$ or $Y\cap X^*$ is empty. Since $G$ is $k$-independence free, it follows that $X^*\cap Y^*\neq\emptyset\neq X\cap Y$ (for example if $X\cap Y^*=\emptyset=X^*\cap Y^*$ then

$Y^* \subseteq V - (X \cup X^*)$, and $(X, X^*)$ and $(Y, Y^*)$ are independent $k$-deficient pairs). Thus $X \cup Y$ is a fragment in $G$. Using (9) we deduce that $X \cup Y$ is tight and $\bar{d}(X \cap Y) = k$.

(b) This follows from (a) using the minimality of $X$.

(c) Suppose $X \not\subseteq Y$ and $d(s, X \cap Y) \geqslant 1$. If $X \cap Y^* \neq \emptyset \neq Y \cap X^*$ then we can use (10) to obtain the contradiction

$$2k + 1 \geqslant \bar{d}(X) + \bar{d}(Y) \geqslant \bar{d}(X \cap Y^*) + \bar{d}(Y \cap X^*) + 2 \geqslant 2k + 2.$$

Thus either $X \cap Y^*$ or $Y \cap X^*$ is empty and, since $G$ is $k$-independence free, $X^* \cap Y^* \neq \emptyset$. Thus $X \cup Y$ is a fragment in $G$. Using (9) we deduce that $X \cup Y$ is dangerous contradicting the maximality of $Y$.

(d) Using (10), we deduce that either $X \cap Y^*$ or $Y \cap X^*$ is empty and, since $G$ is $k$-independence free, $X \cap Y \neq \emptyset \neq X^* \cap Y^*$. We can now use (9) to deduce that $\bar{d}(X \cap Y) \leqslant k + 1$. □

Using Lemma 3.1 we deduce

**Corollary 3.2.** *If $G + s$ is a $k$-critical extension of a $k$-independence free graph $G$ then $d(s) = t(G)$. Furthermore, there exists a unique minimal tight set in $G + s$ containing $x$ for each $x \in N(s)$.*

**Proof.** Let $\mathcal{F}$ be a family of tight sets which cover $N(s)$ such that $\sum_{X \in \mathcal{F}} |X|$ is as small as possible. Since every edge incident to $s$ is $k$-critical, such a family exists. We show that the members of $\mathcal{F}$ are pairwise disjoint. Choose $X, Y \in \mathcal{F}$ and suppose that $X \cap Y \neq \emptyset$. By Lemma 3.1(a) we may replace $X$ and $Y$ in $\mathcal{F}$ either by $X \cup Y$, or by $X \cap Y^*$ and $Y \cap X^*$. Both alternatives contradict the minimality of $\sum_{X \in \mathcal{F}} |X|$. Since the members of $\mathcal{F}$ are pairwise disjoint, tight, and cover $N(s)$, we have $d(s) = \sum_{X \in \mathcal{F}} (k - n_G(X)) \leqslant t(G)$. The inequality $d(s) \geqslant t(G)$ follows easily from (7). Thus $d(s) = t(G)$, as required.

The second assertion of the corollary follows immediately from criticality and Lemma 3.1(b). □

**Lemma 3.3.** *Let $G + s$ be a $k$-critical extension of a $k$-independence free graph $G$ and $x_1, x_2 \in N(s)$. Then the pair $sx_1, sx_2$ is not $k$-admissible for splitting in $G + s$ if and only if there exists a dangerous set $W$ in $G + s$ with $x_1, x_2 \in W$.*

**Proof.** Suppose the lemma is false. Using Lemma 2.11 we may assume without loss of generality that there exists a tight set $X_1$ in $G + s$ such that $x_1 \in X_1$ and $x_2 \in N_G(X_1)$. Let $X_2$ be the minimal tight set in $G + s$ containing $x_2$. Since $x_2 \in N(s) \cap (X_2 - X_1^*)$, it follows from Lemma 3.1(a) that $X_1 \cup X_2$ is a tight, and hence dangerous, set in $G + s$ containing $x_1, x_2$. □

**Theorem 3.4.** *Let $G + s$ be a $k$-critical extension of a $k$-independence free graph $G$ and $x_0 \in N(s)$.*

(a) *There is no $k$-admissible split in $G + s$ containing $sx_0$ if and only if either: $d(s) = b(G)$; or $d(s)$ is odd and there exist maximal dangerous sets $W_1, W_2$ in $G + s$ such that*

$N(s) \subseteq W_1 \cup W_2, x_0 \in W_1 \cap W_2, d(s, W_1 \cap W_2) = 1, d(s, W_1 \cap W_2^*) = (d(s)-1)/2 = d(s, W_1^* \cap W_2)$, *and* $W_1 \cap W_2^*$ *and* $W_2 \cap W_1^*$ *are tight.*

(b) *If there is no admissible split containing* $sx_0$ *and* $3 \neq d(s) \neq b(G)$ *then there is an admissible split containing* $sx_1$ *for all* $x_1 \in N(s) - x_0$.

**Proof.** Note that since $G + s$ is a $k$-critical extension, $d(s) \geqslant 2$.

(a) Using Lemma 3.3, we may choose a family of dangerous sets $\mathcal{W} = \{W_1, W_2, \ldots, W_r\}$ in $G + s$ such that $x_0 \in \cap_{i=1}^r W_i$, $N(s) \subseteq \cup_{i=1}^r W_i$ and $r$ is as small as possible. We may assume that each set in $\mathcal{W}$ is a maximal dangerous set in $G + s$. If $r = 1$ then $N(s) \subseteq W_1$ and

$$\bar{d}(W_1^*) = n_G(W_1^*) \leqslant n_G(W_1) \leqslant k + 1 - d(s, W_1) \leqslant k - 1,$$

since $W_1$ is dangerous. This contradicts the fact that $G+s$ is $(k, s)$-connected. Hence $r \geqslant 2$.

**Claim 3.5.** *Let* $W_i, W_j \in \mathcal{W}$. *Then* $W_i \cap W_j^* \neq \emptyset \neq W_j \cap W_i^*$ *and* $d(s, W_i - W_j^*) = 1 = d(s, W_j - W_i^*)$.

**Proof.** Suppose $W_i \cap W_j^* = \emptyset$. Since $G$ is $k$-independence free, it follows that $W_i^* \cap W_j^* \neq \emptyset$ and hence $W_i \cup W_j$ is a fragment of $G$. The minimality of $r$ now implies that $W_i \cup W_j$ is not dangerous, and hence $\bar{d}(W_i \cup W_j) \geqslant k + 2$. Applying (9) we obtain

$$2k + 2 \geqslant \bar{d}(W_i) + \bar{d}(W_j) \geqslant \bar{d}(W_i \cap W_j) + \bar{d}(W_i \cup W_j) \geqslant 2k + 2.$$

Hence equality holds throughout. Thus $\bar{d}(W_i \cap W_j) = k$ and, since $x_0 \in W_i \cap W_j$, $W_i \cap W_j$ is tight.

Choose $x_i \in N(s) \cap (W_i - W_j)$ and let $X_i$ be the minimal tight set in $G + s$ containing $x_i$. Since $x_i \in N(s) \cap X_i \cap W_i$, it follows from Lemma 3.1(c) that $X_i \subseteq W_i$. Since $G$ is $k$-independence free, $X_i \not\subseteq N(W_j)$. The assumption that $W_i \cap W_j^* = \emptyset$ now implies that $X_i \cap W_i \cap W_j \neq \emptyset$. Applying Lemma 3.1(b), we deduce that $X_i \cup (W_i \cap W_j)$ is tight. Now, $X_i \cup (W_i \cap W_j)$ and $W_j$ contradict Lemma 3.1(c) since $x_0 \in W_i \cap W_j$ and $W_j$ is a maximal dangerous set. Hence, we must have $W_i \cap W_j^* \neq \emptyset \neq W_j \cap W_i^*$. The second part of the claim follows from (10) and the fact that $x_0 \in W_i \cap W_j$. □

Suppose $r = 2$. Using Claim 3.5, we have $d(s) = 1 + d(s, W_1 \cap W_2^*) + d(s, W_2 \cap W_1^*)$. Without loss of generality we may suppose that $d(s, W_1 \cap W_2^*) \leqslant d(s, W_2 \cap W_1^*)$. Then

$$\begin{aligned}\bar{d}(W_2^*) &= d(s, W_1 \cap W_2^*) + n_G(W_2^*) \leqslant d(s, W_2 \cap W_1^*) + n_G(W_2) \\ &= \bar{d}(W_2) - 1 \leqslant k.\end{aligned}$$

Thus equality must hold throughout. Hence $d(s, W_1 \cap W_2^*) = d(s, W_2 \cap W_1^*) = (d(s)-1)/2$, $d(s)$ is odd, $W_1 \cap W_2^*$ and $W_2 \cap W_1^*$ are tight and the second alternative in (a) holds.

Finally, we suppose that $r \geqslant 3$. Choose $W_i, W_j, W_h \in \mathcal{W}, x_i \in (N(s) \cap W_i) - (W_j \cup W_h)$. Then Claim 3.5 implies that $x_i \in W_i \cap W_j^* \cap W_h^*$, and hence $W_i \cap W_j^* \cap W_h^* \neq \emptyset$. Since $G+s$ is $k$-critical, we may choose a maximal tight set $X_0$ in $G + s$ with $x_0 \in X_0$. Lemma 3.1(c) implies that $X_0 \subseteq W_t$ for all $1 \leqslant t \leqslant r$. Since $x_h \in W_i^* \cap W_j^* \cap W_h$ we have $W_i^* \cap W_j^* \neq \emptyset$. We can use (9) to deduce that $W_i \cap W_j$ is tight. Since $X_0 \subseteq W_i \cap W_j$, the maximality of $X_0$

now implies that $W_i \cap W_j = X_0$ for all $1 \leqslant i < j \leqslant r$. Applying Lemma 2.15 we deduce that $K = N_G(X_0)$ is a $(k-1)$-shredder in $G$ with $b_G(K) = d(s)$. Since the $(k,s)$-connectivity of $G+s$ implies that $b(G) \leqslant d(s)$, we have $b(G) = d(s)$.

(b) Using (a) we have $d(s)$ is odd and there exist maximal dangerous sets $W_1, W_2$ in $G+s$ such that $N(s) \subseteq W_1 \cup W_2$, $x_0 \in W_1 \cap W_2$, $d(s, W_1 \cap W_2) = 1$, $d(s, W_1 \cap W_2^*) = d(s, W_1^* \cap W_2) = (d(s)-1)/2 \geqslant 2$, and $W_1 \cap W_2^*$ and $W_1^* \cap W_2$ are tight. Suppose $x_1 \in N(s) \cap W_1 \cap W_2^*$ and there is no admissible split containing $sx_1$. Then applying (a) to $x_1$ we find maximal dangerous sets $W_3, W_4$ with $x_1 \in W_3 \cap W_4$ and $d(s, W_3 \cap W_4) = 1$. Using Lemma 3.1(c) we have $W_1 \cap W_2^* \subseteq W_3$ and $W_1 \cap W_2^* \subseteq W_4$. Thus, $W_1 \cap W_2^* \subseteq W_3 \cap W_4$ and $d(s, W_3 \cap W_4) \geqslant 2$. This contradicts the fact that $d(s, W_3 \cap W_4) = 1$. □

We can use this splitting result to determine $a_k(G)$ when $G$ is $k$-independence free. We first solve the case when $b(G)$ is large compared to $d(s)$.

**Lemma 3.6.** *Let $G+s$ be a $k$-critical extension of a $k$-independence free graph $G$ and $K$ be a $(k-1)$-shredder in $G$. If $d(s) \leqslant 2b(K) - 2$ then $d(s, K) = 0$.*

**Proof.** Let $b(K) = b$. Suppose $x \in N(s) \cap K$ and let $X$ be the minimal tight set in $G+s$ containing $x$. Let $\mathcal{L} = \{X_1, X_2, \ldots, X_r\}$ be the leaf components of $K$. Since $d(s) \leqslant 2b-2$ we have $r \geqslant 2$. Choose $X_i \in \mathcal{L}$ and $x_i \in N(s) \cap X_i$. Then $X_i$ is tight. Since $x \in K = N_G(X_i)$ we have $X \not\subseteq X_i^*$. Using Lemma 3.1(b), we deduce that $X \cup X_i$ is tight, $n_G(X \cap X_i) = k$ and $d(s, X \cap X_i) = 0$. Hence, $x_i \notin X$ and $N(X) \cap X_i \neq \emptyset$. Since this holds for all $X_i \in \mathcal{L}$ and $x \in X \cap K$, we have

$$|N(X) \cap (X_1 \cup X_2 \ldots X_r)| \geqslant r. \tag{14}$$

Furthermore, since $X \cap X_2 \neq \emptyset$ and $X \cap X_2 \subseteq X \cap X_1^*$ we have $X \cap X_1^* \neq \emptyset$. Using (10) and the fact that $d(s, X - X_1^*) \geqslant 1$ since $x \in X \cap N_G(X_1)$, it follows that $X^* \cap X_1 = \emptyset$. Using symmetry we deduce that $X^* \cap X_i = \emptyset$ for all $X_i \in \mathcal{L}$.

Since $X_1 \cup X_2$ is dangerous and $x_1, x_2 \notin X^*$, we can use Lemma 3.1(d) to deduce that $\bar{d}(X \cap (X_1 \cup X_2)) \leqslant k+1$. Using the facts that $n_G(X \cap X_1) = k = n_G(X \cap X_2)$, $N_G(X \cap (X_1 \cup X_2)) = N_G(X \cap X_1) \cup N_G(X \cap X_2)$, and $N_G(X \cap X_i) \cap X_i \neq \emptyset$ for each $i \in \{1,2\}$, we have $|N_G(X \cap X_i) \cap X_i| = 1$ for each $i \in \{1,2\}$ and $K = N_G(X \cap X_1) \cap N_G(X \cap X_2)$. Thus $x \in N_G(X \cap X_1)$, $K \subseteq X \cup N_G(X)$ and $X^* \cap K = \emptyset$. Since $X^* \cap X_i = \emptyset$ for all $X_i \in \mathcal{L}$, $X^* \cap Y \neq \emptyset$ for some non-leaf component $Y$ of $G - K$. Using (14) and the facts that $N_G(X^* \cap Y) \subseteq (N_G(X) \cap Y) \cup (N_G(X) \cap K)$ and $n_G(X) \leqslant k-1$, we deduce that $n_G(X^* \cap Y) \leqslant k-1-r$. Since $G+s$ is $(k,s)$-connected we have $d(s, Y) \geqslant d(s, X^* \cap Y) \geqslant r+1$. Thus

$$\begin{aligned} d(s) &= d(s, Y) + d(s, X_1 \cup X_2 \ldots X_r) + d(s, (Y_1 \cup Y_2 \ldots Y_{b-r}) - Y) + d(s, K) \\ &\geqslant (r+1) + r + 2(b-r-1) + 1 \geqslant 2b. \end{aligned}$$

This contradicts the hypothesis that $d(s) \leqslant 2b-2$. □

**Lemma 3.7.** *Let $G+s$ be a $k$-critical extension of a $k$-independence free graph $G$ such that $b(G)+1 \leqslant d(s) \leqslant 2b(G)-2$. Then there exists an admissible split at $s$ such that, for the resulting graph $G'+s$, we have $b(G') = b(G) - 1$.*

**Proof.** Let $b(G) = b$ and let $K$ be a $(k-1)$-shredder in $G$ with $b_G(K) = b$ and, subject to this condition, with the maximum number $r$ of leaves in $G + s$. Let $C_1, C_2, \ldots, C_r$ be the leaf components of $K$ and let $N(s) \cap C_i = \{x_i\}$ for $1 \leqslant i \leqslant r$. Since $d(s) \leqslant 2b(G) - 2$ we have $r \geqslant 2$. Since $d(s) \geqslant b(G) + 1$ and $r \geqslant 2$, we may use Theorem 3.4 to deduce without loss of generality that there is an admissible split in $G + s$ containing $sx_1$. Choose $sw$ such that $sx_1, sw$ is an admissible split in $G + s$. Splitting $sx_1, sw$ we obtain $G' + s$ where $d_{G'+s}(s) = d_{G+s}(s) - 2$ and $G' = G + x_1 w$.

Suppose $b(G') = b(G)$. Then $G$ has a $(k-1)$-shredder $K'$ with $b_G(K') = b(G)$ such that $x_1, w$ belong to the same component $C'$ of $G - K'$. (Note that $\{x_1, w\} \cap K' = \emptyset$ by Lemma 3.6.) We shall prove that such a $K'$ cannot exist in $G$.

Suppose $x_1, x_2, \ldots, x_r \in V(C')$. Since $w$ is also contained in $C'$ we have $d(s, C') \geqslant r+1$. Since $d(s) \leqslant 2b - 2$ it follows that $K'$ has at least $r + 1$ leaf components, contradicting the maximality of $r$. Hence, we may assume without loss of generality that

$$x_2 \notin C'. \tag{15}$$

Thus $K'$ separates $x_1$ and $x_2$. Since, by Lemma 2.14, the subgraph of $G$ induced by $C_1 \cup C_2 \cup K$ contains $k - 1$ openly disjoint $x_1x_2$-paths, we have

$$K' \subseteq C_1 \cup C_2 \cup K. \tag{16}$$

**Claim 3.8.** *$K$ and $K'$ are meshing local separators.*

**Proof.** Arguing by contradiction we assume that $K$ and $K'$ do not mesh. Let $C'_2$ be the component of $G - K'$ containing $x_2$. Since, every $x_1w$-path in $G$ contains a vertex of $K$ we have $C' \cap K \neq \emptyset$. Also since $G$ has $(k-1)$ $x_1x_2$-paths by Lemma 2.14, both $C'$ and $C'_2$ are essential $K'$-components. Since $K$ and $K'$ do not mesh, we have $C'_2 \cap K = \emptyset$. Hence, $C'_2$ is a connected subgraph of $G - K$. Since $x_2 \in V(C'_2)$, this implies that $C'_2 \subseteq C_2$ and $K' \cap C_2 \neq \emptyset$ (since $K \neq K'$). Since $K'$ does not mesh $K$, we have $C_1 \cap K' = \emptyset$. Thus $C_1$ is a connected subgraph of $G - K'$. Since $x_1 \in V(C')$, it follows that $C_1 \subseteq C'$. Since $N(C_1) = K$ we have $K - C' \subseteq K'$. Let $C'_1$ be a leaf component of $K'$ distinct from $C'_2$. Since $x_1, w \in V(C')$, $C'$ is not a leaf component of $K'$ and hence $C'_1 \neq C'$. The assumption that $K$ and $K'$ do not mesh and the fact that $C'$ is an essential $K'$ component intersecting $K$ now gives $K \cap C'_1 = \emptyset$. Thus $C'_1$ is a connected subgraph of $G - K$.

Since $C'_1$ and $C'_2$ are leaf components of $K'$, Lemma 2.14 implies that there are $(k-1)$ openly disjoint paths in $C'_1 \cup C'_2 \cup K'$ from each vertex of $C'_1$ to $x_2$. Since $K \cap C' \neq \emptyset$, we have $|K \cap (C'_1 \cup C'_2 \cup K')| \leqslant k - 2$. Thus $C'_1$ is contained in the same component of $G - K$ as $x_2$, and hence $C'_1 \subseteq C_2$. But $x_2$ is the only $s$-neighbour in $C_2$. Thus $d(s, C'_1) = 0$, a contradiction. □

**Claim 3.9.** $r = 2$.

**Proof.** Suppose $r \geqslant 3$. By Lemma 3.6, $x_1, x_2 \notin K'$. By Lemma 2.14, the subgraph of $G$ induced by $C_1 \cup C_2 \cup K$ contains $k - 1$ openly disjoint $x_1x_2$-paths. Since $K$ and $K'$ mesh by Claim 3.8, $K' \cap C_3 \neq \emptyset$, so $|K' \cap (C_1 \cup C_2 \cup K)| \leqslant k - 2$. Hence, at least one of the above $k - 1$ openly disjoint $x_1x_2$-paths avoids $K'$. This contradicts (15). □

We can now complete the proof of the lemma. Let $C_w$ be the component of $G - K$ containing $w$. Since $sx_1, sw$ is an admissible split and $C_1$ is a leaf component of $K$, it follows that $C_w$ is not a leaf component of $K$. Using (16), we deduce that $C_w$ is a connected subgraph of $G - K'$ and hence $C_w \subseteq C'$. Since $d(s, C_w) \geqslant 2$ and $x_1 \in N(s) \cap (C' - C_w)$ we have $d(s, C') \geqslant 3$. Since $d(s) \leqslant 2b - 2$, it follows that $K'$ has at least three leaf components. This contradicts the maximality of $r$ by Claim 3.9. Thus, $K'$ does not exist and we have $b(G') = b(G) - 1$. □

**Lemma 3.10.** *Let $G + s$ be a $k$-critical extension of a $k$-independence free graph $G$ and $p$ be an integer such that $0 \leqslant p \leqslant \frac{1}{2}d(s) - 1$. Then there exists a sequence of $p$ admissible splits at $s$ if and only if $p \leqslant d(s) - b(G)$.*

**Proof.** We first suppose that there exists a sequence of $p$ admissible splits at $s$ in $G$. Let the resulting graph be $G_1 + s$. Then $d_{G_1+s}(s) = d_G(s) - 2p$ and $b(G_1) \geqslant b(G) - p$. Since $G_1 + s$ is $(k, s)$-connected we must have $d_{G_1+s}(s) \geqslant b(G_1)$ and hence $p \leqslant d(s) - b(G)$.

