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Briot–Bouquet differential superordinations and sandwich theorems

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Abstract

Briot–Bouquet differential subordinations play a prominent role in the theory of differential subordinations. In this article we consider the dual problem of Briot–Bouquet differential superordinations. Let β and γ be complex numbers, and let Ω be any set in the complex plane **C**. The function p analytic in the unit disk **U** is said to be a *solution* of the *Briot–Bouquet differential superordination* if

$$\Omega \subset \left\{ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \;\middle|\; z \in U \right\}.$$

The authors determine properties of functions p satisfying this differential superordination and also some generalized versions of it.

In addition, for sets Ω_1 and Ω_2 in the complex plane the authors determine properties of functions p satisfying a Briot–Bouquet sandwich of the form

$$\Omega_1 \subset \left\{ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \;\middle|\; z \in U \right\} \subset \Omega_2.$$

Generalizations of this result are also considered.

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1. Introduction

We begin by introducing the two important classes of functions considered in this article.

Let $\mathbf{H} = \mathbf{H}(\mathbf{U})$ denote the class of functions analytic in $\mathbf{U}$. For n a positive integer and $a \in \mathbf{C}$, let

$$\mathbf{H}[a,n] = \{ f \in \mathbf{H} \mid f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \}.$$

Let $\mathbf{Q}$ denote the set of functions f that are analytic and injective on the set $\overline{\mathbf{U}} \setminus \mathbf{E}(f)$, where

$$\mathbf{E}(f) = \Big\{ \varsigma \in \partial U \;\Big|\; \lim_{z \to \varsigma} f(z) = \infty \Big\},$$

and are such that $f'(\varsigma) \neq 0$ for $\varsigma \in \partial\mathbf{U} \setminus \mathbf{E}(f)$. The subclass of $\mathbf{Q}$ for which $f(0) = a$ is denoted by $\mathbf{Q}(a)$.

Most of the functions considered in this article, and conditions on them are defined uniformly in the unit disk $\mathbf{U}$. Because of this we shall omit the requirement "$z \in \mathbf{U}$" in most of the definitions and results.

Many of the inclusion results that follow can be written very neatly in terms of subordination and superordination. We recall these definitions. Let $f, F \in \mathbf{H}$ and let F be univalent in $\mathbf{U}$. The function F is said to be *superordinate to* f, or f is *subordinate to* F, written $f \prec F$, if $f(0) = F(0)$ and $f(\mathbf{U}) \subset F(\mathbf{U})$.

Let β and γ be complex numbers, let Ω_2 and Δ_2 be sets in the complex plane, and let p be analytic in the unit disk $\mathbf{U}$. In a series of articles the authors and many others [7, pp. 80–119] have determined conditions so

$$\left\{ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \;\middle|\; z \in \mathbf{U} \right\} \subset \Omega_2 \quad \Rightarrow \quad p(\mathbf{U}) \subset \Delta_2. \tag{1}$$

The differential operator on the left is known as the *Briot–Bouquet differential operator.* The main concern in this subject has been to find the smallest set Δ_2 in $\mathbf{C}$ for which (1) holds. This particular differential implication has a surprising number of applications in univalent function theory.

In this article we consider the dual problem of determining conditions so that

$$\Omega_1 \subset \left\{ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \;\middle|\; z \in \mathbf{U} \right\} \quad \Rightarrow \quad \Delta_1 \subset p(\mathbf{U}). \tag{2}$$

In particular, we are interested in determining the largest set Δ_1 in $\mathbf{C}$ for which (2) holds.

If the sets Ω and Δ in (1) and (2) are simply connected domains not equal to $\mathbf{C}$, then it is possible to rephrase these expressions very neatly in terms of subordination and superordination in the forms:

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h_2(z) \quad \Rightarrow \quad p(z) \prec q_2(z), \tag{1$'$}$$

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \quad \Rightarrow \quad q_1(z) \prec p(z). \tag{2$'$}$$

The left side of (1′) is called a *Briot–Bouquet differential subordination*, and the function q_2 is called a *dominant* of the differential subordination. The *best dominant,* which provides a sharp result, is the dominant that is subordinate to all other dominants. Many results and applications on these topics can be found in [7, pp. 80–119].