We next suppose that $p \leqslant d(s) - b(G)$. We shall show by induction on $p$ that $G + s$ has a sequence of $p$ admissible splits at $s$. If $p = 0$ then there is nothing to prove. Hence, we may assume $p \geqslant 1$. Since $p \leqslant \frac{1}{2}d(s) - 1$ we have $d(s) \geqslant 4$. By Theorem 3.4 there is an admissible split at $s$. Let the resulting graph be $G_2 + s$. If $p - 1 \leqslant d_{G_2+s}(s) - b(G_2)$ then we are done by induction. Hence, we may assume that $p \geqslant d_{G_2+s}(s) - b(G_2) + 2 \geqslant d_G(s) - b(G)$. Hence, $p = d_G(s) - b(G)$. Since $p \leqslant \frac{1}{2}d_G(s) - 1$, we have $d_G(s) \leqslant 2b(G) - 2$. By Lemma 3.7 there exists an admissible split at $s$ such that the resulting graph $G_3 + s$ satisfies $b(G_3) = b(G) - 1$. It now follows by induction that $G_3 + s$ has a sequence of $p - 1$ admissible splits at $s$. □

**Lemma 3.11.** *Let $G + s$ be a $k$-critical extension of a $k$-independence free graph $G$. If $d(s) \leqslant 2b(G) - 2$ then $a_k(G) = b(G) - 1$.*

**Proof.** Suppose $d(s) = b(G)$. Let $K$ be a $(k - 1)$-shredder in $G$ with $b(K) = b(G)$. Then all components of $G - K$ are leaf components. Let $F$ be the edge set of a tree $T$ on the vertices of $N(s)$. We shall show that $G + F$ is $k$-connected. If not, then we can partition $V$ into three sets $\{X, Y, Z\}$ such that $|Z| = k - 1$ and no edge of $G + F$ joins $X$ to $Y$. Each pair of vertices of $N(s)$ are joined by $k$ openly disjoint paths in $G + F$, consisting of $(k - 1)$ paths in $G$ (which exist by Lemma 2.14) and one path in $T$. Thus either $X$ or $Y$ is disjoint from $N(s)$. Assuming $X \cap N(s) = \emptyset$, we have $\bar{d}(X) = n(X) \leqslant k - 1$, contradicting the fact that $G + s$ satisfies (7). Hence, $G + F$ is a $k$-connected augmentation of $G$ with $b(G) - 1$ edges.

Henceforth, we may assume that $d(s) > b(G)$. By Lemma 3.7, there exists an admissible split at $s$ such that, for the resulting graph $G' + s$, we have $b(G') = b(G) - 1$. Since $G' + s$ is a $k$-critical extension of $G'$, the lemma follows by induction on $d_{G+s}(s) - b(G)$. □

**Theorem 3.12.** *If $G$ is $k$-independence free then $a_k(G) = \max\{\lceil t(G)/2 \rceil, b(G) - 1\}$.*

**Proof.** Let $G + s$ be a $k$-critical extension of $G$. By Corollary 3.2, $d(s) = t(G)$. If $d(s) \leqslant 3$ then $a_k(G) = \lceil t(G)/2 \rceil$ by Lemma 2.10. Hence, we may suppose that $d(s) \geqslant 4$. If $d(s) \leqslant 2b(G) - 2$ then $a_k(G) = b(G) - 1$ by Lemma 3.11. Hence, we may suppose that $d(s) \geqslant 2b(G) - 1$.

By Lemma 3.10, there exists a sequence of $\lfloor d(s)/2 \rfloor - 1$ admissible splits at $s$. Let the resulting graph be $G' + s$. Then $G' + s$ is a $k$-critical extension of $G'$, $d_{G'+s}(s) \leqslant 3$, and $a_k(G') = \lceil d_{G'+s}(s)/2 \rceil$ by Lemma 2.10. This gives the required augmenting set $F$ for $G$ with $|F| = \lceil d_{G+s}(s)/2 \rceil = \lceil t(G)/2 \rceil$. □

## 4. Augmenting connectivity by one

Throughout this section, we assume that $G = (V, E)$ is a $(k-1)$-connected graph on at least $k+1$ vertices. We shall show that if $a_k(G)$ is large compared to $k$, then $a_k(G) = \max\{b(G) - 1, \lceil t(G)/2 \rceil\}$. Our proof uses Theorem 3.12 and some results from [15]. With the following result we can verify the desired min–max equality when $b(G) - 1 \geqslant \lceil t(G)/2 \rceil$.

**Theorem 4.1** ([15]). *Suppose $G$ is a $(k-1)$-connected graph such that $b(G) \geqslant k$ and $b(G) - 1 \geqslant \lceil t(G)/2 \rceil$. Then $a_k(G) = b(G) - 1$.*

We will apply Theorem 4.1 to graphs which do not satisfy $b(G) - 1 \geqslant \lceil t(G)/2 \rceil$ using the following concept. A set $F$ of new edges is *saturating* for $G$ if $t(G + F) = t(G) - 2|F|$. Thus an edge $e = uv$ is *saturating* if $t(G + e) = t(G) - 2$.

**Lemma 4.2.** *If $F$ is a saturating set of edges for a $(k-1)$-connected graph $G$ with $b(G + F) - 1 = \lceil t(G + F)/2 \rceil \geqslant k - 1$ then $a_k(G) = \lceil t(G)/2 \rceil$.*

**Proof.** By Theorem 4.1 the graph $G + F$ can be made $k$-connected by adding a set $F'$ of $\lceil t(G + F)/2 \rceil$ edges. Since $F$ is saturating, we have $t(G) = t(G + F) + 2|F|$. Therefore, the set $F \cup F'$ is an augmenting set for $G$ of size $\lceil t(G)/2 \rceil$. Since $a_k(G) \geqslant \lceil t(G)/2 \rceil$, the lemma follows. □

We shall show that if $a_k(G)$ is large, then we can find a saturating set of edges $F$ for $G$ so that $G + F$ is $k$-independence free. In order to do this we need to measure how close $G$ is to being $k$-independence free. We use the following concepts. Since $G$ is $(k-1)$-connected, we have $n_G(X) = k - 1$ for every $k$-deficient fragment of $G$. Following [15], we call the (inclusionwise) minimal $k$-deficient fragments in $G$ the *$k$-cores* of $G$. A $k$-core $B$ is *active* in $G$ if there exists a $(k-1)$-cut $K$ with $B \subseteq K$. Otherwise $B$ is said to be *passive*. Let $\alpha(G)$ and $\pi(G)$ denote the numbers of active, respectively, passive, $k$-cores of $G$. Since $G$ is $(k-1)$-connected, the definition of $k$-independence implies that $G$ is $k$-independence free if and only if $\alpha(G) = 0$. The following characterization of active $k$-cores also follows easily from the above definitions.

**Lemma 4.3.** *Let $B$ be a $k$-core in $G$. Then $B$ is active if and only if $\kappa(G - B) = k - |B| - 1$.*

A set $S \subseteq V$ is a *$k$-deficient fragment cover* for $G$ if $S \cap T \neq \emptyset$ for every $k$-deficient fragment $T$. Clearly, $S$ is a $k$-deficient fragment cover for $G$ if and only if $S$ covers every $k$-core of $G$. Note that $S$ is a minimal $k$-deficient fragment cover for $G$ if and only if the extension $G + s$ obtained by joining $s$ to each vertex of $S$ is $k$-critical. We shall need the following results from [15].

**Lemma 4.4.** (a) *Every minimal augmenting set for $G$ induces a forest.*

(b) *For every $k$-deficient fragment cover $S$ for $G$, there exists a minimal augmenting set $F$ for $G$ with $V(F) \subseteq S$.*

(c) *If $F$ is a minimal augmenting set for $G$, $e = xy \in F$, and $H = G + F - e$, then $H$ has precisely two $k$-cores $X, Y$. Furthermore $X \cap Y = \emptyset$; $x \in X$, $y \in Y$; for any edge $e' = x'y'$ with $x' \in X$, $y' \in Y$, the graph $H + e'$ is $k$-connected; and, for every $k$-deficient fragment $Z$ in $H$, we have $X \subseteq Z$ or $Y \subseteq Z$.*

**Proof.** Assertion (a) is given in [15, p. 16].

To prove (b), note that since $S$ covers all $k$-deficient fragments, $G$ becomes $k$-connected when we add all edges between the vertices of $S$.

Assertion (c) follows from [15, Lemma 3.2]. □

Based on these facts we can prove the following lemma.

**Lemma 4.5.** *Let $S$ be a minimal $k$-deficient fragment cover for $G$ and let $F$ be a minimal augmenting set with $V(F) \subseteq S$. Let $d_F(v) = 1$ and let $e = uv$ be the leaf of $F$ incident with $v$. Let $X$ and $Y$ be the $k$-cores of $G + F - e$ and suppose that for a set $F'$ of edges we have $\kappa(x, y, G + F') \geqslant k$ for some vertices $x \in X$, $y \in Y$. Then $S - \{v\}$ is a $k$-deficient fragment cover of $G + F'$.*

**Proof.** Without loss of generality we may assume that $u \in X$ and $v \in Y$. By the minimality of $S$, there exists a $k$-core $Z$ of $G$ such that $Z \cap S = \{v\}$. Since $Z$ is also $k$-deficient in $G + F - e$, it must contain a $k$-core of $G + F - e$, so $Y \subseteq Z$ by Lemma 4.4(c). Now, since $Y$ is also $k$-deficient in $G$ and $Z$ is a $k$-core in $G$, we must have $Z = Y$ and $Y \cap S = \{v\}$. For a contradiction suppose that there is a $k$-deficient fragment $P$ in $G + F'$ which is not covered by $S - \{v\}$. Then $P \cap S = \{v\}$ and so $P$ is also $k$-deficient in $G + F' + F - e$ and in $G + F - e$. Thus, by Lemma 4.4(c), $Y \subseteq P$ and $y \in P$ hold. Furthermore, since $G + F' + F - e + xy$ is $k$-connected by Lemma 4.4(c), we must have $x \notin P \cup N(P)$ in $G + F' + F - e$. Thus $x \notin P \cup N(P)$ holds in $G + F'$ as well. This contradicts the fact that $\kappa(x, y, G + F') \geqslant k$. □

We need some further results from [15].

**Lemma 4.6** ([15, Lemma 2.1, Claim I(a)]). *Suppose $t(G) \geqslant k$. Then the $k$-cores of $G$ are pairwise disjoint and the number of $k$-cores of $G$ is equal to $t(G)$. Furthermore, if $t(G) \geqslant k+1$, then for each $k$-core $X$, there is a unique maximal $k$-deficient fragment $S_X \subseteq V$ with the properties that $X \subseteq S_X$, and $S_X \cap Y = \emptyset$ for every $k$-core $Y$ of $G$ with $X \neq Y$. In addition, for two different $k$-cores $X, Y$ we have $S_X \cap S_Y = \emptyset$.*

**Lemma 4.7** ([15, Lemma 2.2]). *Let $K$ and $L$ be distinct $(k-1)$-cuts in $G$ with $b(K) \geqslant k$. Then $L$ intersects precisely one component $D$ of $G - K$.*

**Lemma 4.8.** Suppose $t(G) \geqslant k+1$. *Let $K$ be a $(k-1)$-shredder in $G$ with $b(K) \geqslant k$. Then*

(a) *if $C = S_X$ for some component $C$ of $G - K$ and for some $k$-core $X$ then $X$ is passive,*

(b) *if some component $D$ of $G-K$ contains precisely two $k$-cores $X, Y$ and no edge of $G$ joins $S_X$ to $S_Y$ then both $X$ and $Y$ are passive.*

**Proof.** (a) Suppose that $X$ is active and let $L$ be a $(k-1)$-cut with $X \subseteq L$. Since $b(K) \geqslant k$, we have $L \subset K \cup C$ by Lemma 4.7. Since $G$ is $(k-1)$-connected and $L \neq K$, $G-L-C$ is connected. Hence, $G-L$ has a component $C''$ with $C'' \subset C$. Therefore, $C$ contains a (minimal) $k$-deficient set $X'$ with $X \cap X' = \emptyset$, contradicting $C = S_X$.

(b) Suppose $X$ is active and let $L$ be a $(k-1)$-cut with $X \subseteq L$. As in the proof of (a), this implies that $G-L$ has a component $C$ with $C \subseteq D-L$. Since $D$ contains precisely two $k$-cores, $Y \subset C$ and hence, since $S_Y$ is the unique maximal $k$-deficient fragment containing $Y$ which is disjoint from every $k$-core, $C \subseteq S_Y$ must hold. On the other hand, since $C$ is a component of $G-L$, we have $X \subseteq N(C)$ and so $X \cap N(S_Y) \neq \emptyset$. This contradicts our assumption that no edge of $G$ joins $S_X$ to $S_Y$. □

Recall that an edge $e = uv$ is *saturating* if $t(G+e) = t(G)-2$. We say that two $k$-cores $X, Y$ form a *saturating pair* if there is a saturating edge $e = xy$ with $x \in X$, $y \in Y$ and otherwise that the pair $X, Y$ is *non-saturating*. If $t(G) \geqslant k+2$ and $X, Y$ are a saturating pair, then every edge $xy$ with $x \in X$ and $y \in Y$ is saturating. (To see this suppose that $e = xy$ is not saturating. Then $t(G+e) \geqslant t(G)-1 \geqslant k+1$ and hence the $k$-cores of $G+e$ are pairwise disjoint by Lemma 4.6. This implies that all $k$-cores of $G$ other than $X, Y$ are $k$-cores of $G+e$ and that there is a $k$-core $S$ in $G+e$ which is disjoint from all $k$-cores of $G$ other than $X, Y$. Since $S$ is a $k$-core in $G+e$, $S$ is $k$-deficient in $G$. We may assume that $S \cap X \neq \emptyset$. By applying (4) to $S$ and $X$ and using the minimality of $X$ we can deduce that $X \subseteq S$. Since $X, Y$ is a saturating pair, this implies $S \cap Y^* \neq \emptyset$ and $Y \cap S^* \neq \emptyset$. By applying (5) to $S$ and $Y$ we obtain that $Y \cap S^*$ is $k$-deficient in $G$. Since $S$ is $k$-deficient in $G+e$, we must have $y \in S \cup N_G(S)$ and hence $Y \cap S^*$ is a proper subset of $Y$. This contradicts the minimality of $Y$.) We shall need the following characterization of saturating pairs.

**Lemma 4.9** ([15, p. 13–14]). *Let $t(G) \geqslant k+2$ and suppose that two $k$-cores $X, Y$ do not form a saturating pair. Then one of the following holds*: (a) $X \subseteq N(S_Y)$, (b) $Y \subseteq N(S_X)$, (c) *there exists a $k$-deficient fragment $M$ with $S_X, S_Y \subset M$, which is disjoint from every $k$-core other than $X, Y$.*

For a $k$-core $X$ let $\nu(X)$ be the number of $k$-cores $Y$ $(Y \neq X)$ for which the pair $X, Y$ is non-saturating. The following lemma implies that an active $k$-core cannot belong to many non-saturating pairs.

**Lemma 4.10.** *Suppose $t(G) \geqslant k+2$ and let $X$ be an active $k$-core in $G$. Then $\nu(X) \leqslant 2k-3$.*

**Proof.** Let $\mathcal{Y}$ be the set of cores $Y$ $(Y \neq X)$ for which $X, Y$ is a non-saturating pair, and let $\mathcal{Y}' = \{Y_1, Y_2, \ldots, Y_r\}$ be the set of those cores from $\mathcal{Y}$ for which Lemma 4.9(c) holds (with respect to $X$). For each $Y_i$, $1 \leqslant i \leqslant r$, let $M_i$ be a $k$-deficient fragment which is disjoint from every $k$-core other than $X, Y_i$. Consider two sets $M_i, M_j$, $1 \leqslant i < j \leqslant r$. Since $t(G) \geqslant k+2$, $M_i \cap M_j$ is a $k$-deficient fragment, and hence $S_X = M_i \cap M_j$ must hold. This implies that each vertex of $V - S_X$ belongs to at most one set $M_i$.

For a contradiction suppose that $v(X) \geqslant 2k-2$. Let $K = N(S_X)$ and let $\mathcal{Y}'' = \{Y_i \in \mathcal{Y}' : M_i \cap K = \emptyset\}$. Since $|K| = k-1$ and $v(X) \geqslant 2k-2$, it follows from Lemmas 4.6 and 4.9, that $|\mathcal{Y}''| \geqslant k-1$.

Since $X$ is active, Lemma 4.8(a) implies that $b(K) \leqslant k-1$. Thus, since the vertex set of one of the components of $G-K$ is $S_X$, and $|\mathcal{Y}''| \geqslant k-1$, there is a component $D$ of $G-K$ which contains at least two sets $Y_i, Y_j$ from $\mathcal{Y}''$. Consider $M_i$. Since $S_X \subset M_i$ and $K \cap M_i = \emptyset$, we have $K \subset N(M_i)$. Since $Y_j \subset D$, we have $D - M_i \neq \emptyset$, and hence $D \cap N(M_i) \neq \emptyset$. Hence $n(M_i) \geqslant |K| + 1 = k$, contradicting the fact that $M_i$ is a $k$-deficient fragment. □

For every passive $k$-core $B_i$ $(1 \leqslant i \leqslant \pi(G))$ let $\mathcal{F}_i = \{X \subset V \ : \ X$ is $k$-deficient in $G$, $B_i \subseteq X$, the subgraph $G[X]$ is connected, and $X$ contains at most $4k-8$ active $k$-cores$\}$. Let $M_i = \cup_{X \in \mathcal{F}_i} X$ and let $T(G) = \cup_{i=1}^{\pi(G)} (M_i \cup N(M_i))$.

**Lemma 4.11.** *Let $B_i$ be a passive $k$-core for some $1 \leqslant i \leqslant \pi(G)$ and let $\mathcal{X} = \{X_1, \ldots, X_t\}$ be a minimal family of members of $\mathcal{F}_i$ for which $\cup_{j=1}^{t} X_j = M_i$. Then $t \leqslant k$ and $n(M_i) \leqslant k(k-1)$. Moreover, if $\alpha(G) \geqslant 5k-8$, then $M_i$ intersects at most $k(4k-8)$ active $k$-cores.*

**Proof.** First we prove that $t \leqslant k$. For a contradiction suppose that $t \geqslant k+1$. By the minimality of the family $\mathcal{X}$ we have that $\hat{X}_j = X_j - \cup_{r \neq j} X_r$ is non-empty for all $1 \leqslant j \leqslant t$. Note that the sets $\hat{X}_j$ are pairwise disjoint. By applying (4) to a pair $X_r, X_j \in \mathcal{X}$, and using the facts that $X_r \cap X_j \neq \emptyset$ since $B_i \subseteq X_r \cap X_j$, that $t \geqslant k+1$, and that $G$ is $(k-1)$-connected, we deduce that $X_r \cap X_j$ is $k$-deficient in $G$. Since $B_i \subseteq X_r$ for each $X_r \in \mathcal{X}$, a similar argument shows that $P = \cup_{j \neq r} (X_r \cap X_j)$ is also $k$-deficient. Note that $M_i - P = \cup_{j=1}^{t} \hat{X}_j$, so $|M_i - P| \geqslant t \geqslant k+1$. Since $X_r = \hat{X}_r \cup (P \cap X_r)$ and $G[X_r]$ is connected, there exists a neighbour of $P$ in $\hat{X}_r$. Since the sets $\hat{X}_r$ are pairwise disjoint, these neighbours are distinct. Hence $n(P) \geqslant t \geqslant k+1$, contradicting the fact that $P$ is $k$-deficient. Thus $t \leqslant k$. Since each neighbour of $M_i$ is a neighbour of some set in $\mathcal{X}$, and $\mathcal{X}$ consists of $k$-deficient fragments, we have $n(M_i) \leqslant k(k-1)$.

To see the second part of the statement suppose that for some $X_r \in \mathcal{X}$ and for some active $k$-core $A$ we have $X_r \cap A \neq \emptyset$ and $X_r - A \neq \emptyset \neq A - X_r$. Since $\alpha(G) \geqslant 5k-8$, $X_r$ contains at most $4k-8$ active $k$-cores, and the (active) $k$-cores are pairwise disjoint, we have $|V - (X_r \cup A)| \geqslant k-1$. Now (4) implies that $X_r \cap A$ is $k$-deficient, a contradiction. Thus every active $k$-core $A$ for which $A \cap M_i \neq \emptyset$ satisfies $A \subset X_r$ for some $X_r \in \mathcal{X}$. Hence, the definition of $\mathcal{F}_i$ implies that $M_i$ intersects at most $k(4k-8)$ active $k$-cores. □

We shall use the following lemmas to find a saturating set $F$ for $G$ such that $G+F$ has many passive cores. Informally, the idea is to pick a properly chosen active $k$-core $B$ and, by adding a set $F$ of at most $2k-2$ saturating edges between the active $k$-cores of $G$ other than $B$, make $\kappa(G+F-B) \geqslant k - |B| = r$. By Lemma 4.3, this will make $B$ passive, and will not eliminate any of the passive $k$-cores of $G$. We shall increase the connectivity of $G-B$ by choosing a minimal $r$-deficient fragment cover $S$ for $G-B$ of size at most $k-1$ and then iteratively add one or two edges so that the new graph has an $r$-deficient fragment cover properly contained in $S$. Thus, after at most $k-1$ such steps (and adding at most $2k-2$ edges) we shall make $B$ passive. The first lemma tells us how to choose the active $k$-core $B$.

**Lemma 4.12.** *Suppose $\pi(G) \leqslant 4(k-1)$ and $\alpha(G) \geqslant 20k(k-1)^2$. Then there exists an active $k$-core $B$ with $B \cap T(G) = \emptyset$.*

**Proof.** Since $\alpha(G) \geqslant 20k(k-1)^2 \geqslant 5k-8$, Lemma 4.11 implies that for any passive $k$-core $B_i$, the set $M_i$ intersects at most $k(4k-8)$ active $k$-cores, and $N(M_i)$ intersects at most $k(k-1)$ active $k$-cores. Thus $T(G)$ intersects at most $\pi(G)(k(5k-9)) < 4(k-1)k(5k-5) = 20k(k-1)^2$ active $k$-cores. Since $\alpha(G) \geqslant 20k(k-1)^2$, the lemma follows. □

**Lemma 4.13.** *Suppose $\pi(G) \leqslant 4(k-1)$ and $\alpha(G) \geqslant 8k^3 + 6k^2 - 23k - 16$. Let $B$ be an active $k$-core in $G$, $H = G - B$, $r = k - |B|$, and $S$ be a minimal $r$-deficient fragment cover of $H$. Suppose every $r$-deficient fragment $Z$ of $H$ contains an active $k$-core of $G$. Then there exists a saturating set of edges $F$ for $G$ such that $|F| \leqslant 2$, $F$ is not incident with $B$, and either $\pi(G+F) > \pi(G)$; or $\pi(G+F) = \pi(G)$, $B$ is an active $k$-core in $G+F$, and $H+F$ has an $r$-deficient fragment cover $S'$ which is properly contained in $S$.*

**Proof.** Since $B$ is active, $\kappa(H) = k - 1 - |B| = r - 1$.

By Lemma 4.4 there exists a minimal $r$-augmenting set $F^*$ for $H$ such that $F^*$ is a forest and $V(F^*) \subseteq S$. Let $d_{F^*}(v) = 1$ and let $e = uv$ be a leaf of $F^*$. By Lemma 4.4(c), there exist precisely two $r$-cores $Z, W$ in $H + F^* - e$ with $u \in Z, v \in W$. Then $Z, W$ are $r$-deficient in $H$. By an hypothesis of the lemma, there exist active $k$-cores $X, Y$ of $G$ with $X \subseteq Z$ and $Y \subseteq W$.

Suppose $X$ and $Y$ form a saturating pair in $G$. We may choose a saturating edge $xy$ for $G$ with $x \in X$ and $y \in Y$. Then $xy \notin E$ and, since $\kappa(G) = k-1$, we have $\kappa(x, y, G+xy) \geqslant k$ and $\kappa(x, y, H+xy) \geqslant r$. Hence either $\pi(G+xy) > \pi(G)$; or every active $k$-core of $G$ other than $X, Y$ remains active in $G + xy$. If the second alternative holds then $B$ remains active in $G + xy$ and, by Lemma 4.5, $S' = S - v$ is an $r$-deficient fragment cover in $H + xy$.

Hence, we may assume that $X, Y$ is not a saturating pair in $G$. By Lemma 4.9 either

(i) there exists a $k$-deficient fragment $M$ in $G$ with $S_X \cup S_Y \subseteq M$ which is disjoint from every $k$-core other than $X, Y$, or
(ii) $Y \subseteq N_G(S_X)$ or $X \subseteq N_G(S_Y)$.

Choose $x \in X$ and $y \in Y$ arbitrarily and let $P_1, P_2, \ldots, P_{k-1}$ be $k-1$ openly disjoint $xy$-paths in $G$. Let $Q = \cup_{i=1}^{k-1} V(P_i)$. It is easy to see that if some edge of $G$ joins $S_X$ to $S_Y$, then one of the paths, say $P_1$, satisfies $V(P_1) \subseteq S_X \cup S_Y$. On the other hand, if no edge of $G$ joins $S_X$ to $S_Y$, then (ii) cannot hold. Hence (i) holds and, either one of the paths, say $P_1$, satisfies $V(P_1) \subseteq M$, or each of the $k-1$ paths intersects $N_G(M)$. In the latter case, since $n_G(M) = k-1$, we have $|N_G(M) \cap P_i| = 1$, $V(P_i) \subseteq M \cup N_G(M)$ for all $1 \leqslant i \leqslant k-1$, and hence $N_G(M) \subset Q$ and $Q \subset M \cup N_G(M)$. We shall handle these two cases separately.

*Case* 1: No edge of $G$ joins $S_X$ to $S_Y$, (i) holds, and we have $N_G(M) \subset Q \subset M \cup N_G(M)$.

Let $C_1, C_2, \ldots, C_p$ be the components of $G - N_G(M)$. Using the properties of $M$ ($M$ intersects exactly two $k$-cores, $M$ is the union of one or more components of $G - N_G(M)$, and $N_G(M) = k-1$) we can see that either, one component $C_i$ contains $S_X$ and $S_Y$ and is disjoint from every $k$-core of $G$ other than $X, Y$ and $M = V(C_i)$, or each of $S_X$ and $S_Y$ corresponds to a component of $G - N_G(M)$ and $M = S_X \cup S_Y$.

Since $X$ and $Y$ are active $k$-cores, Lemma 4.8, with $K = N_G(M)$, implies that $p \leqslant k-1$. Since $\alpha(G) \geqslant (k-2)(2k+2) + k + 3$, $G$ has at least $(k-2)(2k+2) + 1$ active $k$-cores

disjoint from $B$, $X$, $Y$, and $N_G(M)$. Thus some component $C_j$ of $G - N_G(M)$ is disjoint from $M$ and contains at least $2k+3$ active $k$-cores distinct from $B$. By Lemma 4.10, there exists a saturating edge $xa_1$ with $a_1 \in A_1$ for some active $k$-core $A_1 \subset C_j$, $A_1 \neq B$. If $\pi(G + xa_1) \geqslant \pi(G) + 1$ then we are done. Otherwise all the active $k$-cores in $G$ other than $X$, $A_1$ remain active in $G + xa_1$. Applying Lemma 4.10 again, we may pick a saturating edge $ya_2$ with $a_2 \in A_2$ for some active $k$-core $A_2$ of $G + xa_1$, with $A_2 \subset C_j$, $A_2 \neq B$.

We have $\kappa(x, y, G + xa_1 + ya_2) \geqslant k$, since there is a path from $x$ to $y$, using the edges $xa_1$, $ya_2$, and vertices of $C_j$ only, and thus this path is openly disjoint from $Q$ (since $Q \subseteq M \cup N_G(M)$). Hence $\kappa(x, y, H + xa_1 + ya_2) \geqslant r$. Thus by Lemma 4.5, $S' = S - v$ is an $r$-deficient set cover in $H + xa_1 + ya_2$.

*Case* 2: Either $V(P_1) \subseteq S_X \cup S_Y$ or (i) holds and $V(P_1) \subseteq M$.

Let us call a component $D$ of $G - Q$ *essential* if $D$ intersects an active $k$-core other than $X$, $Y$ or $B$. Let $D_1, D_2, \ldots, D_p$ be the essential components of $G - Q$. We say that a component $D_i$ is *attached to* the path $P_j$ if $N_G(D_i) \cap V(P_j) \neq \emptyset$ holds. Let $R = S_X \cup S_Y$ if $V(P_1) \subseteq S_X \cup S_Y$ holds and let $R = M$ if $V(P_1) \subseteq M$. Then, $R$ is disjoint from every active $k$-core other than $X$, $Y$.

**Claim 4.14.** *At most* $2k-2$ *essential components are attached to* $P_1$.

**Proof.** Focus on an essential component $D$ which is attached to $P_1$ and let $w \in W \cap D$ for some active $k$-core $W \neq X, Y, B$ which has a vertex in $D$. There exists a path $P_D$ from $w$ to a vertex of $P_1$ whose inner vertices are in $D$. Since $w \notin R$ and $V(P_1) \subseteq R$, we have $D \cap N_G(R) \neq \emptyset$. The claim follows since the essential components are pairwise disjoint and $n(R) \leqslant 2k-2$. □

Suppose that one of the paths $P_i$ intersects at least $4k+4$ active $k$-cores in $G$ other than $X$, $Y$ or $B$. For every such active $k$-core $A$ intersecting $P_i$ choose a representative vertex $a \in A \cap P_i$. Since the $k$-cores are pairwise disjoint, the representatives are pairwise distinct. Order the active $k$-cores intersecting $P_i$ following the ordering of their representatives along the path $P_i$ from $x$ to $y$. By Lemma 4.10, we may choose a saturating edge $xa_1$ in $G$, where $a_1$ is among the $2k+2$ rightmost representatives and $a_1$ belongs to an active $k$-core $A_1$. If $\pi(G + xa_1) \geqslant \pi(G) + 1$ then we are done. Otherwise all the active $k$-cores of $G$ other than $X$, $A_1$ remain active in $G + xa_1$. Again using Lemma 4.10, we may choose a saturating edge $ya_2$ in $G + xa_1$, where $a_2$ is among the $2k+2$ leftmost representatives. By the choice of $a_1$ and $a_2$ there exist two openly disjoint paths from $x$ to $y$ in $G + xa_1 + ya_2$ using vertices of $V(P_i)$ only. Thus $\kappa(x, y, G + xa_1 + ya_2) \geqslant k$. Hence, by Lemma 4.5, $S' = S - v$ is an $r$-deficient set cover in $H + xa_1 + ya_2$.

Thus, we may assume that each path $P_i$ intersects at most $4k+3$ active $k$-cores in $G$ other than $X$, $Y$ or $B$. Hence there are at least

$$\begin{aligned}\alpha(G) - 3 - (k-1)(4k+3) &\geqslant (8k^3 + 6k^2 - 23k - 19) - (k-1)(4k+3)\\ &= (2k+2)(4k^2 - 3k - 8)\end{aligned}$$

active $k$-cores other than $B$ contained in $G - Q$. Note that since $k$-cores are minimal $k$-deficient fragments, they induce connected subgraphs in $G$. Hence, each $k$-core contained in $G - Q$ is contained in a component of $G - Q$. If some component of $G - Q$ contains at

least $2k+3$ active $k$-cores of $G$ other than $B$ then the lemma follows as in Case 1. Hence, we may assume that there are at least $4k^2-3k-8$ essential components in $G-Q$, each containing an active $k$-core distinct from $X$, $Y$, and $B$.

Using Claim 4.14, we deduce that there are at least $4k^2-3k-8-(2k-2)=(4k+3)(k-2)+1$ essential components $D_i$ with all their attachments on $P_2, P_3, \ldots, P_{k-1}$, each containing an active core other than $X$, $Y$, $B$. Since $G$ is $(k-1)$-connected, $n(D_i)\geqslant k-1$ and hence $D_i$ has at least two attachments on at least one of the paths $P_2, P_3, \ldots, P_{k-1}$. Relabelling the components $D_1, \ldots, D_p$ and the paths $P_2, \ldots, P_{k-1}$ if necessary, we may assume that $D_i$ has at least two attachments on $P_{k-1}$ for $1\leqslant i\leqslant 4k+4$.

Let $z_i$ be the leftmost attachment of $D_i$ on $P_{k-1}$. Without loss of generality we may assume that $z_1, z_2, \ldots, z_{4k+4}$ occur in this order on $P_{k-1}$ as we pass from $x$ to $y$. By Lemma 4.10, there exists a saturating edge $ya_i$ where $a_i\in A_i$ for some active $k$-core $A_i\subseteq D_i$, where $A_i\neq B$ and $1\leqslant i\leqslant 2k+2$. If $\pi(G+ya_i)\geqslant\pi(G)+1$ then we are done. Otherwise every active $k$-core in $G$ other than $Y$, $A_i$ remains active in $G+ya_i$. Using Lemma 4.10 again, there exists a saturating edge $xa_j$ where $a_j\in A_j$ for some active $k$-core $A_j\subseteq D_j$, where $A_j\neq B$ and $2k+3\leqslant j\leqslant 4k+4$. Note that $z_i$ is either to the left of $z_j$ or $z_i=z_j$. Hence, using the fact that $D_j$ has at least two attachments on $P_{k-1}$ and by the choice of $z_i, z_j$, there exist two openly disjoint paths in $G+xa_j+ya_i$, using vertices from $V(P_{k-1})\cup D_i\cup D_j$ only. Therefore $\kappa(x, y, G+xa_j+ya_i)\geqslant k$, and we are done as above. This completes the proof of the lemma. □

**Lemma 4.15.** *Suppose $\pi(G)\leqslant 4(k-1)$ and $\alpha(G)\geqslant 20k(k-1)^2$. Then there exists a saturating set of edges $F$ for $G$ such that $|F|\leqslant 2k-2$ and $\pi(G+F)\geqslant\pi(G)+1$.*

**Proof.** Let $B$ be an active $k$-core in $G$ with $B\cap T(G)=\emptyset$. Such a set exists by Lemma 4.12. Let $H=G-B$, and $r=k-|B|$. Since $B$ is active, $\kappa(H)=r-1$. Every $r$-deficient fragment $X$ in $H$ is $k$-deficient in $G$ and $N_G(B)\cap X\neq\emptyset$. Hence $N_G(B)$ is an $r$-deficient fragment cover of $H$. Let $S\subseteq N_G(B)$ be a minimal $r$-deficient fragment cover of $H$. Since $B$ is $k$-deficient in $G$, we have $|S|\leqslant n_G(B)=k-1$.

We shall prove by induction on $i$ that, for $0\leqslant i\leqslant k-1$, there exists a saturating set of edges $F_i$ for $G$ such that $|F_i|\leqslant 2i$, $F_i$ is not incident with $B$, and either $\pi(G+F_i)\geqslant\pi(G)+1$; or $\pi(G+F_i)=\pi(G)$, $B$ is an active $k$-core of $G+F_i$, and $H+F_i$ has an $r$-deficient fragment cover $S_i\subseteq S$ with $|S_i|\leqslant|S|-i$. The lemma will follow since the second alternative cannot hold with $|S_i|=0$ (since this would imply that $H+F_i$ is $r$-connected and hence that $B$ is passive in $G+F_i$).

The statement is trivially true for $i=0$ taking $F_i=\emptyset$. Hence, suppose that there exists a set $F_i$ satisfying the above statement for some $0\leqslant i\leqslant k-2$. If $\pi(G+F_i)\geqslant\pi(G)+1$ then we can put $F_{i+1}=F_i$. Hence we may suppose that $\pi(G+F_i)=\pi(G)$, $B$ is an active $k$-core of $G+F_i$, and $H+F_i$ has an $r$-deficient fragment cover $S_i\subseteq S$ with $|S_i|\leqslant|S|-i$. We would like to apply Lemma 4.13 to $B$ and $G+F_i$. To do this we must show that $G+F_i$, $B$ and $S_i$ satisfy the hypotheses of this lemma. We have $\pi(G+F_i)=\pi(G)\leqslant 4(k-1)$. Thus, $\alpha(G+F_i)=\alpha(G)-2|F_i|\geqslant 8k^3+6k^2-23k-16$.

The last property we need to verify is that every $r$-deficient fragment $Z$ in $G+F_i-B$ contains at least one active $k$-core of $G+F_i$. Since $F_i$ is a saturating set for $G$, and since the $k$-cores of $G$ are pairwise disjoint, each $k$-core of $G+F_i$ is a $k$-core of $G$. Furthermore,

since $\pi(G+F_i)=\pi(G)$, if $A$ is an active $k$-core of $G$ and $A$ is a $k$-core of $G+F_i$ then $A$ is an active $k$-core of $G+F_i$. Since $Z$ is $r$-deficient in $G+F_i-B$, it is $k$-deficient in $G+F_i$. Thus $Z$ contains at least one core in $G+F_i$. If $Z$ contains an active $k$-core in $G+F_i$, then we are done, so suppose that every $k$-core of $G+F_i$ in $Z$ is passive. Let $B_j$ be such a $k$-core. Then $B_j$ is a passive $k$-core in $G$ so $G[B_j]$ is connected. Let $C$ be the component of $G[Z]$ containing $B_j$ and let $Z'=V(C)$. Since $Z$ is $k$-deficient in $G$, $Z'$ is $k$-deficient in $G$, and $B\subseteq N_G(Z')$. Since $B\cap T(G)=\emptyset$ and $B\subseteq N_G(Z')$, it follows that $Z'\notin\mathcal{F}_j$ and hence $Z'$ contains at least $4k-7$ active $k$-cores in $G$. Since $|F_i|\leqslant 2(k-2)=2k-4$ and each edge of $F_i$ is incident to at most two $k$-cores of $G$, it follows that there exists an active $k$-core $A$ in $G$ with $A\subset Z'$ which is still an (active) $k$-core in $G+F_i$, contradicting the assumption that every $k$-core of $G+F_i$ in $Z$ is passive. Hence $G+F_i$, $B$ and $S_i$ satisfy the hypotheses of Lemma 4.13. Thus, there exists a saturating set of edges $F$ for $G+F_i$ such that $|F|\leqslant 2$, $F$ is not incident with $B$, and either $\pi(G+F_i+F)>\pi(G+F_i)=\pi(G)$; or $\pi(G+F_i+F)=\pi(G+F_i)=\pi(G)$ and $G+F_i+F-B$ has an $r$-deficient fragment cover $S_{i+1}$ which is properly contained in $S_i$. Hence, the inductive statement holds with $F_{i+1}=F_i\cup F$. □

**Lemma 4.16.** *Suppose $t(G)\geqslant 20k(k-1)^2+(4k-3)(4k-4)$. Then there exists a saturating set of edges $F$ for $G$ such that $G+F$ is $k$-independence free and $t(G+F)\geqslant 2k-1$.*

**Proof.** Since every graph is 1-independence free and every connected graph is 2-independence free, we may suppose that $k\geqslant 3$. If $\pi(G)\leqslant 4(k-1)$ then we may apply Lemma 4.15 recursively $4k-3-\pi(G)$ times to $G$ to find a saturating set of edges $F_1$ for $G$ such that $\pi(G+F_1)\geqslant 4k-3$. If $\pi(G)\geqslant 4k-3$ we set $F_1=\emptyset$. Applying Lemma 4.10 to $G+F_1$, we can add saturating edges joining pairs of active $k$-cores until the number of active $k$-cores is at most $2k-2$. Thus there exists a saturating set of edges $F_2$ for $G+F_1$ such that $\alpha(G+F_1+F_2)\leqslant 2k-2$ and $\pi(G+F_1+F_2)\geqslant 4k-3$. Applying Lemma 4.10 to $G+F_1+F_2$, we can add saturating edges joining pairs consisting of one active and one passive $k$-core until the number of active $k$-cores decreases to zero. Thus there exists a saturating set of edges $F_3$ for $G+F_1+F_2$ such that $\alpha(G+F_1+F_2+F_3)=0$ and $\pi(G+F_1+F_2+F_3)\geqslant 2k-1$. □

The main theorem of this section is the following.