In a recent paper [6] the authors have introduced the dual concept of a differential superordination. In light of those results we call the left side of (2′) a *Briot–Bouquet differential* *superordination,* and the function q_1 is called a *subordinant* of the differential superordination. The *best subordinant*, which provides a sharp result is the subordinant which is superordinate to all other subordinants. Some other recent results related to (2′) can be found in [1] and [2].

In this article we will combine (1′) and (2′) to obtain conditions so that the *Briot–Bouquet sandwich*

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h_2(z) \tag{3}$$

implies that $q_1(z) \prec p(z) \prec q_2(z)$. This result, implication (2′), and generalizations of these results will be given in Section 3. First we provide the lemmas needed to complete the proofs in Section 3.

2. Preliminaries

The first lemma provides a simple criterion for finding a subordinant and best subordinant of a first-order differential superordination.

Lemma A. [8, Theorem 5] *Let h be analytic in* $\mathbf{U}$*,* $q \in \mathbf{H}[a,n]$*,* $\varphi : \mathbf{C}^2 \to \mathbf{C}$*, and suppose that*

$$\varphi\big(q(z), tzq'(z)\big) \in h(\mathbf{U}), \tag{4}$$

for $z \in \mathbf{U}$*, and* $0 < t \leqslant 1/n \leqslant 1$*. If* $p \in \mathbf{Q}(a)$ *and* $\varphi(p(z), zp'(z))$ *is univalent in* $\mathbf{U}$*, then*

$$h(z) \prec \varphi\big(p(z), zp'(z)\big) \quad \Rightarrow \quad q(z) \prec p(z).$$

Furthermore, if $\varphi(q(z), zq'(z)) = h(z)$ *has a univalent solution* $q \in \mathbf{Q}(a)$*, then* q *is the best subordinant.*

A function $L(z,t)$, with $z \in \mathbf{U}$ and $t \geqslant 0$, is a subordination chain if $L(\cdot,t)$ is analytic and univalent in $\mathbf{U}$ for all $t \geqslant 0$, $L(z,\cdot)$ is continuously differentiable on $\mathbf{R}^+$ for all $z \in \mathbf{U}$, and $L(z,s) \prec L(z,t)$, when $0 \leqslant s \leqslant t$ [9, p. 157]. The following lemma provides a sufficient condition for $L(z,t)$ to be a subordination chain.

Lemma B. ([7, p. 4] and [9, p. 159]) *The function* $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$*, with* $a_1(t) \neq 0$*, for* $t \geqslant 0$*, and* $\lim_{t\to\infty} |a_1(t)| = \infty$*, is a subordination chain if*

$$\operatorname{Re}\left[\frac{z(\partial L/\partial z)}{\partial L/\partial t}\right] > 0, \tag{5}$$

for $z \in \mathbf{U}$ *and* $t \geqslant 0$*.*

The next lemma provides subordinants and best subordinants of a differential superordination by applying the theory of subordination chains.

Lemma C. [8, Theorem 7] *Let* $q \in \mathbf{H}[a,1]$*, let* $\varphi : \mathbf{C}^2 \to \mathbf{C}$*, and let h be defined by*

$$\varphi\big(q(z), zq'(z)\big) = h(z). \tag{6}$$

If $L(z,t) = \varphi(q(z), tzq'(z))$ *is a subordination chain and* $p \in \mathbf{H}[a,1] \cap \mathbf{Q}$*, then*

$$h(z) \prec \varphi\big(p(z), zp'(z)\big) \quad \Rightarrow \quad q(z) \prec p(z).$$

Furthermore, if $\varphi(q(z), zq'(z)) = h(z)$ *has a univalent solution* $q \in \mathbf{Q}(a)$*, then* q *is the best subordinant.*