**Theorem 4.17.** *If $a_k(G)\geqslant 20k^3$ then*

$$a_k(G)=\max\{\lceil t(G)/2\rceil, b(G)-1\}.$$

**Proof.** Since every graph is 1-independence free and every connected graph is 2-independence free, the result follows from Theorem 3.12 if $k\leqslant 2$. Hence we may suppose that $k\geqslant 3$. Let $G+s$ be a $k$-critical extension of $G$. By Lemma 2.10 we have $d(s)\geqslant a_k(G)+1\geqslant 20k^3>k+1$. Hence, by [15, Lemmas 3.4, 3.5] we have $t(G)=d(s)\geqslant 20k^3$. (This equality will also follow from Lemma 5.2 in Section 5.) If $b(G)-1\geqslant\lceil t(G)/2\rceil$ then $a_k(G)=b(G)-1$ by Theorem 4.1 and we are done. Thus, we may assume that $\lceil t(G)/2\rceil>b(G)-1$ holds. We shall show that $a_k(G)=\lceil t(G)/2\rceil$. By Lemma 4.16, there exists a saturating

set of edges $F$ for $G$ such that $G+F$ is $k$-independence free and $t(G+F)\geqslant 2k-1$. Note that adding a saturating edge to a graph $H$ reduces $\lceil t(H)/2\rceil$ by exactly one and $b(H)$ by at most one. Thus, if $\lceil t(G+F)/2\rceil\leqslant b(G+F)-1$, then there exists $F'\subseteq F$ such that $\lceil t(G+F')/2\rceil = b(G+F')-1$ and the theorem follows by applying Lemma 4.2. Hence we may assume that $\lceil t(G+F)/2\rceil > b(G+F)-1$. Since $G+F$ is $k$-independence free, we can apply Theorem 3.12 to deduce that $a_k(G+F)=\lceil t(G+F)/2\rceil$. Using (1) and the fact that $t(G)=t(G+F)+2|F|$ we have $a_k(G)=\lceil t(G)/2\rceil$, as required. □

Theorem 4.17 gives an affirmative answer to a conjecture of the second author, [16, p. 300].

## 5. Unsplittable extensions

In this section, we consider a $k$-critical extension $G+s$ of an $l$-connected graph $G$ on at least $k+1$ vertices in which $d(s)$ is large. We show that $d(s)=t(G)$ and characterize when there is no admissible split containing a given edge at $s$.

**Lemma 5.1.** *Let $X, Y\subset V$ be two sets with $X\cap Y\neq\emptyset$. Suppose $d(s)\geqslant(k-l)(k-1)+4$.*

(a) *If $X$ and $Y$ are tight then $X\cup Y$ is tight and $\bar{d}(X\cap Y)=k$.*
(b) *If $X$ is tight and $Y$ is dangerous then $X\cup Y$ is dangerous.*
(c) *If $d(s)\geqslant(k-l+1)(k-1)+4$ and $X$ and $Y$ are dangerous then $X^*\cap Y^*\neq\emptyset$.*

**Proof.** We prove (a). Let $X, Y$ be tight sets with $X\cap Y\neq\emptyset$. By (9) we have

$$2k=\bar{d}(X)+\bar{d}(Y)\geqslant\bar{d}(X\cap Y)+\bar{d}(X\cup Y). \tag{17}$$

Clearly, $X\cap Y$ is a fragment and hence $\bar{d}(X\cap Y)\geqslant k$ by (7). Using (17) we have $\bar{d}(X\cup Y)\leqslant k$. Thus if $X^*\cap Y^*\neq\emptyset$ then $X\cup Y$ is also a fragment and hence is tight and $\bar{d}(X\cap Y)=k$.

Suppose $X^*\cap Y^*=\emptyset$. Since $\bar{d}(X\cup Y)\leqslant k$, we have $n(X\cup Y)\leqslant k-d(s,X\cup Y)$. Since $G$ is $l$-connected and $G+s$ is $k$-critical, $d(s,v)\leqslant k-l$ for all $v\in V$. Thus

$$\begin{aligned} d(s) &\leqslant d(s,X\cup Y)+d(s,N(X\cup Y))\leqslant d(s,X\cup Y)+(k-l)n(X\cup Y)\\ &\leqslant d(s,X\cup Y)+(k-l)(k-d(s,X\cup Y))\\ &=(k-l)k-(k-l-1)d(s,X\cup Y).\end{aligned}$$

Since $k-l-1\geqslant 0$ and $d(s,X\cup Y)\geqslant 1$, this gives $d(s)\leqslant(k-l)(k-1)+1$, contradicting the hypothesis on $d(s)$.

The proofs of (b) and (c) are similar, using the fact that $d(s,X\cup Y)\geqslant 2$ in (b) and (c). □

The following lemma shows that $d(s)=t(G)$ when $d(s)$ is large.

**Lemma 5.2.** *If $d(s)\geqslant(k-l)(k-1)+4$ then $d(s)=t(G)$.*

**Proof.** Let $\mathcal{F}$ be a family of tight sets which cover $N(s)$ such that $|\mathcal{F}|$ is as small as possible. Since every edge incident to $s$ is critical, such a family exists. We show that the members of $\mathcal{F}$ are pairwise disjoint. Choose $X, Y \in \mathcal{F}$ and suppose that $X \cap Y \neq \emptyset$. By Lemma 5.1(a), $X \cup Y$ is also tight. So replacing $X$ and $Y$ in $\mathcal{F}$ by $X \cup Y$ we contradict the minimality of $|\mathcal{F}|$.

Since the members of $\mathcal{F}$ are pairwise disjoint, tight, and cover $N(s)$, we have $d(s) = \sum_{X\in\mathcal{F}}(k-n(X)) \leqslant t(G)$. The inequality $d(s) \geqslant t(G)$ follows easily from (7). Thus $d(s) = t(G)$, as required. □

**Lemma 5.3.** *Let $sx_0$ be a designated edge of a $k$-critical extension $G+s$ of $G$ and suppose that there are $q \geqslant (k-l+1)(k-1)+4$ edges $sy$ $(y \neq x_0)$ incident to $s$ for which the pair $sx_0, sy$ is not admissible. Then there exists a $(k-1)$-shredder $K$ in $G$ such that $K$ has $q+1$ leaf components $C_0, C_1, \ldots, C_q$ in $G+s$, where $X_0 = V(C_0)$ is the maximal tight set containing $x_0$ and $K = N_G(X_0)$.*

**Proof.** Let $X_0$ be the maximal tight set in $G+s$ containing $x_0$. Note that the set $X_0$ is uniquely determined by Lemma 5.1(a). Let $\mathcal{T} = \{X_1, \ldots, X_m\}$ be the set of all maximal tight sets which intersect $N(X_0)$. Note that $X_i \cap X_j = \emptyset$ for $0 \leqslant i < j \leqslant m$ by Lemma 5.1(a). Thus we have $d(s, \cup_{i=0}^{m} X_i) = d(s, X_0) + d(s, \cup_{i=1}^{m} X_i)$.

Since each $X_i \in \mathcal{T}$ contains a neighbour of $X_0$ and $X_0$ is tight, we have $m \leqslant n(X_0) = k - d(s, X_0)$. Since each $X_i \in \mathcal{T}$ is tight and $G$ is $l$-connected, we have $d(s, X_i) \leqslant k-l$. So

$$\begin{aligned} d(s, \cup_{i=0}^{m} X_i) &\leqslant d(s, X_0) + (k-l)(k-d(s, X_0)) \\ &= k(k-l) - d(s, X_0)(k-l-1). \end{aligned} \tag{18}$$

Let $M = \{y \in N(s) - x_0 : sx_0, sy \text{ is not admissible}\}$. Since there exist $q \geqslant (k-l+1)(k-1)+4$ edges incident to $s$ which are not admissible with $sx_0$, we can use (18) to deduce that $R := M - \cup_{i=0}^{m} X_i \neq \emptyset$. By Lemma 2.11 and by the choice of $\mathcal{T}$ there exists a family of maximal dangerous sets $\mathcal{W} = \{W_1, \ldots, W_r\}$ such that $x_0 \in W_i$ for all $1 \leqslant i \leqslant r$ and $R \subseteq \cup_{j=1}^{r} W_i$. Let us assume that $\mathcal{W}$ is chosen so that $r$ is as small as possible. By Lemma 5.1(b), $X_0 \subseteq W_i$ for all $1 \leqslant i \leqslant r$. Since $d(s, W_i - X_0) \leqslant k+1-l-d(s, X_0)$, we can use (18) and the fact that $q \geqslant (k-l+1)(k-1)+4$ to deduce that $r \geqslant 2$. For $W_i, W_j \in \mathcal{W}$ we have $W_i^* \cap W_j^* \neq \emptyset$ by Lemma 5.1(c). Since $W_i \cup W_j$ is not dangerous by the maximality of $W_i$, we may apply (9) to obtain

$$k+1+k+1 \geqslant \bar{d}(W_i) + \bar{d}(W_j) \geqslant \bar{d}(W_i \cap W_j) + \bar{d}(W_i \cup W_j) \geqslant k+k+2. \tag{19}$$

Thus, equality holds throughout and $W_i \cap W_j$ is tight. Since $X_0$ is a maximal tight set and $X_0 \subseteq W_i \cap W_j$ we have $X_0 = W_i \cap W_j$. Furthermore, since we have equality in (19), we can use (8) to deduce that $W_j \cap N(W_i) \subseteq N(W_i \cap W_j)$. So $W_j \cap N(W_i) \subseteq N(X_0)$ and, similarly, $W_i \cap N(W_j) \subseteq N(X_0)$. Hence $N(s) \cap W_i \cap N(W_j) \subseteq \cup_{i=0}^{m} X_i$. (Note that every $z \in N(s) \cap N(X_0)$ is contained in one of the $X_i$'s by the criticality of $G+s$.) So by the choice of $\mathcal{W}$, $R \cap W_i \cap W_j^* \neq \emptyset$ and $R \cap W_j \cap W_i^* \neq \emptyset$ follows.

By (10),

$$\begin{aligned} 2k+2 = \bar{d}(W_i) + \bar{d}(W_j) &\geqslant \bar{d}(W_i \cap W_j^*) + \bar{d}(W_i^* \cap W_j) \\ &\quad + d(s, W_i - W_j^*) + d(s, W_j - W_i^*) \geqslant 2k+2 \end{aligned}$$

and so we have equality throughout. Thus all edges from $s$ to $W_i$, other than the single edge $sx_0$, end in $W_i \cap W_j^*$ and $d(s, X_0) = 1$. Hence $R \cap W_j \cap W_i^* = (R \cap W_j) - x_0$. Since $d(s, W_i \cup W_j) = \bar{d}(W_i \cup W_j) - n_G(W_i \cup W_j) \leqslant k + 2 - l$, we have $d(s, (W_i \cup W_j) - X_0) \leqslant k + 2 - l - d(s, X_0)$. We can now use (18) and the fact that $q \geqslant (k-l+1)(k-1)+4$ to deduce that $r \geqslant 3$. Thus $\emptyset \neq (R \cap W_j) - x_0 \subseteq W_j \cap W_i^* \cap W_h^*$ holds for all distinct $i, j, h \in \{1, \ldots, r\}$. Applying Lemma 2.15 we deduce that $K = N_G(X_0)$ is a $(k-1)$-shredder with $r+1$ leaf components $C_0, C_1, \ldots, C_r$ in $G + s$, where $V(C_0) = X_0$ and $V(C_i) = W_i - X_0$ for $1 \leqslant i \leqslant r$.

We complete the proof of the lemma by showing that $M = R$ and hence that $r = q$. Suppose that $M \neq R$. Then $\mathcal{T} \neq \emptyset$ and we may choose $X_1 \in \mathcal{T}$. Since $X_1 \cap N(X_0) \neq \emptyset$, we have $X_1 \cap K \neq \emptyset$. Since $X_1 \cap R = \emptyset$, $N(X_1) \cap C_i \neq \emptyset$ for $0 \leqslant i \leqslant r$. Using $r = |R| \geqslant q - d(s, \cup_{i=0}^{m} X_i)$, and the facts that $d(s, X_0) = 1$, and $q \geqslant (k-l+1)(k-1)+4$, we may use (18) to deduce that $r \geqslant k+2$. This contradicts the fact that $X_1$ is tight since $\bar{d}(X_1) \geqslant n_G(X_1) \geqslant r+1$. □

## 6. Graphs containing shredders with many components

We show in this section, that if $\hat{b}(G)$ and $t(G)$ are large compared to $k$ and $\hat{b}(G) - 1 \geqslant \lceil t(G)/2 \rceil$ then $a_k(G) = \hat{b}(G) - 1$. We need several new observations on $(k-1)$-shredders. We assume throughout this section that $G + s$ is a $k$-critical extension of an $l$-connected graph $G$, and that $K$ is a $(k-1)$-shredder of $G$ satisfying $d(s) \leqslant 2\hat{b}(K) - 2$.

**Lemma 6.1.** *Suppose $\hat{b}(K) \geqslant 4k + 3(k-l) - 1$. Then*

(a) *the number of components $C$ of $G - K$ with $d(s, C) \geqslant 3$ is at most $b(K) - 2k - 1$,*
(b) *$|N(s) \cap K| \leqslant 1$, and*
(c) *if $d(s, x) = j \geqslant 1$ for some $x \in K$ then $k - d_G(x) = j$.*

**Proof.** Let $w$ be the number of components $C$ of $G - K$ with $d(s, C) \geqslant 3$. Then $d(s) \geqslant 3w + (b(K) - w)$. Thus

$$\begin{aligned} 2w &\leqslant d(s) - b(K) \leqslant 2\hat{b}(K) - 2 - b(K) = 2b(K) + 2\delta(K) - 2 - b(K) \\ &= 2b(K) + 3\delta(K) - 2 - \hat{b}(K). \end{aligned}$$

Since $\delta(K) \leqslant k - l$ and $\hat{b}(K) \geqslant 4k + 3(k-l) - 1$, we have $w \leqslant b(K) - 2k - 1$. This proves (a).

Since $G + s$ is a critical extension of $G$, each vertex in $N(s)$ is contained in a tight set of $G + s$. Thus (b) will follow from the next claim.

**Claim 6.2.** *At most one vertex of $K$ belongs to a tight set in $G + s$.*

**Proof.** Suppose that there exist two distinct vertices $x_1, x_2 \in K$ and tight sets $Y_1, Y_2$ in $G+s$ such that $x_1 \in Y_1$, $x_2 \in Y_2$. Let $Y = Y_1 \cup Y_2$ and let $\mathcal{D} = \{C : C \text{ is a component of } G - K, C \cap (Y \cup N(Y)) \neq \emptyset\}$. We have $|\mathcal{D}| \leqslant 2k$, since $\bar{d}(Y) \leqslant \bar{d}(Y_1) + \bar{d}(Y_2) \leqslant 2k$ and for every

$C \in \mathcal{D}$ either $C - Y \neq \emptyset$, in which case $N(Y) \cap C \neq \emptyset$ holds, or $C \subset Y$, in which case $d(s, C \cap Y) \geqslant 1$ holds by (7).

Since $\hat{b}(K) \geqslant 4k + 3(k-l) - 1$ we have $b(K) \geqslant 4k + 2(k-l) - 1$. Thus we may choose a component $C'$ of $G - K$ such that $C' \notin \mathcal{D}$. Then $C' \cap N(Y) = \emptyset$ and hence $x_1, x_2 \notin N(C')$. Hence $n(C') \leqslant k - 3$ and $d(s, C') \geqslant 3$. Since we have at least $b(K) - 2k$ choices for $C'$, this contradicts (a). □

To prove (c), we choose a tight set $X$ containing $x$. By Claim 6.2, $X \cap K = \{x\}$. If $X = \{x\}$ then, since $X$ is tight, we have $d(s, x) = k - d_G(x)$, as required. Thus we may suppose that $X - K \neq \emptyset$. By Lemma 2.4, $d(s, x) = 1$.

We first consider the case when $X$ intersects two distinct components $C_1, C_2$ of $G - K$. Since $N_G(C_1 \cap X) \subseteq C_1 \cup K$ and $N_G(C_1 \cap X) \subseteq N_G(X) \cup \{x\}$, we have

$$\bar{d}(X) \geqslant \bar{d}(C_1 \cap X) - 1 + d(s, x) + d(s, C_2 \cap X) + |N_G(C_2 \cap X) \cap C_2|.$$

If $C_2 \subseteq X$ then $d(s, C_2 \cap X) \geqslant 1$, and if $C_2 \not\subseteq X$ then $|N_G(C_2 \cap X) \cap C_2| \geqslant 1$. Since $d(s, x) = 1$ and $\bar{d}(C_1 \cap X) \geqslant k$, we deduce that $\bar{d}(X) \geqslant k + 1$. This contradicts the fact that $X$ is tight.

Thus, $X$ intersects a unique component $C$ of $G - K$. Let $M = C \cap X$. Then $N_G(M) \subseteq C \cup K$. Since $(N_G(M) - \{x\}) \cup N_G(x) \subseteq N_G(X)$, we may use (7) to obtain

$$\begin{aligned} k = \bar{d}(X) &\geqslant \bar{d}(M) - 1 + d(s, x) + |N_G(x) - (M \cup N_G(M))| \\ &\geqslant k - 1 + d(s, x) + |N(x) - M - N(M)|. \end{aligned}$$

This implies that $N_G(x) \subseteq M \cup N_G(M)$. Therefore $\hat{b}(K) \leqslant b(K) + 1$, and $x \notin N_G(C')$ for every component $C' \neq C$ of $G - K$. Hence $d(s, C') \geqslant k - n_G(C') \geqslant 2$. For $C$ we have $d(s, C) \geqslant 1$ by (7). This gives $d(s) \geqslant 2(b(K) - 1) + 1 + d(s, x) = 2b(K) \geqslant 2\hat{b}(K) - 2$. Thus equality must hold throughout $\hat{b}(K) = b(K) + 1$ and $\delta(K) = 1$. Since $N(s) \cap K = \{x\}$ by (b), we have $k - d_G(x) = \delta(K) = 1 = d(s, x)$. □

We shall use the following construction to augment $G$ with $\hat{b}(G) - 1$ edges in the case when $d(s, K) = 0$ and $b(K) = \hat{b}(G) = b$. Let $C_1, \ldots, C_b$ be the components of $G - K$ and let $w_i = d_{G+s}(s, C_i)$, $1 \leqslant i \leqslant b$. Note that $w_i \geqslant 1$ by (7). Since $d(s) \leqslant 2b - 2$, there exists a tree $T$ on $b$ vertices $C_1, C_2, \ldots, C_b$ with degree sequence $d_1, \ldots, d_b$ such that $d_i \geqslant w_i$, for $1 \leqslant i \leqslant b$. (It will be clear from the context whether the label $C_i$ refers to a component of $G - K$ or a vertex of $T$.) Let $F$ be a set of edges joining vertices of $N_{G+s}(s)$ with $d_F(v) \geqslant d_{G+s}(s, v)$ for every $v \in V(G)$ and such that the graph obtained from $(V - K, F)$ by contracting $C_1, C_2, \ldots, C_b$ to single vertices is $T$. Thus $|F| = |E(T)| = b - 1$. We shall say that $G + F$ is a *forest augmentation* of $G$ with respect to $K$ and $G + s$, and prove that $G + F$ is $k$-connected. Note that since $d_{G+s}(s, K) = 0$, there are no $k$-deficient fragments of $G$ contained in $K$ by (7).

**Lemma 6.3.** *Suppose $d(s, K) = 0$ and let $G + F$ be a forest augmentation of $G$ with respect to $K$ and $G + s$. If $X$ is a $k$-deficient fragment in $G + F$ then $|X \cap K| \geqslant 2$.*

**Proof.** We proceed by contradiction. Suppose $X$ is a $k$-deficient fragment in $G + F$ with $|X \cap K| \leqslant 1$. Let $X^* = V - X - N_{G+F}(X)$. Replacing $X$ by $X^*$ if necessary, we may

assume that

$$|X^* \cap K| \geqslant |X \cap K|. \tag{20}$$

We first suppose that $L \subseteq X$ for some leaf $L$ of $T$. Since $d(s, L) \leqslant d_T(L)$, $L$ is a leaf component of $K$ in $G+s$. Hence $K \subseteq X \cup N_G(X)$ by Lemma 2.14. It follows that $X^* \cap K = \emptyset$. Hence $X \cap K = \emptyset$ by (20) and $K \subseteq N_G(X)$. If $X$ properly intersects some component $C_i \neq L$ of $G - K$ then $n_G(X) \geqslant k$ follows, contradicting the fact that $X$ is $k$-deficient in $G+F$. Since $X^* \neq \emptyset$, there exists a component $C$ of $G-K$ for which $C \cap X = \emptyset$. Choose a path $P$ from $L$ to $C$ in $T$. Let $C'$ be the first component for which the edge on $P$ which enters $C'$ corresponds to an edge in $F$ which connects $X$ to $V - X$. For this component we have $|N_{G+F}(X) \cap C'| \geqslant 1$, so $n_{G+F}(X) \geqslant |K| + 1 = k$, as required. Thus we may assume that

$$L \cap X \neq L \quad \text{for each leaf } L \text{ of } T. \tag{21}$$

Choose a component $D$ of $G - K$ such that $D \cap X \neq \emptyset$ and let $R$ be the set of edges of $F$ which are incident with $X \cap D$. Let $e_1, \ldots, e_r$ be the edges incident to $D$ in $T$, which correspond to the edges in $R$. Choose $r$ longest paths $P_1, \ldots, P_r$ in $T$ starting at $D$ and containing the edges $e_1, \ldots, e_r$. Let $A$ be the set of all paths $P_j$, $1 \leqslant j \leqslant r$, which contain an edge $C_s C_t$ corresponding to an edge $u_j v_j$ in $F$ with $u_j \in C_s \cap X$ and $v_j \in C_t - X$. For every such path we have $v_j \in N_{G+F}(X)$. Let $A' = \{v_j : P_j \in A\}$. Let $B$ be the set of paths $P_j$, $1 \leqslant j \leqslant r$, which do not belong to $A$ and choose $P_j \in B$. Since the first edge of $P_j$ corresponds to an edge in $F$ which is incident to $D \cap X$, every edge of $P_j$ corresponds to an edge of $F$ joining two vertices of $X$. In particular, the last edge of $P_j$ is incident to a leaf $L_j$ of $T$ which is distinct from $D$ and for which $X \cap L_j \neq \emptyset$. Since $X \cap L_j \neq L_j$ by (21), we may choose a vertex $w_j \in N_G(X) \cap L_j$. Let $B' = \{w_j : P_j \in B\}$. Clearly, $|A| = |A'|$, $|B| = |B'|$ and $|A| + |B| = r$. The above observations imply that

$$A' \cup B' \cup (N_G(D \cap X) - (X \cap K)) \cup (N_G(X \cap K) - X) \subseteq N_{G+F}(X). \tag{22}$$

Since $G+s$ is $(k, s)$-connected, $r \geqslant k - n_G(D \cap X)$. Since $A'$, $B'$, $N_G(D \cap X)$ are pairwise disjoint, we may deduce that, if $X \cap K = \emptyset$, then $X$ is not $k$-deficient in $G + F$. Hence $X \cap K = \{x\}$ for some $x \in K$.

Let $L$ be a leaf of $T$ distinct from $D$. Then $L$ is a leaf component of $K$ in $G + s$ so $N_G(x) \cap L \neq \emptyset$. Hence either $(N_G(x) \cap L) - X \neq \emptyset$, or $X \cap L \neq \emptyset$ and, by (21), $N_G(X) \cap L \neq \emptyset$. It follows that, in both cases, we may choose $y \in N_G(X) \cap L$. Thus

$$A' \cup B' \cup (N_G(D \cap X)) \cup \{y\} \subseteq N_{G+F}(X).$$

Clearly $y \notin N_G(D \cap X)$. Since $X$ is $k$-deficient in $G + F$, we must have $y \in A' \cup B'$. Thus

$$L \cap (A' \cup B') \neq \emptyset \quad \text{for each leaf } L \text{ of } T \text{ distinct from } D. \tag{23}$$

The definitions of $A'$, $B'$ now imply that the paths $P_j$, $1 \leqslant j \leqslant r$, cover $T$, and hence that each edge of $F$ which is incident with $D$, is incident with $D \cap X$. Since $V(F) = N_{G+s}(s)$, we have $N_{G+s}(s) \cap D \subseteq X$. Since $D$ can be any component of $G - K$ which intersects $X$ we may deduce that

$$\text{If } D \cap X \neq \emptyset \quad \text{for some component } D \text{ of } G - K \quad \text{then } N_{G+s}(s) \cap D \subseteq X. \tag{24}$$

Suppose $C$ is a component of $G-K$ with $C\cap X=\emptyset$. Then (23) implies that $C$ is a leaf of $T$ and $A'\cap C\neq\emptyset$. Furthermore, the argument used in the derivation of (23) gives $A'\cap C=\{y\}=N_G(x)\cap L$. Since $y\in A'\subseteq N_{G+s}(s)$, $y$ is the unique neighbour of $s$ in $C$. Thus

$$\text{If}\quad C\cap X=\emptyset\quad\text{for some component } C \text{ of } G-K \text{ then } N_{G+s}(s)\cap C\subseteq N_G(x). \tag{25}$$

Properties (24) and (25) imply that $N_{G+s}(s)\subseteq X\cup N_G(X)$. Thus $N_{G+s}(s)\cap X^*=\emptyset$ and $\bar{d}(X^*)\leqslant n_G(X)<k$. This contradicts the $(k,s)$-connectivity of $G+s$ and completes the proof of Lemma 6.3. □

**Lemma 6.4.** *Suppose $d(s,K)=0$ and $\hat{b}(K)=b(K)\geqslant 4k+3(k-l)-1$. Let $G+F$ be a forest augmentation of $G$ with respect to $K$ and $G+s$. Then $G+F$ is $k$-connected.*

**Proof.** We proceed by contradiction. Let $X$ be a $k$-deficient fragment in $G+F$. Then $X^*$ is also $k$-deficient so by Lemma 6.3, $|X\cap K|\geqslant 2$ and $|X^*\cap K|\geqslant 2$. Since $|V-(K\cup X\cup X^*)|\leqslant V-(X\cup X^*)|\leqslant k-1$, there are at least $b_G(K)-(k-1)$ components $C$ of $G-K$ which are contained in $X\cup X^*$. There is no edge from $X$ to $X^*$ in $G+F$, so for each such component either $C\subseteq X$ or $C\subseteq X^*$ holds. Thus we have $N_G(C)\subseteq K-X^*$ or $N_G(C)\subseteq K-X$, and so $n_G(C)\leqslant k-3$. Hence $d_G(s,C)\geqslant 3$ by (7). This contradicts Lemma 6.1(a). □

Our final step is to show how to augment $G$ with $\hat{b}(K)-1$ edges when $d(s,K)\neq 0$. In this case, Lemma 6.1(b) implies that there is exactly one vertex $x\in K$ which is adjacent to $s$. We use the next lemma to split off all edges from $s$ to $x$ and hence reduce to the case when $d(s,K)=0$.

**Lemma 6.5.** *Suppose $d(s,x)\geqslant 1$ for some $x\in K$ and $d(s)\geqslant(k+1)(k-l+1)$. Then there exists a sequence of $d(s,x)$ admissible splits at $s$ which split off all edges from $s$ to $x$.*

**Proof.** We have $d(s,x)\leqslant k-l$. Suppose we get stuck after splitting off some copies of $sx$, i.e. we obtain a graph $H+s$ where some edge $sx$ cannot be split off. Since $d_{H+s}(s)\geqslant d_{G+s}(s)-2(k-l-1)\geqslant(k-l+1)(k-1)+4$, we can use Lemma 5.3 to deduce that there is a $(k-1)$-shredder $K'$ in $H$ with $b_H(K')=d_{H+s}(s)$ and with $x$ in one of the components of $H-K'$. Let $u,v$ be two neighbours of $s$ in $H$ distinct from $x$ and let $C_u$ and $C_v$ be the components of $H-K'$ containing $u$ and $v$, respectively. By Lemma 2.14, there exist $k-1$ openly disjoint paths between $u$ and $v$ in $H$ containing only vertices of $C_u$, $C_v$ and $K'$, and hence avoiding $x$. Since all edges of $E(H)-E(G)$ are incident with $x$, these paths exist in $G$ as well.

Since $b_G(K)\geqslant\hat{b}_G(K)-(k-l)\geqslant(d_{G+s}(s)+2)/2-(k-l)\geqslant k+1\geqslant d_{G+s}(s,V-x)-d_{H+s}(s,V-x)+2$, and each component of $G-K$ contains a neighbour of $s$ in $G$, we can choose the two neighbours $u,v$ of $s$ in $H+s$ to belong to different components in $G-K$. But for such a choice of $u,v$ there do not exist $k-1$ disjoint paths from $u$ to $v$ in $G-x$, contradicting the above claim. □

We can now prove our augmentation result for graphs $G$ for which $\hat{b}(G)$ is large.

**Theorem 6.6.** *Suppose that $G$ is $l$-connected, $\hat{b}(G) \geqslant 4k+4(k-l)-1$, $t(G) \geqslant (k+1)(k-l+1)$ and $\hat{b}(G)-1 \geqslant \lceil t(G)/2 \rceil$. Then $a_k(G) = \hat{b}(G)-1$.*

**Proof.** Let $G+s$ be a $k$-critical extension of $G$. Then $d(s) = t(G)$ by Lemma 5.2. Let $K$ be a $(k-1)$-shredder in $G$ with $\hat{b}(K) = \hat{b}(G)$. Then $2\hat{b}(K)-2 \geqslant t(G) = d(s)$. Suppose $d(s, K) = 0$. Then $\hat{b}(G) = b(K)$. Let $G+F$ be a forest augmentation of $G$ with respect to $K$ and $G+s$. Then $|F| = b(G)-1$ and by Lemma 6.4, $G+F$ is the required $k$-augmentation of $G$. Hence we may assume that $d(s, K) \geqslant 1$.

Applying Lemma 6.1(c), we deduce that $\delta_G(K) = d_{G+s}(s, K) = d_{G+s}(s, x)$ for some $x \in K$. By Lemma 6.5, we can construct a graph $H+s$ by performing a sequence of $d_G(s, x)$ admissible splits at $s$ which split off all edges from $s$ to $x$ in $G+s$. Since we only split edges incident to $x \in K$ to form $H+s$, we have $G-K = H-K$ and so $b_G(K) = b_H(K)$. Hence

$$\begin{aligned} d_{H+s}(s) &= d_{G+s}(s) - 2d_{G+s}(s, x) = d_{G+s}(s) - 2\delta_G(K) \\ &\leqslant 2\hat{b}_G(K) - 2 - 2\delta_G(K) = 2b_G(K) + 2\delta_G(K) - 2 - 2\delta_G(K) \\ &= 2b_G(K) - 2 = 2b_H(K) - 2. \end{aligned}$$

Thus we have $d_{H+s}(s) \leqslant 2b_H(K)-2$, and $d_{H+s}(s, K) = 0$. Also note that the splittings add a set $F_0$ of $\delta_G(K)$ new edges to $G$ to form $H$, and that $b_H(K) = b_G(K) \geqslant b^*_G(K) - (k-l) \geqslant 4k+3(k-l)-1$. Let $H+F_1$ be a forest augmentation of $H$ with respect to $K$ and $H+s$. Then $|F_1| = b_H(K)-1 = b_G(K)-1$, and $H+F_1$ is $k$-connected by Lemma 6.4. Thus, $G+F_0+F_1 = H+F_1$ is the required $k$-augmentation of $G$ with $\delta_G(K)+b_G(K)-1 = \hat{b}_G(K)-1$ edges. □

We will apply Theorem 6.6 to graphs which do not satisfy $\hat{b}(G)-1 \geqslant \lceil t(G)/2 \rceil$ using saturating edges. Recall that a set $F$ of new edges is saturating for $G$ if $t(G+F) = t(G)-2|F|$.

**Lemma 6.7.** *If $F$ is a saturating set of edges for an $l$-connected graph $G$ with $\hat{b}(G+F) \geqslant 4k+4(k-l)-1$, $t(G+F) \geqslant (k+1)(k-l+1)$, and $\hat{b}(G+F)-1 = \lceil t(G+F)/2 \rceil$, then $a_k(G) = \lceil t(G)/2 \rceil$.*

**Proof.** By Theorem 6.6 the graph $G+F$ can be made $k$-connected by adding a set $F'$ of $\lceil t(G+F)/2 \rceil$ edges. Since $F$ is saturating, we have $t(G) = t(G+F)+2|F|$. Therefore, the set $F \cup F'$ is an augmenting set for $G$ of size $\lceil t(G)/2 \rceil$. Since $a_k(G) \geqslant \lceil t(G)/2 \rceil$, the lemma follows. □

## 7. Augmenting connectivity by at least two

Throughout this section, we assume that $G = (V, E)$ is an $l$-connected graph on at least $k+1$ vertices and that $l \leqslant k-2$. We shall show that if $a_k(G)$ is large compared to $k$, then $a_k(G) = \max\{\hat{b}(G)-1, \lceil t(G)/2 \rceil\}$. Our proof uses Theorems 4.17 and 6.6. We shall show that if $a_k(G)$ is large then either we can add a saturating set of edges $F$ so that $G+F$ is $(k-1)$-connected, or else $G$ has a $(k-2)$-shredder with many components. If the latter

occurs then we show directly that we can make $G$ $k$-connected by adding $\lceil t(G)/2 \rceil$ edges. We will occasionally consider two different extensions of the same graph $H$. To distinguish between them we shall label one of them as $H + s$ and the other as $H \oplus s$.

Let $G + s$ be a $k$-critical extension of $G$. Construct a $(k-1)$-critical extension $G \oplus s$ of $G$ from $G + s$ by deleting a set of edges incident to $s$. Let $f = (k-l+1)(k-1)+4$ be the bound on the number of non-admissible pairs containing a fixed edge given by Lemma 5.3.

**Lemma 7.1.** *If $d_{G+s}(s) \geqslant f(k-l+1)/(k-l)$ then $d_{G+s}(s) - d_{G\oplus s}(s) \geqslant d_{G+s}(s)/(k-l+1)$.*

**Proof.** If $d_{G\oplus s}(s) \leqslant f$ then the lemma is trivial. Otherwise by Lemma 5.2(a) there exists a family $\mathcal{F}$ of pairwise disjoint $(k-1)$-deficient fragments in $G$ such that $d_{G\oplus s}(s) = \sum_{\mathcal{F}}(k-1-n(X))$. Since $G + s$ is $(k,s)$-connected we have $d_{G+s}(s) \geqslant \sum_{\mathcal{F}}(k-n(X))$. Hence $d_{G+s}(s) \geqslant d_{G\oplus s}(s) + |\mathcal{F}|$. Since $d_{G\oplus s}(s, X) \leqslant k-l$ for each $X \in \mathcal{F}$, we have $|\mathcal{F}| \geqslant d_{G\oplus s}(s)/(k-l)$. Thus, $d_{G+s}(s) \geqslant d_{G\oplus s}(s) + d_{G\oplus s}(s)/(k-l) = (k-l+1)d_{G\oplus s}(s)/(k-l)$. Hence, $d_{G+s}(s) - d_{G\oplus s}(s) \geqslant d_{G+s}(s)/(k-l+1)$. □

We next perform a sequence of $(k-1)$-admissible splits at $s$ in $G \oplus s$ and obtain $G_1 \oplus s$. We do this according to the following rules. If $d_{G\oplus s}(s) \leqslant 2f$ then we put $G_1 \oplus s = G \oplus s$. If $d_{G\oplus s}(s) \geqslant 2f+1$ then we perform $(k-1)$-admissible splits until either $d_{G_1\oplus s}(s) \leqslant 2f$, or $d_{G_1\oplus s}(s) \geqslant 2f+1$ and there is no $(k-1)$-admissible split at $s$ in $G_1 \oplus s$. We then add all the edges of $(G+s) - (G \oplus s)$ to $G_1 \oplus s$ and obtain $G_1 + s$. We shall refer to the edges of $(G+s) - (G \oplus s)$ as *new edges* of $G_1 + s$.

**Lemma 7.2.** *If $d_{G+s}(s) \geqslant f(k+l-1)$ then $G_1 + s$ is a $k$-critical extension of $G_1$.*

**Proof.** Suppose $G_1 + s$ is not $(k,s)$-connected. If $d_{G_1\oplus s}(s) \leqslant f$ then $G_1 \oplus s = G \oplus s$ and $G_1 + s = G + s$, contradicting the assumption that $G + s$ is $(k,s)$-connected. Hence, $d_{G_1\oplus s}(s) \geqslant f+1$. Choose a minimal fragment $X$ of $G_1$ such that $\bar{d}_{G_1+s}(X) < k$. Since $\bar{d}_{G_1\oplus s}(X) \geqslant k-1$ we have $\bar{d}_{G_1+s}(X) = k-1 = \bar{d}_{G_1\oplus s}(X)$ and no new edge of $G_1 + s$ is incident with $X$. Since $\bar{d}_{G+s}(X) \geqslant k$, there exists an edge $sx$ in $G + s$ with $x \in X$. Then $sx \in E(G \oplus s)$, since no new edge is incident with $X$. Hence, $sx$ is $(k-1)$-critical in $G \oplus s$ so there exists a minimal tight set $Y$ with $x \in Y$ and $\bar{d}_{G\oplus s}(Y) = k-1$. Hence $\bar{d}_{G_1\oplus s}(Y) = k-1$. Working in $G_1 \oplus s$ we may use Lemma 5.1(a) to deduce that $\bar{d}_{G_1\oplus s}(X \cap Y) = k-1$. Since there are no new edges incident to $X$, this gives $\bar{d}_{G_1+s}(X \cap Y) = k-1$. Now the minimality of $X$ implies that $X \subseteq Y$. Since $\bar{d}_{G\oplus s}(Y) = \bar{d}_{G_1\oplus s}(Y)$, we now deduce that $\bar{d}_{G\oplus s}(X) = \bar{d}_{G_1\oplus s}(X)$. Thus $\bar{d}_{G\oplus s}(X) = k-1$ and the minimality of $Y$ gives $X = Y$. Since no new edge is incident with $X$ this gives $\bar{d}_{G+s}(Y) = \bar{d}_{G\oplus s}(Y) = k-1$. Thus $Y$ is $k$-deficient in $G + s$, contradicting the fact that $G + s$ is $(k,s)$-connected.

Criticality of $G_1 + s$ follows from the criticality of $G + s$, since splitting off pairs of edges from $s$ cannot increase $\bar{d}(X)$ for any $X \subseteq V$. □

Using Lemma 5.3, we can deduce that either $d_{G_1\oplus s}(s)$ is small or else there exists a $(k-2)$-shredder $K$ in $G_1$ such that $G_1 - K$ has $d_{G_1\oplus s}(s)$ components. In the first case, we show that there exists a sequence of $k$-admissible splits in $G_1 + s$ such that, in the resulting graph $G_1' + s$, $G_1'$ is $(k-1)$-connected and then apply Theorems 6.6 and 4.17.