3. Main results

Theorem 1. *Let h be convex in $\mathbf{U}$, with $h(0)=a$, and let Θ and Φ be analytic in a domain $\mathbf{D}$. Let $p\in\mathbf{H}[a,1]\cap\mathbf{Q}$ and suppose that $\Theta[p(z)]+zp'(z)\Phi[p(z)]$ is univalent in $\mathbf{U}$. If the differential equation*

$$\Theta\big[q(z)\big]+zq'(z)\Phi\big[q(z)\big]=h(z) \tag{7}$$

has a univalent solution q that satisfies $q(0)=a$, $q(\mathbf{U})\subset\mathbf{D}$, and

$$\Theta\big[q(z)\big]\prec h(z), \tag{8}$$

then

$$h(z)\prec\Theta\big[p(z)\big]+zp'(z)\Phi\big[p(z)\big] \quad\Rightarrow\quad q(z)\prec p(z). \tag{9}$$

The function q is the best subordinant.

Proof. We can assume that h, p and q satisfy the conditions of this theorem on the closed disk $\overline{\mathbf{U}}$, and that $q'(\varsigma)\neq 0$ for $|\varsigma|=1$. If not, then we can replace h, p and q with $h(\rho z)$, $p(\rho z)$ and $q(\rho z)$, where $0<\rho<1$. These new functions have the desired properties on $\overline{\mathbf{U}}$, and we can use them in the proof of the theorem. Theorem 1 would then follow by letting $\rho\to 1$.

We will use Lemma A to prove this result. If we let $\varphi(r,s)=\Theta[r]+s\Phi[r]$, then (7) becomes $\varphi(q(z),zq'(z))=h(z)$, and we have

$$\varphi\big(q(z),tzq'(z)\big)=\Theta\big[q(z)\big]+tzq'(z)\Phi\big[q(z)\big].$$

By applying (7) this simplifies to

$$\varphi\big(q(z),tzq'(z)\big)=(1-t)\Theta\big[q(z)\big]+th(z).$$

From (8) and the convexity of $h(\mathbf{U})$ we conclude that $\varphi(q(z),tzq'(z))\in h(\mathbf{U})$ for $0\leqslant t\leqslant 1$. Hence condition (4) of Lemma A is satisfied and the conclusions of this theorem follow. □

In the special case when $\Theta[w]=w$ and $\Phi[w]=[\beta w+\gamma]^{-1}$ we obtain the following result for the Briot–Bouquet differential superordination.

Corollary 1.1. *Let $\beta,\gamma\in\mathbf{C}$, and let h be convex in $\mathbf{U}$, with $h(0)=a$. Suppose that the differential equation*

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=h(z) \tag{10}$$

has a univalent solution q that satisfies $q(0)=a$, and $q(z)\prec h(z)$. If $p\in\mathbf{H}[a,1]\cap\mathbf{Q}$ and $p(z)+zp'(z)[\beta p(z)+\gamma]^{-1}$ is univalent in $\mathbf{U}$, then

$$h(z)\prec p(z)+\frac{zp'(z)}{\beta p(z)+\gamma} \quad\Rightarrow\quad q(z)\prec p(z). \tag{11}$$

The function q is the best subordinant.

For conditions and examples for which the Briot–Bouquet differential equation (10) has univalent solutions see [4] and [7, p. 91].

There is a complete analog of Theorem 1 for differential subordinations, which is given in [5, p. 189] and [7, p. 125]. We can combine that result with Theorem 1 and obtain the following sandwich theorem.

Theorem 2. *Let h_1 and h_2 be convex in* $\mathbf{U}$*, with $h_1(0) = h_2(0) = a$, and let Θ and Φ be analytic in a domain D. Let $p \in \mathbf{H}[a,1] \cap \mathbf{Q}$ and suppose that $\Theta[p(z)] + zp'(z)\Phi[p(z)]$ is univalent in* $\mathbf{U}$*. If the differential equations*

$$\Theta\big[q_i(z)\big] + zq_i'(z)\Phi\big[q_i(z)\big] = h_i(z)$$

have univalent solutions q_i that satisfy $q_i(0) = a$, $q_i(\mathbf{U}) \subset \mathbf{D}$, and

$$\Theta\big[q_i(z)\big] \prec h_i(z),$$

for $i = 1, 2$, then

$$h_1(z) \prec \Theta\big[p(z)\big] + zp'(z)\Phi\big[p(z)\big] \prec h_2(z) \quad \Rightarrow \quad q_1(z) \prec p(z) \prec q_2(z).$$

The functions q_1 and q_2 are the best subordinant and best dominant, respectively.