We accomplish this by ensuring that $\kappa(x, y, G'_1) \geqslant k-1$ for every $x, y \in N_{G_1 \oplus s}(s)$. This is possible since there are many new edges and hence $d_{G_1+s}(s)$ is large compared to $d_{G_1 \oplus s}(s)$. We proceed incrementally using the lemmas below. In the second case, we show directly that we can make $G$ $k$-connected by adding $\lceil t(G)/2 \rceil$ edges.

Henceforth, we shall assume that $G'_1 + s$ is obtained from $G_1 + s$ by performing a sequence of $k$-admissible splits and that $T \subseteq V$ is a cover of all $(k-1)$-deficient fragments of $G'_1$. (In proving the theorem we will take $T = N_{G_1 \oplus s}(s)$.) Let $|T| = \tau$.

**Lemma 7.3.** *If $\kappa(u, v, G'_1) \geqslant k-1$ for all $u, v \in T$ then $G'_1$ is $(k-1)$-connected.*

**Proof.** Suppose $G'_1$ has a fragment $X$ with $n(X) \leqslant k-2$. Then we may choose $u \in X \cap T$ and $v \in X^* \cap T$, contradicting the fact that $\kappa(u, v) \geqslant k-1$. □

**Lemma 7.4.** *Let $sz, sw \in E(G'_1 + s)$ and suppose that the pair $sz, sw$ is not $k$-admissible. If $\kappa(z, w, G'_1) \leqslant k-2$ then there are at most $f$ pairs of edges $sz, sx$ which are not $k$-admissible in $G'_1 + s$.*

**Proof.** Let $R = \{sx : sz, sx \text{is not } k-\text{admissible in} G'_1 + s\}$. Suppose that $r = |R| > f$. Then by Lemma 5.3, there is a $(k-1)$-shredder $K$ in $G'_1$ with $r+1$ leaf components in $G'_1 + s$ such that $z$ as well as each vertex $x$, $sx \in R$, is in one of these components. By Lemma 2.14, $\kappa(z, x) \geqslant k-1$ for every such $x$. Taking $x = w$ gives a contradiction. □

**Lemma 7.5.** *Suppose that $d_{G'_1+s}(s) \geqslant (f+1)(2(k-2)(f+2)+\tau) + (k-2)(k-l-2)$. Choose $u, v \in T$ and suppose that $\kappa(u, v, G'_1) = m \leqslant k-2$. Then there exists a sequence of at most two $k$-admissible splits such that, for the resulting graph $G''_1 + s$, we have $\kappa(u, v, G''_1) = m+1$.*

**Proof.** Let $X_u$ and $X_v$ be the smallest sets which contain $u$ and $v$, respectively, separate $u$ and $v$, and have precisely $m$ neighbours. It is well-known that these unique smallest separators exist. Since $n_{G'_1}(X_u) = n_{G'_1}(X_v) = m \leqslant k-2$, there exist vertices $x \in X_u \cap N_{G'_1+s}(s)$ and $y \in X_v \cap N_{G'_1+s}(s)$. It is also known that there exist $m$ paths $P_1, \ldots, P_m$ from $u$ to $v$, and two paths $P_0$ and $P_{m+1}$, one from $u$ to $x$ and the other from $v$ to $y$ such that all these $m+2$ paths are vertex-disjoint apart from at $u$ and $v$. (Note that $u = x$ or $v = y$ is possible.) We may assume, without loss of generality, that $N_{G'_1+s}(s) \cap (V(P_0) - x) = \emptyset$ and $N_{G'_1+s}(s) \cap (V(P_{m+1}) - y) = \emptyset$. Let $Q = \cup_{i=1}^{m} V(P_i) - \{u, v\}$. If the pair $sx, sy$ is $k$-admissible, we have $\kappa(u, v, G'_1 + xy) \geqslant m+1$, as required. If not, we need to choose $k$-admissible pairs in a more complicated way, as in the proof of Lemma 4.13.

Suppose there exists a path $P_i$ $(1 \leqslant i \leqslant m)$ with $d_{G'_1+s}(s, V(P_i)) \geqslant 2f + (k-l) + 1$. By Lemma 7.4 we may choose an admissible pair $sx, sa$ in $G'_1 + s$ such that $a$ is a neighbour of $s$ on $P_i$ as close to $v$ as possible. Lemma 7.4 implies that there are at most $f$ edges from $s$ to $P_i(a, v]$. If $\kappa(u, v, G'_1 + xa) \geqslant m+1$ then we are done. Otherwise we may split $sy, sb$ in $G'_1 + s + xa$, where $b$ a neighbour of $s$ on $P_i$ as close to $u$ as possible. Lemma 7.4 implies that there are at most $f$ edges from $s$ to $P_i[u, b)$. Since $d(s, w) \leqslant k-l$ for each $w \in V(P_i)$, the vertices $x, b, a, y$ appear on $P_i$ in this order. Hence, there exist two vertex-disjoint $uv$-paths on vertex set $V(P_i) \cup V(P_0) \cup V(P_{m+1})$, showing $\kappa(u, v, G'_1 + xa + yb) \geqslant m+1$,

as required. Thus we may assume that no such path exists and hence $d_{G'_1+s}(s, V-Q) > d_{G'_1+s}(s) - m(2f+k-l) \geqslant (f+1)(2(k-2)(f+1)+\tau)$.

Let $H$ be the graph obtained from $G'_1 - Q$ by deleting any edges joining $u$ and $v$. Let $C_0, C_1, \ldots, C_{p+1}$ be the components of $H$ which each contain at least one neighbour of $s$, where $u, x \in V(C_0)$ and $v, y \in V(C_{p+1})$. Suppose $d(s, C_j) \geqslant f+2$ for some $1 \leqslant j \leqslant p$. We may perform a $k$-admissible split $sx, sa$ for some $a \in C_j$, and then a $k$-admissible split $sy, sb$ in $G'_1 + s + sa$ for some $b \in C_j$. These admissible pairs exist by Lemma 7.4. It is easy to see that $\kappa(u, v, G'_1 + xa + yb) \geqslant m+1$, as required. Thus we may assume that no such component exists. Similarly, if $d(s, C_0) \geqslant f+1$, then we may split $sy, sc$ for some $c \in C_0$ which is admissible with $sy$ in $G'_1 + s$, and we again have $\kappa(u, v, G'_1 + yc) \geqslant m+1$, as required. A similar construction holds if $d(s, C_{p+1}) \geqslant f+1$. Hence we have at least $d(s, V-Q)/(f+1) \geqslant 2(k-2)(f+1)+\tau$ components in $H$, each containing at least one neighbour of $s$.

Since each component $C_i$ with $n_{G'_1}(C) \leqslant k-2$ must contain a vertex from $T$, and $u, v \in T$, there are at least $2(k-2)(f+1)$ components $C_i$, $1 \leqslant i \leqslant p$, with at least $k-1$ attachments on $Q$. Since $m \leqslant k-2$, we have at least $2f+2$ components $D_1, \ldots, D_r$ which have two attachments on the same path, $P_1$ say. We now proceed as in the final part of the proof of Lemma 4.13. Let $a_j$ be the attachment of $D_j$ on $P_1$ closest to $u$ for $1 \leqslant j \leqslant r$. We first pick a $D_i$ where $a_i$ is among the $f+1$ attachment vertices $a_j$ closest to $u$ on $P_1$ and we choose a $k$-admissible pair $sy, sb$ with $b \in D_i$. This pair exists by Lemma 7.4. Then we pick a $D_h$ where $a_h$ is among the $f+1$ attachment vertices $a_j$ closest to $v$ on $P_1$ and we choose a $k$-admissible pair $sx, sa$ with $a \in D_h$. This pair exists by Lemma 7.4. Note that $a_i$ either occurs before $a_h$ on $P_1$ or $a_i = a_h$. Hence, using the fact that the components $D_j$ have at least two attachments on $P_1$ and by the choice of $a_i, a_h$, there exist two openly disjoint $uv$-paths in $G'_1 + xa + yb$, using vertices from $V(P_1) \cup V(P_0) \cup V(P_{m+1}) \cup D_i \cup D_h$ only. Therefore $\kappa(u, v, G'_1 + xa + yb) \geqslant m+1$, as required. □

Applying this lemma iteratively to all pairs of vertices in $T$, starting with $G'_1 + s = G_1 + s$ and using the fact that $f$ is a decreasing function of $l$, we obtain

**Corollary 7.6.** *Suppose that*

$$d_{G_1+s}(s) \geqslant (f+1)(2(k-2)(f+2)+\tau) + (k-2)(k-l-2) + 2\tau^2(k-l-1).$$

*Then there exists a sequence of at most $\tau^2(k-l-1)$ $k$-admissible splits such that, for the resulting graph $G'_1 + s$, we have $\kappa(G'_1) \geqslant k-1$.*

**Theorem 7.7.** *If $G$ is $l$-connected and $a_k(G) \geqslant 10(k-l+2)^3(k+1)^3$ then $a_k(G) = \max\{\hat{b}(G)-1, \lceil t(G)/2 \rceil\}$.*

**Proof.** We have $d_{G+s}(s) = t(G) \geqslant a_k(G)+1 \geqslant 10(k-l+2)^3(k+1)^3$ by Lemmas 2.10 and 5.2. If $\hat{b}(G)-1 \geqslant \lceil t(G)/2 \rceil$ then $a_k(G) = \hat{b}(G)-1$ by Theorem 6.6 and we are done. Thus we may assume that $\lceil t(G)/2 \rceil \geqslant \hat{b}(G)$ holds. We shall show that $a_k(G) = \lceil t(G)/2 \rceil$. We construct $G \oplus s$, $G_1 \oplus s$, and $G_1 + s$ as above. By Lemma 7.2, $G_1$ is obtained from $G$ by adding a saturating set $F$ of edges. Note that adding a saturating edge to a graph $G_0$ reduces $\lceil t(G_0)/2 \rceil$ by exactly one and $\hat{b}(G_0)$ by at most one. Thus, if $\lceil t(G+F)/2 \rceil \leqslant \hat{b}(G+F)-1$,

then there exists $F' \subseteq F$ such that $\lceil t(G+F')/2\rceil = \hat{b}(G+F')-1$ and the theorem follows by applying Lemma 6.7. Hence, we may assume that $\lceil t(G_1)/2\rceil \geqslant \hat{b}(G_1)-1$. We have

$$t(G_1) = d_{G_1+s}(s) \geqslant d_{G+s}(s) - d_{G\oplus s}(s) \geqslant 10(k-l+2)^2(k+1)^3 \tag{26}$$

by Lemma 7.1. Using Lemma 5.3, we either have $d_{G_1\oplus s}(s) \leqslant 2f$ or else $d_{G_1\oplus s}(s) \geqslant 2f+1$ and there exists a $(k-2)$-shredder $K$ in $G_1$ such that $b_{G_1}(K) = d_{G_1\oplus s}(s)$.

*Case* 1: $d_{G_1\oplus s}(s) \leqslant 2f$.

Let $T = N_{G_1\oplus s}(s)$. Then $|T| = \tau \leqslant 2f$. Corollary 7.6 and the fact that $f \leqslant (k-l+1)(k+1)-2$ imply that there exists a sequence of at most $4(k-l+1)^3(k+1)^2$ $k$-admissible splits in $G_1+s$ such that, for the resulting graph $G_1'+s$, we have $\kappa(G_1') \geqslant k-1$. Note that $d_{G_1'+s}(s) \geqslant 2(k-l+2)^2(k+1)^3$, by (26). Thus there exists a saturating set of edges $F$ for $G$ such that $G_1' = G+F$ is $(k-1)$-connected and $t(G+F) \geqslant 2(k-l+2)^2(k+1)^3$. As above, we may assume that $\lceil t(G+F)/2\rceil \geqslant \hat{b}(G+F)-1 \geqslant b(G+F)-1$ (otherwise we are done by Lemma 6.7). Since $G+F$ is $(k-1)$-connected, we can apply Theorem 4.17 to deduce that $a_k(G+F) = \lceil t(G+F)/2\rceil$. Using (1) and the fact that $t(G) = t(G+F)+2|F|$ we have $a_k(G) = \lceil t(G)/2\rceil$, as required.

*Case* 2: $d_{G_1\oplus s}(s) \geqslant 2f+1$ and there is no $(k-1)$-admissible split at $s$ in $G_1 \oplus s$.

By Lemma 5.3, there exists a $(k-2)$-shredder $K$ in $G_1$ such that $b_{G_1}(K) = d_{G_1\oplus s}(s)$ and hence each component of $G_1 - K$ is a leaf component. Using Lemma 2.14, and the fact that $N_{G_1\oplus s}(s)$ covers all $(k-1)$-deficient fragments $X$ in $G_1$, we deduce:

**Claim 7.8.** *$G_1$ is $(k-2)$-connected.*

Since $G_1+s$ is $k$-critical, Claim 7.8 and Lemma 2.4 imply:

**Claim 7.9.** *For all $v \in V$ we have $d_{G_1+s}(s,v) \leqslant 2$. Furthermore, $d_{G_1+s}(s,v) = 2$ if and only if $d_{G_1}(v) = k-2$.*

Let $G_2+s$ be the graph obtained from $G_1+s$ by splitting off as many $k$-admissible pairs of edges $sx, sy$ as possible in $G_1+s$ such that $x$ and $y$ belong to the same component of $G_1-K$. Then $G_2+s$ is a $k$-critical extension of $G_2$. Let $C_1, C_2, \ldots, C_r$ be the components of $G_2-K$. Note that these components have the same vertex sets as the components of $G_1-K$ and hence

$$r = d_{G_1\oplus s}(s) \geqslant 2f+1. \tag{27}$$

Let $d_{G_2+s}(s, C_i) = d_i$. Relabelling if necessary, we have $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_r$.

**Claim 7.10.** $d_{G_2+s}(s, K) = 0$.

**Proof.** Suppose $G_2+s$ has an edge $sx$ with $x \in K$. By criticality there exists a fragment $X$ of $G_2$ such that $x \in X$ and $\bar{d}_{G_2+s}(X) = k$. Since, by Claim 7.8, $x \in N_{G_1}(C_i)$ for all $1 \leqslant i \leqslant r$, we have $x \in N_{G_2}(C_i)$. Hence either $N_{G_2}(X) \cap C_i \neq \emptyset$, or $C_i \subseteq X$ and $d_{G_2+s}(s, X \cap C_i) \geqslant 1$, for all $1 \leqslant i \leqslant r$. Thus $\bar{d}_{G_2+s}(X) \geqslant r > k$, a contradiction. □

Using Lemma 6.7 we may suppose that

$$\hat{b}(G_2) \leqslant \lceil t(G_2)/2 \rceil = \lceil d_{G_2+s}(s)/2 \rceil. \tag{28}$$

**Claim 7.11.** $d_1 \leqslant (\sum_{i=2}^{r} d_i) - 1$.

**Proof.** Suppose $d_1 \geqslant (\sum_{i=2}^{r} d_i)$. Since $d_1 + (\sum_{i=2}^{r} d_i) = d_{G_2+s}(s) \geqslant r \geqslant 2f + 1$ by Claim 7.10 and (27), we have $d_1 \geqslant f + 1$. Since there is no $k$-admissible pair of edges joining $s$ to $C_1$ in $G_2 + s$, it follows from Lemma 5.3 that there is a $(k-1)$-shredder $\hat{K}$ in $G_2$ with each of the $d_1$ neighbours of $s$ in $C_1$ in distinct components of $G_2 - \hat{K}$ and at least one other component containing the remaining neighbours of $s$ in $G_2 + s$. Thus $b(G_2) \geqslant d_1 + 1$, and $\hat{b}(G_2) \geqslant b(G_2) \geqslant d_1 + 1 \geqslant (d_{G_2+s}(s)/2) + 1$. This contradicts (28). □

**Claim 7.12.** *Suppose $X$ is a fragment in $G_2$ with $|X \cap K| \leqslant |X^* \cap K|$.*

(a) *If $n_{G_2}(X) = k - 2$, then either $X = C_{i_1} \cup C_{i_2} \cup \ldots \cup C_{i_p}$ for some $\{i_1, i_2, \ldots, i_p\} \subset \{1, 2, \ldots, r\}$; or $X = Z_i \subset C_i$ for some $1 \leqslant i \leqslant r$.*

(b) *If $n_{G_2}(X) = k - 1$, then either $X = Z_{i_1} \cup C_{i_2} \cup \ldots \cup C_{i_p}$ for some $\{i_1, i_2, \ldots, i_p\} \subseteq \{1, 2, \ldots, r\}$ and $Z_{i_1} \subseteq C_{i_1}$; or $X = Z_{i_1} \cup Z_{i_2}$ for some $1 \leqslant i_1 < i_2 \leqslant r$, $Z_{i_1} \subseteq C_{i_1}$, $Z_{i_2} \subseteq C_{i_2}$, and $n_{G_2}(Z_{i_1}) = k - 2 = n_{G_2}(Z_{i_2})$.*

**Proof.** Suppose $X \cap K \neq \emptyset$. Then $X^* \cap K \neq \emptyset$. Since $N_{G_2}(C_i) = K$ by Claim 7.8, it follows that $C_i \not\subseteq X$ and $C_i \not\subseteq X^*$ for all $1 \leqslant i \leqslant r$, and hence that $n_{G_2}(X) = |V - (X \cup X^*)| \geqslant r > k$. Thus we may suppose that $X \cap K = \emptyset$. Let

$$S = \{i \ : \ X \cap C_i \text{ is a proper subset of } C_i, 1 \leqslant i \leqslant r\}.$$

Since the claim holds when $S = \emptyset$ we may suppose that $|S| \geqslant 1$. Let $Z_i = X \cap C_i$ for $i \in S$. By Claim 7.8, $n_{G_2}(Z_i) \geqslant k - 2$. Hence, $|N_{G_2}(X) \cap (K \cup C_i)| \geqslant k - 2$ and $|N_{G_2}(X) \cap C_i| \geqslant 1$ for all $i \in S$. The claim now follows using the hypothesis of (a) and (b) that $n_{G_2}(X) = k - 2$ and $n_{G_2}(X) = k - 1$, respectively. □

**Claim 7.13.** *For each $i$, $1 \leqslant i \leqslant r$, there exists a unique minimal subset $Y_i \subseteq V(C_i)$ such that $n_{G_2}(Y_i) = k - 2$.*

**Proof.** The existence of such a set follows from the fact that $n_{G_2}(C_i) = k - 2$. To prove uniqueness we suppose to the contrary that $X_1$ and $X_2$ are two minimal subsets of $C_i$ satisfying $n_{G_2}(X_1) = k - 2 = n_{G_2}(X_2)$. Then $n_{G_1}(X_1) = k - 2 = n_{G_1}(X_2)$, since $G_1$ is $(k-2)$-connected by Claim 7.8, and the operation used in going from $G_1$ to $G_2$ (splitting off pairs of edges from $s$) cannot decrease $n(X_i)$. Let $sw$ be the unique edge of $G_1 \oplus s$ from $s$ to $C_i$. Since $G_1 \oplus s$ is $(k-1, s)$-connected, we must have $w \in X_1 \cap X_2$. Since $X_1 \cup X_2 \subseteq C_i$, $X_1 \cup X_2$ is a fragment of $G_2$, and hence we have $n_{G_2}(X_1 \cup X_2) \geqslant k - 2$, by Claim 7.8. Submodularity of $n_{G_2}$, now implies that $n_{G_2}(X_1 \cap X_2) \leqslant k - 2$, contradicting the minimality of $X_1$ and $X_2$. □

For each $i$, $1 \leqslant i \leqslant r$, choose two distinct edges $sy_i, sy_i'$ in $G_2 + s$ with $y_i, y_i' \in Y_i$. Note that these edges exist by the $(k, s)$-connectivity of $G_2$. Furthermore, by Claim 7.9, $y_i = y_i'$, if and only if $Y_i = \{y_i\}$ and $d_{G_2}(y_i) = k - 2$.

We are now ready to construct the required augmentation of $G$. Let $G_2 \oplus s$ be the graph obtained from $G_2 + s$ by adding an extra edge from $s$ to $C_2$ if $d_{G_2+s}(s)$ is odd. Thus $d_{G_2\oplus s}(s) = 2\lceil t(G_2)/2\rceil$ is even. First, we try to define a good augmenting set by a method similar to forest augmentation. Since we need to increase the connectivity of $G_2$ by two, we now look for a loopless 2-connected multigraph $H$ on $r$ vertices whose degree sequence is $d_1, d'_2, \ldots, d_r$, where $d'_2 = d_{G_2\oplus s}(s, C_2)$ (so $d'_2$ is either $d_2$ or $d_2 + 1$, depending on whether $d_{G_2+s}(s)$ is even or odd). If such a multigraph exists, it leads to a good augmenting set in a natural way, as we shall see in Subcase 2.1. However, such a graph may not exist, as the following example shows. Let $G$ be obtained from $K_{r,k-2}$ by replacing some vertex $v$ in the $r$-set by a copy of $K_{k-1,4}$ and then connecting each vertex of the $(k-2)$-set to each vertex of the $(k-1)$-set. It can be seen that the degree sequence defined by the corresponding extension $G_2 \oplus s$ of $G$ is $4, 2, 2, \ldots, 2$. There is no loopless 2-connected multigraph with this degree sequence. When such a multigraph does not exist, we need a somewhat more involved method to define the augmenting set. This will be described in Subcase 2.2.

*Subcase* 2.1: There exists a loopless 2-connected multigraph $H$ on $r$ vertices with degree sequence $d_1, d'_2, \ldots, d_r$.

Let $F$ be a set of edges joining the components of $G_2 - K$ such that $d_F(v) = d_{G_2\oplus s}(s, v)$ for all $v \in V$ and such that the graph obtained from $(V - K, F)$ by contracting each component $C_i$ to a single vertex $c_i$, is $H$. Since $H$ is 2-connected, each vertex $c_i \in V(H)$ has at least two distinct neighbours in $H$, and thus each component $C_i$ is joined to at least two other components by edges of $F$. Furthermore, since $H$ is loopless, each edge of $F$ is incident with two distinct components of $G_2 - K$. Let $y_i, y'_i$ be the neighbours of $s$ in $C_i$ as defined after Claim 7.13. Since we are free to interchange the end vertices of the edges of $F$ within each component, we may choose $F$ to have the additional property that, for each $1 \leqslant i \leqslant r$, the two edges of $F$ incident to $y_i$ and $y'_i$ join $C_i$ to different components of $G_2 - K$. We can now use Claim 7.12 to deduce that $G_2 + F$ is $k$-connected. Suppose to the contrary that $G_2 + F$ has a fragment $X$ with $n_{G_2+F}(X) \leqslant k-1$. Replacing $X$ by $X^*$ if necessary we may assume that $|X \cap K| \leqslant |X^* \cap K|$. By Claim 7.8, $n_{G_2}(X) \geqslant k-2$ and by Claim 7.12, we have one of the following four alternatives.

(a1) $n_{G_2}(X) = k-2$ and $X = C_{i_1} \cup C_{i_2} \cup \ldots \cup C_{i_p}$ for some $p \leqslant r-1$. Suppose $p \leqslant r-2$. Then, the 2-connectivity of $H$ implies that there are two edges of $F$ from $X$ to distinct components $C_{j_1}, C_{j_2}$ disjoint from $X$. Hence $n_{G_2+F}(X) \geqslant k$. Suppose $p = r-1$. There are at least two edges from $X$ to $C_{i_r}$, where $C_{i_r}$ is the unique component of $G_2 - K$ disjoint from $X$. If $C_{i_r}$ has only one vertex then $N_{G_2+F}(X) = V - X$ and $X$ is not a fragment. If all edges of $F$ join $X$ to the same vertex $v \in C_{i_r}$, then we have $n_{G_2}(C_{i_r} - v) \leqslant k-1$ and $d_{G_2+s}(s, C_{i_r} - v) = 0$, contradicting the $(k, s)$-connectivity of $G_2 + s$. Thus at least two edges of $F$ join $X$ to distinct vertices of $C_{i_r}$ and we again have $n_{G_2+F}(X) \geqslant k$.

(a2) $n_{G_2}(X) = k-2$ and $X = Z_i \subset C_i$ for some $1 \leqslant i \leqslant r$. By Claim 7.13, $y_i, y'_i \in X$. Since $y_i, y'_i$ are joined by $F$ to distinct components $C_{j_1}, C_{j_2}$ disjoint from $C_i$, we again have $n_{G_2+F}(X) \geqslant k$.

(b1) $n_{G_2}(X) = k-1$, and $X = Z_{i_1} \cup C_{i_2} \cup \ldots \cup C_{i_p}$ for some $p \leqslant r$ and $Z_{i_1} \subseteq C_{i_1}$. Suppose $2 \leqslant p \leqslant r-1$. Then the 2-connectivity of $H$ implies that there is at least one edge of $F$ from $X - C_{i_1}$ to a component $C_{j_1}$ disjoint from $X$. Hence $n_{G_2+F}(X) \geqslant k$. Suppose $p = r$. Since $G_2 + s$ is $(k,s)$-connected, it has an edge from $s$ to a vertex $v \in X^* \subseteq C_{i_1} - Z_{i_1}$. Since all edges of $F$ are incident to distinct components $v$ is joined by an edge of $F$ to some vertex of $X - C_{i_1}$, and again we have $n_{G_2+F}(X) \geqslant k$. Suppose $p = 1$. Since $G_2 + s$ is $(k,s)$-connected, it has an edge from $s$ to at least one vertex $v \in Z_{i_1}$. Since all edges of $F$ are incident to distinct components, $v$ is joined by an edge of $F$ to some component distinct from $C_{i_1}$, and again we have $n_{G_2+F}(X) \geqslant k$.

(b2) $n_{G_2}(X) = k-1$ and $X = Z_{i_1} \cup Z_{i_2}$ for some $Z_{i_1} \subseteq C_{i_1}$, $Z_{i_2} \subseteq C_{i_2}$, and $n_{G_2}(Z_{i_1}) = k-2 = n_{G_2}(Z_{i_2})$. By Claim 7.13, $y_{i_1}, y'_{i_1} \in Z_{i_1}$. Since $y_{i_1}, y'_{i_1}$ are joined by $F$ to two distinct components $C_{j_1}, C_{j_2}$ disjoint from $C_{i_1}$, at least one of these components is also disjoint from $C_{i_2}$ and we again have $n_{G_2+F}(X) \geqslant k$.

Thus $G_2 + F$ is $k$-connected. Putting $F_0 = E(G_2) - E(G)$, we deduce that $F_0 \cup F$ is the required augmenting set of edges for $G$ of size $\lceil d_{G+s}(s)/2 \rceil = \lceil t(G)/2 \rceil$.

*Subcase* 2.2: There is no loopless 2-connected multigraph on $r$ vertices with degree sequence $d_1, d'_2, \ldots, d_r$.

Hakimi [10] characterized the degree sequences of loopless 2-connected multigraphs, see also [14, Corollary 3.2].

**Theorem 7.14.** *There exists a 2-connected loopless multigraph with degree sequence* $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_r \geqslant 2$ *if and only if* $d_1 + d_2 + \ldots + d_r$ *is even and* $d_1 \leqslant d_2 + d_3 + \ldots + d_r - 2r + 4$.

This characterization implies that in Subcase 2.2 we have either: $d_1 \geqslant d'_2$ and $d_1 \geqslant d'_2 + d_3 + \ldots + d_r - 2r + 5$; or $d_1 = d'_2 - 1$ and $d'_2 \geqslant d_1 + d_3 + \ldots + d_r - 2r + 5$. Since $d_{G_2 \oplus s}(s) = d_1 + d'_2 + d_3 + \ldots + d_r$ and $d_{G_2 \oplus s}(s)$ is even, both alternatives imply that

$$d_{G_2 \oplus s}(s) \leqslant 2d_1 + 2r - 4. \tag{29}$$

We shall use the following concept to find a good augmenting set in this subcase. Let $H_0 = (V, E)$ be a multigraph, $s \in V$, and $m_1, m_2, \ldots, m_q$ be a partition of $d_{H_0}(s)$. Then an $(m_1, m_2, \ldots, m_q)$-*detachment of* $H_0$ *at* $s$ is a multigraph $H_1$ obtained from $H_0$ by 'splitting' $s$ into $q$ vertices $s_1, s_2, \ldots, s_q$ with degrees $m_1, m_2, \ldots, m_q$, respectively. We refer to $s_1, s_2, \ldots, s_q$ as the *pieces* of $s$ in $H_1$. Note that the graph $H$ used in Subcase 2.1 can be viewed as a loopless 2-connected $(d_1, d'_2, d_3 \ldots, d_r)$-detachment at $s$ of the graph $H_0$ consisting of exactly one vertex $s$ incident with $d_{G_2 \oplus s}(s)/2$ loops. Inequality (29) tells us that if this detachment $H$ does not exist, then $d_1$ is 'large' compared to $d_{G_2 \oplus s}(s)$. We modify our approach in this case by finding a loopless 2-connected $(d'_2, d_3, \ldots, d_r)$-detachment of the multigraph obtained from $(G_2 \oplus s) - K - \cup_{i=2}^{r} C_i$ by adding a suitable number of loops to $s$. The pieces of $s$ in the detachment will represent the components $C_2, C_3, \ldots, C_r$. We use the following lemma from [14] to construct the required detachment.

Given a multigraph $H$ and $v_1, v_2, \ldots, v_m \in V(H)$, let $b(v_1, v_2, \ldots, v_m)$ be the number of components of $H - \{v_1, v_2, \ldots, v_m\}$.

**Lemma 7.15** ([14, Corollary 3.3]). *Let $H_0 = (V, E)$ be a multigraph, $s \in V$ and $m_1, m_2, \ldots, m_q$ be a partition of $d(s)$ into at least two positive integers, such that $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_q \geqslant 2$. Let $e(u)$ denote the number of loops incident to each vertex $u$ in $H_0$. Then $H_0$ has a loopless 2-connected $(m_1, m_2, \ldots, m_q)$-detachment at $s$ if and only if*

(a) *$H_0$ is 2-edge-connected,*
(b) *$b(v) + e(v) = 1$ for all $v \in V - s$,*
(c) *$m_2 + m_3 + \cdots + m_q \geqslant b(s) + e(s) + q - 2$, and*
(d) *$d(s, V - v) + e(s) \geqslant q + b(s, v) - 1$ for all $v \in V - s$.*

Let $G_3 + s$ be the multigraph obtained from $(G_2 \oplus s) - K - \cup_{i=2}^{r} C_i$ by adding $p = (d_{G_2 \oplus s}(s) - 2d_1)/2 - 1$ loops at $s$. Note that $p$ is a non-negative integer by Claim 7.11 and the fact that $d_{G_2 \oplus s}(s)$ is even. Applying Lemma 7.15 to $G_3 + s$ we deduce:

**Claim 7.16.** *$G_3 + s$ has a loopless 2-connected $(d_2^*, d_3, \ldots, d_{r-1})$-detachment $H_1$ at $s$, where $d_2^* = d_2' + d_r - 2$.*

**Proof.** We have $d_2^* + d_3 + \ldots + d_{r-1} = d_{G_2 \oplus s}(s) - d_1 - 2 = 2p + d_1 = d_{G_3 + s}(s)$ so $(d_2^*, d_3, \ldots, d_{r-1})$ partitions $d_{G_3+s}(s)$. Since $G_2 \oplus s$ is $(k, s)$-connected and $G_3$ is connected and loopless, it follows that $G_3 + s$ satisfies Lemma 7.15(a) and Lemma 7.15(b). Using $d_i \geqslant 2$ for all $3 \leqslant i \leqslant r - 1$ and (29), we have $d_2' + d_r \leqslant d_{G_2 \oplus s}(s) - d_1 - 2(r - 3) \leqslant d_{G_2 \oplus s}(s) - d_{G_2 \oplus s}(s)/2 + r - 2 - 2(r - 3) = d_{G_2 \oplus s}(s)/2 - r + 4$. Thus $d_3 + \ldots + d_{r-1} = d_{G_2 \oplus s}(s) - d_1 - d_2' - d_r \geqslant d_{G_2 \oplus s}(s) - d_1 - d_{G_2 \oplus s}(s)/2 + r - 4 = 1 + e(s) + r - 4$, proving that Lemma 7.15(c) holds for $G_3 + s$. To show that Lemma 7.15(d) holds focus on a vertex $v$ of $C_1$. Considering the graph $G_2 - (K + v)$ and using Claim 7.9, we have $\hat{b}(G_2) \geqslant b_{G_3}(v) + r - 1 + \beta$, where $\beta = 2$ if $d_{G_2 \oplus s}(s, v) = 2$ and $\beta = 0$, otherwise, since if $d_{G_2 \oplus s}(v) = 2$ then $d_{G_2}(v) = k - 2$. By (28), $\hat{b}(G_2) \leqslant \lceil t(G_2)/2 \rceil = d_{G_2 \oplus s}(s)/2$. Hence $d_{G_2 \oplus s}(s)/2 \geqslant b_{G_3}(v) + r - 1 + \beta$. Thus

$$\begin{aligned} d_{G_3+s}(s, V(C_1) - v) + e(s) &= d_1 - d_{G_2 \oplus s}(s, v) + e(s) \\ &= d_{G_2 \oplus s}(s)/2 - 1 - d_{G_2 \oplus s}(s, v) \\ &\geqslant b_{G_3}(v) + r - 1 + \beta - 1 - d_{G_2 \oplus s}(s, v) \\ &\geqslant (r - 2) + b_{G_3+s}(s, v) - 1, \end{aligned}$$

as required. □

Label the detached vertices of $H_1$ as $c_2, c_3, c_4 \ldots, c_{r-1}$ where $d_{H_1}(c_i) = d_i$ for $3 \leqslant i \leqslant r - 1$ and $d_{H_1}(c_2) = d_2^*$. The edge $e = c_j y_1$ is in $E(H_1)$ for some $2 \leqslant j \leqslant r - 1$. We next subdivide the edge $e$ with a new vertex $c_r$ to form the multigraph $H_2$, and then 'flip' some edges from $c_2$ to $c_r$ in $H_2$ preserving 2-connectivity and increasing the degree of $c_r$ up to $d_r$ while maintaining the property that $y_1$ and $y_1'$ are joined to different pieces of $s$. We use the following result to accomplish this.

**Lemma 7.17** ([14, Corollary 2.17]). *Let $t \geqslant 3$ be an integer. Let $H$ be a loopless 2-connected multigraph, $x, y \in V(H)$ and $xz_i \in E(H - y)$ for $1 \leqslant i \leqslant t$. If $t \geqslant d(y) - d(y, x) + 1$, then $H - xz_i + yz_i$ is loopless and 2-connected for some $i$, $1 \leqslant i \leqslant t$.*

We construct the new multigraph $H_3$ from $H_2$ as follows. If $d_r = 2$ then we put $H_3 = H_2$. If $d_r \geqslant 3$ then we use Lemma 7.17 to find a set of edges $S = \{c_2z_i \in E(H_2) : 1 \leqslant i \leqslant d_r - 2\}$ such that $c_2y_1' \notin S$ and $H_3 = H_2 - S + \{c_rz_i : 1 \leqslant i \leqslant d_r - 2\}$ is 2-connected and loopless. This is possible since $d_{H_2}(c_r) = 2$, $d_{H_2}(c_r, c_2) \leqslant 1$, and $d_{H_2}(c_2) = d_2' + d_r - 2 \geqslant d_r + d_r - 2$. In $H_3$ we have $y_1c_r \in E(H_3)$, $y_1'c_r \notin E(H_3)$, $d_{H_3}(c_i) = d_i$ for $3 \leqslant i \leqslant r$, and $d_{H_3}(c_2) = d_2'$. (Note that we could have used Lemma 7.15 directly to construct a 2-connected loopless detachment with the same degree sequence as $H_3$ from $G_3 + s$ plus one extra loop at $s$. The reason we go via $H_1$ and $H_2$ is to ensure that $y_1$ and $y_1'$ are adjacent to distinct pieces of $s$ in $H_3$.)

Let $F$ be a set of edges joining the components of $G_2 - K$ such that $d_F(v) = d_{G_2 \oplus s}(s, v)$ for all $v \in V - K$ and such that the graph obtained from $(V - K, F)$ by contracting $C_2, \ldots, C_r$ to $c_2, c_3, \ldots, c_r$, respectively, is $H_3$. Since $H_3$ is 2-connected, each vertex $c_i$ in $H_3$ has at least two distinct neighbours. Since $H_3$ is loopless, every edge of $F$ which is incident to a component $C_i$, $2 \leqslant i \leqslant r$, is incident to distinct components of $G_2 - K$. Let $y_i, y_i'$ be the neighbours of $s$ in $C_i$ as defined after Claim 7.13. Since we are free to interchange the end vertices of the edges of $F$ within each component, $C_i$, for $2 \leqslant i \leqslant r$ we may choose $F$ to have the additional property that, for $2 \leqslant i \leqslant r$, the two edges of $F$ incident to $y_i$ and $y_i'$ join $C_i$ to different vertices of $G - K - C_i$, which either belong to different components of $G - K - C_i$, or both belong to $C_1$. Furthermore, since $y_1$ and $y_1'$ are joined to different detached vertices in $H_3$, the two edges of $F$ incident to $y_1$ and $y_1'$ join $C_1$ to different components of $G_2 - K - C_1$.

We can now use Claim 7.12 to deduce that $G_2 + F$ is $k$-connected as in Subcase 2.1. Putting $F_0 = E(G_2) - E(G)$ we deduce that $F_0 \cup F$ is the required augmenting set of edges for $G$ of size $\lceil d_{G+s}(s)/2 \rceil = \lceil t(G)/2 \rceil$. □

## 8. Algorithmic aspects and corollaries

In this section, we discuss the algorithmic aspects of our results and also show that our main theorems imply (partial) solutions to a number of conjectures in this area.

### 8.1. Algorithms

The proofs of our min–max theorems (Theorems 4.17 and 7.7) are algorithmic and lead to a polynomial algorithm which finds an optimal augmenting set with respect to $k$ for any $l$-connected input graph $G$ and target $k \geqslant l + 1$, provided $a_k(G) \geqslant 10(k - l + 2)^3(k + 1)^3$ (or $a_k(G) \geqslant 20k^3$, if $k = l + 1$). As we shall see, the running time in this case can be bounded by $O(n^6)$, even if $k$ is part of the input. Our algorithm for the general case first decides whether $a_k(G)$ is large, compared to $k$, or not. Since, by Lemma 2.10, $a_k(G)$ is large if and only if $d(s)$ is large in a $k$-critical extension $G + s$ of $G$, the first step is to create such an extension. If $a_k(G)$ is small then our algorithm performs an exhaustive search on all possible augmenting sets $F$ with $V(F) \subseteq N(s)$ and outputs the smallest augmenting set which makes $G$ $k$-connected. The number of possibilities depends only on $k$, since $|N(s)|$ is also small. We shall present the algorithm as a sequence of sub-algorithms. Most of the steps of these algorithms are easy to implement in polynomial time by network flow techniques.

*8.1.1. CRITICAL EXTENSION*

*Input*: A graph $G$ and an integer $k \geqslant 1$.

*Output*: A $k$-critical extension $G + s$ of $G$.

*Step* 1: Add a new vertex $s$ to $G$ and $\max\{1, k - d(v)\}$ edges from $s$ to each vertex $v$ of $G$. (This gives a $(k, s)$-connected extension $G \oplus s$ of $G$ by Lemma 2.4.)

*Step* 2: Delete edges incident to $s$ greedily until the remaining graph $G + s$ is a $k$-critical extension. (We check whether each edge deletion preserves $(k, s)$-connectivity using a max-flow computation.)

*8.1.2. EXHAUSTIVE SEARCH*

*Input*: A $k$-critical extension $G + s$ of a graph $G$.

*Output*: An optimal $k$-augmenting set for $G$.

For each set of edges $F$ with $V(F) \subseteq N(s)$, check whether $G + F$ is $k$-connected. Choose the smallest such $k$-augmenting set.

The following lemma implies that the output of EXHAUSTIVE SEARCH is indeed an optimal $k$-augmentation for $G$.