In the special case when $\Theta[w] = w$ and $\Phi[w] = [\beta w + \gamma]^{-1}$ we obtain the following Briot–Bouquet sandwich result.

Corollary 2.1. *Let $\beta, \gamma \in \mathbf{C}$, and let h_i be convex in* $\mathbf{U}$*, with $h_i(0) = a$, for $i = 1, 2$. Suppose that the differential equations*

$$q_i(z) + \frac{zq_i'(z)}{\beta q_i(z) + \gamma} = h_i(z) \tag{12}$$

have a univalent solution q_i that satisfies $q_i(0) = a$, and $q_i(z) \prec h_i(z)$, for $i = 1, 2$. If $p \in \mathbf{H}[a,1] \cap \mathbf{Q}$ and $p(z) + zp'(z)[\beta p(z) + \gamma]^{-1}$ is univalent in $\mathbf{U}$*, then*

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h_2(z) \quad \Rightarrow \quad q_1(z) \prec p(z) \prec q_2(z).$$

The functions q_1 and q_2 are the best subordinant and best dominant, respectively.

If $\beta = 0$ and $\gamma \neq 0$ with $\operatorname{Re}\gamma \geqslant 0$, then (12) has univalent (convex) solutions given by

$$q_i(z) = \frac{\gamma}{z^\gamma} \int_0^z h_i(t) t^{\gamma - 1}\, dt, \tag{13}$$

for $i = 1, 2$. In this case we obtain the following sandwich theorem.

Corollary 2.2. *Let h_1 and h_2 be convex in* $\mathbf{U}$*, with $h_1(0) = h_2(0) = a$. Let $\gamma \neq 0$ with* $\operatorname{Re}\gamma \geqslant 0$*, and let the functions q_i be defined by* (13) *for $i = 1, 2$. If $p \in H[a,1] \cap Q$ and $p(z) + zp'(z)/\gamma$ is univalent in* $\mathbf{U}$*, then*

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\gamma} \prec h_2(z) \quad \Rightarrow \quad q_1(z) \prec p(z) \prec q_2(z). \tag{14}$$

The functions q_1 and q_2 are the best subordinant and best dominant, respectively.

Hallenbeck and Ruscheweyh [3] gave the right differential subordination and conclusion in (14), while the authors [8, Theorem 6] gave the corresponding left differential superordination and conclusion.

Theorem 1 has dealt with finding a subordinant or the best subordinant for a differential superordination for a given h. We next attack the problem from a different direction; first select the subordinant q and then find the appropriate h corresponding to this q.

Theorem 3. *Let Θ and Φ be analytic in a domain* $\mathbf{D}$*, and let q be univalent in* $\mathbf{U}$*, with $q(0)=a$ and $q(\mathbf{U})\subset\mathbf{D}$. Set $Q(z)=zq'(z)\cdot\Phi[q(z)]$, $h(z)=\Theta[q(z)]+Q(z)$ and suppose that*

(i) $\operatorname{Re}\left[\dfrac{\Theta'[q(z)]}{\Phi[q(z)]}\right]>0$, *and*

(ii) $Q(z)$ *is starlike.*

If $p\in\mathbf{H}[a,1]\cap\mathbf{Q}$, $p(\mathbf{U})\subset\mathbf{D}$ and $\Theta[p(z)]+zp'(z)\Phi[p(z)]$ is univalent in $\mathbf{U}$*, then*

$$h(z)\prec\Theta\big[p(z)\big]+zp'(z)\Phi\big[p(z)\big]\quad\Rightarrow\quad q(z)\prec p(z),$$

and q is the best subordinant.

Proof. As we have done before, without loss of generality we can assume that p, q, and h satisfy the conditions of this theorem on the closed disk $\overline{\mathbf{U}}$, and that $q'(\varsigma)\neq 0$ for $|\varsigma|=1$. If we let

$$\varphi(r,s)=\Theta[r]+s\Phi[r]$$

then the function q satisfies the differential equation

$$\varphi\big(q(z),zq'(z)\big)=\Theta\big[q(z)\big]+zq'(z)\Phi\big[q(z)\big]=h(z).$$

We will use Lemma C to prove this result by showing that $L(z,t)\equiv\varphi(q(z),tzq'(z))$ is a subordination chain. The function

$$L(z,t)=\Theta\big[q(z)\big]+tzq'(z)\Phi\big[q(z)\big]=a_1(t)z+a_2(t)z^2+\cdots$$

is analytic in $\mathbf{U}$ for all $t\geqslant 0$, and is continuously differentiable on $[0,\infty)$. A simple calculation shows that

$$a_1(t)=\frac{\partial L}{\partial z}(0,t)=q'(0)\cdot\Phi\big[q(0)\big]\left[\frac{\Theta'[q(0)]}{\Phi[q(0)]}+t\right].\tag{15}$$

Since q is univalent we have $q'(0)\neq 0$, and combining this with condition (i) for $z=0$, from (15) we obtain $a_1(t)\neq 0$, for $t\geqslant 0$. Also from (15) we obtain $\lim_{t\to\infty}|a_1(t)|=\infty$.

Another calculation combined with conditions (i) and (ii) leads to

$$\operatorname{Re}\left[\frac{z(\partial L/\partial z)}{\partial L/\partial t}\right]=\operatorname{Re}\left[\frac{\Theta'[q(z)]}{\Phi[q(z)]}+t\frac{zQ'(z)}{Q(z)}\right]>0,$$

for $z\in\mathbf{U}$ and $t\geqslant 0$. According to Lemma B the function $L(z,t)$ is a subordination chain, and from Lemma C the conclusions of the theorem follow. □

In the special case when $\Theta[w]=w$ and $\Phi[w]=[\beta w+\gamma]^{-1}$ Theorem 3 simplifies to the following result for the Briot–Bouquet differential superordinations.

Corollary 3.1. *Let $\beta,\gamma\in\mathbf{C}$, and let q be univalent in* $\mathbf{U}$*, with $q(0)=a$. Set*

$$h(z)=q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}\tag{16}$$

and suppose that

(i) $\operatorname{Re}[\beta q(z)+\gamma]>0$, *and*

(ii) $\dfrac{zq'(z)}{\beta q(z)+\gamma}$ *is starlike.*

If $p \in \mathbf{H}[a,1] \cap \mathbf{Q}$ *and* $p(z) + zp'(z)[\beta p(z)+\gamma]^{-1}$ *is univalent in* $\mathbf{U}$*, then*

$$h(z) \prec p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \quad \Rightarrow \quad q(z) \prec p(z),$$

and q *is the best subordinant.*

Several previous results of the authors enable us to replace the conditions that q be univalent and that (i) be satisfied in the above result with weaker conditions. To do this we need to introduce the *open door function*. Let c be a complex number such that $\operatorname{Re} c > 0$ and let

$$C = \frac{|c|\sqrt{1+2\operatorname{Re} c} + \operatorname{Im} c}{\operatorname{Re} c}.$$

If $R(z)$ is the univalent function defined in $\mathbf{U}$ by $R(z) = 2Cz/(1-z^2)$, and $b = R^{-1}(c)$, then the *open door function* is defined by

$$R_c(z) = R\left(\frac{z+b}{1+\bar{b}z}\right) = 2C\frac{(z+b)(1+\bar{b}z)}{(1+\bar{b}z)^2 - (z+b)^2}.$$

This function is univalent and maps the unit disk onto the complex plane with slits along the half-lines $\operatorname{Re} w = 0$, and $|\operatorname{Im} w| \geqslant C$. In [7, pp. 86–91] it is shown that if

$$\beta h(z) + \gamma \prec R_{\beta a+\gamma}(z),$$

then differential equation (16) has an analytic solution q that satisfies condition (i) in Corollary 3.1. In addition, condition (ii) implies that this solution q is univalent. Combining these results with Corollary 3.1 we obtain the following improved result.