**Lemma 8.1.** *Let $G + s$ be a $(k, s)$-connected extension of a graph $G$. Then there exists an optimal $k$-augmenting set $F$ for $G$ with $V(F) \subseteq N(s)$.*

**Proof.** Let $S = N(s)$ and let $F$ be an optimal augmenting set with respect to $k$ for which $c(F) = \sum_{uv \in F} |\{u, v\} - S|$ is as small as possible. Suppose $c(F)$ is positive and let $uv \in F$ be an edge with $\{u, v\} - S \neq \emptyset$. Since $F$ is optimal, we have $\kappa(G + F - uv) = k - 1$ and, by Lemma 4.4(c), it follows that $G + F - uv$ has precisely two $k$-cores (i.e. minimal $k$-deficient fragments) $X, Y$. Clearly, $X$ and $Y$ are $k$-deficient fragments in $G$. Thus, since $G + s$ is $(k, s)$-connected, we must have $S \cap X \neq \emptyset \neq S \cap Y$. Lemma 4.4(c) also implies that by taking $F' = F - uv + xy$ for a pair $x, y$ of vertices with $x \in S \cap X$ and $y \in S \cap Y$ we have that $G + F'$ is $k$-connected. Now $|F'| = |F|$ and $c(F') < c(F)$, contradicting the choice of $F$. This proves that $c(F) = 0$ must hold, and hence the required augmenting set exists. □

It follows that, if $a_k(G)$ is small, then we only need to perform $c_k$ $k$-connectivity tests, where $c_k = O(2^{\binom{a_k(G)}{2}})$ depends only on $k$, to find an optimal $k$-augmentation for $G$ using CRITICAL EXTENSION and EXHAUSTIVE SEARCH. If $a_k(G)$ is large then our augmentation algorithm has several sub-algorithms, according to the different subcases in our proofs. In what follows we give a sketch of these algorithms to verify that they can be run in polynomial time. We do not attempt to work out the details of an efficient implementation.

*8.1.3. CORES*

*Input*: A $(k - 1)$-connected graph $G = (V, E)$.

*Output*: The set $\mathcal{C}$ of all $k$-cores and the set $\mathcal{A}$ of all active $k$-cores in $G$.

For each non-adjacent pair $u, v \in V$ such that $\kappa(u, v) = k - 1$, find the minimal (with respect to inclusion) sets $X_u, X_v$ such that $u \in X_u$, $v \in X_v$ and $n(X_u) = k - 1 = n(X_v)$.

Let $\mathcal{C}'$ be the union of the sets $\{X_u, X_v\}$ over all pairs $u, v$, and let $\mathcal{C}$ consist of the minimal members of $\mathcal{C}'$. Let $\mathcal{A} = \{X \in \mathcal{C} : \kappa(G - X) = k - 1 - |X|\}$.

Note that CORES can be used to test if a $(k-1)$-connected graph $G$ is $k$-independence free by checking whether $G$ has any active $k$-cores. We do not know if there is a polynomial algorithm to determine whether an arbitrary graph is $k$-independence free.

Cheriyan and Thurimella [4] give a polynomial algorithm for determining $b_k(G)$ for a $(k-1)$-connected graph $G$ and finding all $(k-1)$-shredders $K$ in $G$ with $b_G(K) = b(G)$. We can use this to give a polynomial algorithm for finding an optimal $k$-augmentation of a $(k-1)$-connected $k$-independence free graph. Note, however, that it is unlikely that there exists an efficient algorithm to determine $b_k(G)$ for an arbitrary graph $G$. This follows since the problem of determining whether $b_k(G) \geqslant k$ for some $1 \leqslant k \leqslant |V|$ is NP-complete by Bauer et al. [1].

*8.1.4. INDEPENDENCE FREE AUGMENTATION*

*Input*: A $(k-1)$-connected $k$-independence free graph $G$.

*Output*: An optimal $k$-augmenting set $F$ for $G$ with $|F| = \max\{b(G) - 1, \lceil t(G)/2 \rceil\}$.

We first construct a $k$-critical extension $G + s$ of $G$ using CRITICAL EXTENSION. We have $d(s) = t(G)$ by Corollary 3.2. We construct the required set $F$ by finding a sequence of admissible splits at $s$ (as in the proofs of Lemmas 3.7, 3.10 and 3.11 and Theorem 3.12) to give $G_1 + s$ with $d_{G_1+s}(s) \in \{3, b(G_1)\}$. We then put $F = F_1 \cup F_2$ where $F_1 = E(G_1) - E(G)$ and $F_2$ is the edge set of a tree with $V(F_2) = N_{G_1+s}(s)$.

We next give algorithms for finding optimal $k$-augmentations for a graph $G$ when $a_k(G)$ is large. The first two algorithms determine whether $G$ has a *dominating shredder*, that is to say a $(k-1)$-shredder $K$ with $\hat{b}_G(K) = \hat{b}(G)$ and $2\hat{b}(K) - 2 \geqslant t(G)$, and find an optimal $k$-augmenting set when $G$ does have such a shredder.

*8.1.5. DOMINATING SHREDDER*

*Input*: A $k$-critical extension $G + s$ of an $l$-connected graph $G$ for which $d_{G+s}(s) \geqslant k(k - l + 1) + 2$.

*Output*: We find a dominating shredder $K$ in $G$ or deduce that no such shredder exists.

We construct a family $\mathcal{K}$ of $(k-1)$-shredders in such a way that $|\mathcal{K}|$ is polynomial in $|V|$ and, if there is a dominating shredder $K$ in $G$, then $K \in \mathcal{K}$. Once we have $\mathcal{K}$, we complete the algorithm by computing $\hat{b}(K')$ for all $K' \in \mathcal{K}$.

For each triple $x, u, v$, where $x \in V$ and $u, v \in N_{G+s}(s) - x$, first we try to split off all copies of the edges $sx$ (if there are any). Suppose that all copies can be split off, and let the resulting graph be $G_x + s$. Then, we try to find a set $\{P_1, P_2, \ldots, P_{k-1}\}$ of openly disjoint $uv$-paths in $G_x$. If we succeed, then we let $Q(x, u, v) = \cup_{i=1}^{k-1} P_i$, $\mathcal{C}(x, u, v) = \{C : C \text{ is a component of } G - Q(x, u, v)\}$,

$$\mathcal{K}_1(x, u, v) = \{N_G(C) : C \in \mathcal{C}(x, u, v) \text{ and } n_G(C) = k - 1\},$$

$$\mathcal{K}_2(x, u, v) = \{N_G(C) \cup \{q\} : C \in \mathcal{C}(x, u, v), n_G(C) = k - 2, q \in Q - \{u, v\}\}.$$

Let $\mathcal{K}$ be the union of the sets $\mathcal{K}_1(x, u, v) \cup \mathcal{K}_2(x, u, v)$ over all choices of $x, u, v$. Clearly, $|\mathcal{K}| \leqslant \binom{n}{3} n^2$.

**Lemma 8.2.** *If $G$ has a dominating shredder $K$ then $K \in \mathcal{K}$.*

**Proof.** Suppose there is a $(k-1)$-shredder $K$ with $d(s) \leqslant 2\hat{b}(G) - 2 = 2\hat{b}(K) - 2$. Then Lemma 6.1 implies that $|N(s) \cap K| \leqslant 1$, and if $x \in N(s) \cap K$ then $d_G(x) = k - d(s,x)$, $\hat{b}(K) = b(K) + d(s,x)$, and we can split off all copies of $sx$ (in any order) by admissible splittings. By splitting off these copies $d(s)$ is reduced by $2d(s,x)$ and $\hat{b}(K)$ is reduced by $d(s,x)$. Hence, $d(s) \leqslant 2b(K) - 2$ holds in the resulting graph $G_x$. This implies that $K$ has at least two leaf components $C, C'$ in $G_x$. By Lemma 2.14 there exist $k-1$ openly disjoint paths from $u \in N(s) \cap C$ to $v \in N(s) \cap C'$. Clearly, $Q \subseteq K \cup C \cup C'$ and $K \subset Q$ hold, where $Q$ is the union of the vertex sets of these paths. Moreover, since $x \in K$, the components of $G - K$ and $G_x - K$ are the same. Lemma 6.1 also implies that $G - K$ has at least $2k + 1 \geqslant 3$ components $D$ with $d_{G+s}(s, D) \leqslant 2$, and hence $n_G(D) \geqslant k - 2$. Thus, there is a component $D'$ of $G - K$, which is a component of $G - Q$, and satisfies that either $K = N_G(D')$ or $K = N_G(D') + q$ for some $q \in Q - \{u, v\}$.

It follows that for some triple $x, u, v$ we have $K \in \mathcal{K}_1(x,u,v) \cup \mathcal{K}_2(x,u,v)$, as required. □

Note that if DOMINATING SHREDDER finds a dominating shredder $K$ when $l = k-1$, then we have $d(s, K) = 0$ and $b_G(K) = \hat{b}(K)$ by Theorem 4.1.

*8.1.6. DOMINATING SHREDDER AUGMENTATION*

*Input*: A $k$-critical extension $G + s$ of an $l$-connected graph $G$ for which $d_{G+s}(s) \geqslant k(k - l + 1) + 2$, and a dominating shredder $K$ for $G$.

*Output*: An optimal augmenting set $F$ for $G$ with $|F| = \hat{b}(G) - 1$.

We construct $F$ by splitting off all edges from $s$ to $K$ and then adding a forest augmentation, as described in Lemma 6.5 and after Lemma 6.1.

*8.1.7. LARGE AUGMENT BY ONE*

*Input*: A $k$-critical extension $G + s$ of a $(k-1)$-connected graph $G = (V, E)$ for which $d_{G+s}(s) \geqslant 20k^3 + 1$.

*Output*: An optimal augmenting set $F$ for $G$ with $|F| = \max\{b(G) - 1, \lceil t(G)/2 \rceil\}$.

We use DOMINATING SHREDDER, DOMINATING SHREDDER AUGMENTATION, CORE, and the proof techniques of Lemmas 4.2, 4.15 and 4.16 to find a saturating set of edges $F_1$ such that either $F_1$ is an optimal $k$-augmenting set for $G$ with $|F| = \max\{b(G) - 1, \lceil t(G)/2 \rceil\}$, or $G + F_1$ is $k$-independence free and has no dominating shredder. In the former case we put $F = F_1$. In the latter case we use INDEPENDENCE FREE AUGMENTATION to find a $k$-augmenting set $F_2$ for $G + F_1$ and put $F = F_1 \cup F_2$.

Note that when we increase the number of passive $k$-cores by making an active $k$-core passive in LARGE AUGMENT BY ONE, we do not need to compute $T(G)$. We choose an arbitrary active $k$-core $B$ and, if we fail to make $B$ passive (which means $B \cap T(G) \neq \emptyset$), then we choose a different active $k$-core.

*8.1.8. LARGE AUGMENT*

*Input*: A $k$-critical extension $G + s$ of a graph $G = (V, E)$ for which $d_{G+s}(s) \geqslant 10(k - l + 2)^3(k+1)^3 + 1$.

*Output*: An optimal augmenting set $F$ for $G$ with $|F| = \max\{\hat{b}(G) - 1, \lceil t(G)/2\rceil\}$.

We use DOMINATING SHREDDER, DOMINATING SHREDDER AUGMENTATION, and the proof techniques of Lemmas 6.7 and 7.5 to find a saturating set of edges $F_1$ such that either $F_1$ is an optimal $k$-augmenting set for $G$ with $|F_1| = \max\{\hat{b}(G) - 1, \lceil t(G)/2\rceil\}$, or $G + F_1$ is $(k-1)$-connected, has no dominating shredder, and $d_{G+s}(s) - 2|F_1| \geqslant 20k^3 + 1$, or $G + F_1$ has a $k$-augmenting set $F_2$ of size $\lceil t(G + F_1)/2\rceil$ (which can be constructed using detachments as in the proof of Case 2 of Theorem 7.7). In the first case we put $F = F_1$. In the second case we use LARGE AUGMENT BY ONE to find a $k$-augmenting set $F_3$ for $G + F_1$ of size $\lceil t(G + F_1)/2\rceil$ and put $F = F_1 \cup F_3$. In the third case we put $F = F_1 \cup F_2$.

*8.1.9. AUGMENT*

*Input*: An $l$-connected graph $G$ and an integer $k > l$.

*Output*: An optimal $k$-augmenting set $F$ for $G$.

Construct a $k$-critical extension $G + s$ for $G$ using CRITICAL EXTENSION. If $k = l + 1$ and $d_{G+s}(s) \geqslant 20k^3 + 1$ then apply LARGE AUGMENT BY ONE. If $l \leqslant k - 2$ and $d_{G+s}(s) \geqslant 10(k - l + 2)^3(k + 1)^3 + 1$ then apply LARGE AUGMENT. Otherwise apply EXHAUSTIVE SEARCH.

As noted above, most of the steps of the above algorithms are easy to implement in polynomial time by network flow techniques. The only exception is finding the required loopless 2-connected detachments as in the proof of Case 2 of Theorem 7.7. We shall not discuss this in this paper but remark that there is a simple algorithm which finds $H$, if it exists, and we also have a similarly simple and efficient algorithm which finds $H_3$, when $H$ does not exist.

Before stating our bound on the running time of our algorithm AUGMENT, we note that by inserting a preprocessing step, which works in linear time, we can make the input graph sparse, and hence reduce the running time, as follows. Let $G = (V, E)$ and $k$ be the input of our problem. Let $n = |V|$ and $m = |E|$. It was shown in [3,19] that $G = (V, E)$ has a spanning subgraph $G' = (V, E')$ with $|E'| \leqslant k(n - 1)$ satisfying $\kappa(u, v, G') \geqslant \min\{k, \kappa(u, v, G)\}$ for each pair $u, v \in V$. It can be seen that by replacing $G$ by $G'$ we do not change the $k$-deficient fragments (or their deficiencies) and that for any augmenting set $F$ the graph $G + F$ is $k$-connected if and only if $G' + F$ is $k$-connected. Thus we can work with $G'$ and assume that $m = O(kn)$. Note also that $d(s) = O(kn)$ in any $k$-critical extension $G + s$ of $G$. By using these facts and efficient network flow algorithms for the basic operations (such as finding admissible splittings, checking whether an edge is $k$-critical, etc) we can conclude with the following theorem.

**Theorem 8.3.** *Given an $l$-connected graph $G$ and a positive integer $k$, our algorithm AUGMENT finds an optimal $k$-augmenting set. If $a_k(G) \geqslant 10(k-l+2)^3(k+1)^3$ then the running time is $O(kn^5)$. Otherwise the running time is $O(c_k n^3)$.*

We close this subsection by noting that we can also use the theory developed in this paper to derive a near-optimal algorithm for the vertex connectivity augmentation problem which is similar to the one given in [13].

*8.1.10. NEAR-OPTIMAL AUGMENT*

*Input*: An $l$-connected graph $G$ and an integer $k > l$.

*Output*: A $k$-augmenting set $F$ for $G$ with $|F| \leqslant a_k(G) + \frac{1}{2}k(k-l+1) + 1$.

Construct a $k$-critical extension $G + s$ for $G$ using CRITICAL EXTENSION. We first suppose that $d_{G+s}(s) \geqslant k(k-l+1)+2$. We use DOMINATING SHREDDER to determine if $G$ has a dominating shredder. If it does then we use DOMINATING SHREDDER AUGMENTATION, to find an optimal $k$-augmenting set for $G$. If $G$ does not have a dominating shredder then, by Lemma 5.3, we can split off edges from $s$ such that, in the resulting graph $G_1 + s$, we have either $d_{G_1+s}(s) > k(k-l+1)+2$ and $G_1$ has a dominating shredder, or $k(k-l+1)+1 \leqslant d_{G_1+s}(s) \leqslant k(k-l+1)+2$. In the former case we can use DOMINATING SHREDDER and DOMINATING SHREDDER AUGMENTATION, to find an optimal $k$-augmenting set for $G$. In the latter case, Lemma 2.7 implies that we may construct a minimal augmenting set $F_1$ for $G_1$ with $V(F_1) \subseteq N_{G_1+s}(s)$. Let $F = F_1 \cup F_2$, where $F_2 = E(G_1) - E(G)$. Lemma 2.9 implies that $|F_2| \leqslant d_{G_1+s}(s) - 1$. Since $t(G) = d_{G+s}(s)$ and $t(G_1) = d_{G_1+s}(s)$ we have $|F| \leqslant \frac{1}{2}t(G) + \frac{1}{2}d_{G_1+s}(s) \leqslant a_k(G) + \frac{1}{2}k(k-l+1) + 1$. Finally, if $d_{G+s}(s) < k(k-l+1)+2$, then we construct a minimal augmenting set $F$ for $G$ with $V(F) \subseteq N_{G+s}(s)$. Lemma 2.10 implies that $|F| \leqslant a_k(G) + \frac{1}{2}d_{G+s}(s) \leqslant a_k(G) + \frac{1}{2}k(k-l+1) + 1$.

The running time of NEAR-OPTIMAL AUGMENT is $O(n^6)$.

## 8.2. Corollaries

Our main results (Theorems 4.17 and 7.7) imply (partial) solutions to several related conjectures. The extremal version of the connectivity augmentation problem is to find, for given parameters $n, k, t$, the smallest integer $m$ for which every $k$-connected graph on $n$ vertices can be made $(k+t)$-connected by adding $m$ new edges. Several special cases of this problem were solved in [17] and it was conjectured that (at least if $n$ is large enough compared to $k$) the extremal value of $m$ for $t \geqslant 2$, $k \geqslant 2$ is $\lceil nt/2 \rceil$ (or $\lfloor nt/2 \rfloor$, depending on the parities of $n, k, t$). Since $\hat{b}(G) - 1 \leqslant n$, the min–max equality of Theorem 7.7 shows that if $n$ is large enough and $t \geqslant 2$ then $a_k(G)$ is maximized if and only if $G$ is (almost) $k$-regular. This proves the conjecture (when $n$ is large compared to $k$), by noting that such (almost) regular graphs exist for $k \geqslant 2$.

A different version of this problem, when the graphs to be augmented are $k$-regular, was studied in [9]. It was conjectured there that if $G$ is a $k$-regular $k$-connected graph on $n$ vertices, and $n$ is even and large compared to $k$, then $G$ can be made $(k+1)$-connected by adding $n/2$ edges. If $G$ is $k$-regular, $b(K) \leqslant k$ for any cut of size $k$. Thus if $n$ is large enough, we have $\max\{b(G)-1, \lceil t(G)/2 \rceil\} = n/2$. Now the conjecture follows from Theorem 4.17.

A similar question is whether $a_k(T) = \lceil (\sum_{v \in V(T)} (k - d(v))^+)/2 \rceil$ holds when graph $T$ is a tree, where $x^+ = \max\{0, x\}$ for all integers $x$. It is known that the minimum number of edges needed to make a tree $k$-edge-connected (or an arborescence $k$-edge- or $k$-vertex-connected) is determined by the sum of the (out)degree-deficiencies of its vertices. As above, using the fact that $\hat{b}(G) - 1 \leqslant n$, Theorem 7.7 implies (when $n$, and hence also $a_k(T)$, is large compared to $k$) that if $k \geqslant 3$ then $a_k(T) = \lceil t(T)/2 \rceil$. That is, $a_k(T)$ is determined by the total deficiency of a family of pairwise disjoint subsets of $V(T)$. Since $T$ is a tree, each

member $X$ of this family induces a forest. This implies that there exists a vertex $v \in X$ with $k - d(v) \geqslant k - n(X)$. Therefore, we can find a family consisting of singletons with the same total deficiency. This yields an affirmative answer to our question provided $k \geqslant 3$ and $n$ is large compared to $k$. Note that the answer is negative for $k = 2$.

Frank and Jordán [8, Corollary 4.8] prove that every $(k-1)$-connected graph $G = (V, E)$ can be made $k$-connected by adding a set $F$ of new edges such that $(V, F)$ consists of vertex-disjoint paths. They conjectured that such an $F$ can be found among the optimal augmenting sets as well. We can verify this, provided $a_k(G)$ is large enough. In this case, we may use the min–max formula of Theorem 4.17. If $a_k(G) = \lceil t(G)/2 \rceil$ then an optimal augmenting set is a collection of vertex-disjoint paths of length one or two. If $a_k(G) = b(G) - 1$, then a careful analysis of the forest augmentation method shows that we can find an optimal augmenting set $F$ satisfying $d_F(v) \leqslant 2$ for all $v \in V$. Since $F$ is a forest, it induces vertex-disjoint paths, as claimed.

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