Corollary 3.2. *Let* $h \in \mathbf{H}(\mathbf{U})$ *with* $h(0) = a$*, let* $\beta, \gamma \in \mathbf{C}$ *with* $\operatorname{Re}[\beta a + \gamma] > 0$*, and suppose that*

(i) $\beta h(z) + \gamma \prec R_{\beta a+\gamma}(z)$.

Let q *be the analytic solution of the Briot–Bouquet differential equation*

$$h(z) = q(z) + \frac{zq'(z)}{\beta q(z)+\gamma}$$

and suppose that

(ii) $\dfrac{zq'(z)}{\beta q(z)+\gamma}$ *is starlike.*

If $p \in \mathbf{H}[a,1] \cap \mathbf{Q}$ *and* $p(z) + zp'(z)[\beta p(z)+\gamma]^{-1}$ *is univalent in* $\mathbf{U}$*, then*

$$h(z) \prec p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \quad \Rightarrow \quad q(z) \prec p(z),$$

and q *is the best subordinant.*

There is a complete analog of Theorem 3 for differential subordinations, which is given in [5, p. 190] and [7, p. 132]. We can combine that result with Theorem 3 and obtain the following sandwich theorem.

Theorem 4. *Let Θ and Φ be analytic in a domain* **D**, *and let q_1 and q_2 be univalent in* **U**, *with $q_i(0) = a$ and $q_i(\mathbf{U}) \subset \mathbf{D}$, for $i = 1, 2$. Set $Q_i(z) = zq_i'(z) \cdot \Phi[q_i(z)]$, $h_i(z) = \Theta[q_i(z)] + Q_i(z)$ and suppose that*

(i) $\mathrm{Re}\left[\dfrac{\Theta'[q_i(z)]}{\Phi[q_i(z)]}\right] > 0$, *and*

(ii) $Q_i(z)$ *is starlike.*

If $p \in \mathbf{H}[a, 1] \cap \mathbf{Q}$, $p(\mathbf{U}) \subset \mathbf{D}$ and $\Theta[p(z)] + zp'(z)\Phi[p(z)]$ is univalent in **U**, *then*

$$h_1(z) \prec \Theta\big[p(z)\big] + zp'(z)\Phi\big[p(z)\big] \prec h_2(z) \quad \Rightarrow \quad q_1(z) \prec p(z) \prec q_2(z).$$

The functions q_1 and q_2 are the best subordinant and best dominant, respectively.

For the special case of the Briot–Bouquet differential operator this result becomes:

Corollary 4.1. *For $i = 1, 2$ let $h_i \in \mathbf{H}(\mathbf{U})$ with $h_i(0) = a$. Let $\beta, \gamma \in \mathbf{C}$ with* $\mathrm{Re}[\beta a + \gamma] > 0$, *and suppose that*

(i) $\beta h_i(z) + \gamma \prec R_{\beta a+\gamma}(z)$.

Let q_i be analytic solutions of the Briot–Bouquet differential equation

$$h_i(z) = q_i(z) + \frac{zq_i'(z)}{\beta q_i(z) + \gamma}$$

for $i = 1, 2$, and suppose that

(ii) $\dfrac{zq_i'(z)}{\beta q_i(z) + \gamma}$ *is starlike.*

If $p \in \mathbf{H}[a, 1] \cap \mathbf{Q}$ and $p(z) + zp'(z)[\beta p(z) + \gamma]^{-1}$ is univalent in **U**, *then*

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h_2(z) \quad \Rightarrow \quad q_1(z) \prec p(z) \prec q_2(z).$$

The functions q_1 and q_2 are the best subordinant and best dominant, respectively.

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