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A new asymptotic series for the Gamma function ☆

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Abstract

The famous Stirling's formula says that $\Gamma(s+1)=\sqrt{2\pi s}(s/e)^s e^{\gamma(s)}=\sqrt{2\pi}(s/e)^s e^{\theta(s)/12s}$. In this paper, we obtain a novel convergent asymptotic series of $\gamma(s)$ and proved that $\theta(s)$ is increasing for $s>0$.

MSC: 33B15; 05A16

Keywords: Gamma function; Asymptotic series; Stirling formula

1. Introduction

The Gamma function, one of the most famous functions in both mathematics and applied sciences,

$$s! = \Gamma(s+1) = \int_0^{\infty} x^s e^{-x}\, dx, \qquad s>0 \tag{1}$$

can be analytically expanded to the whole complex plane excluding non-positive integers. It is well-known that the Gamma function has the following famous Stirling asymptotic series

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(see [1, p. 167, 6, pp. 111–112])

$$\ln(s-1)! = \frac{1}{2}\ln 2\pi + \left(s-\frac{1}{2}\right)\ln s - s + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_n}{2n(2n-1)s^{2n-1}}, \tag{2}$$

where B_m are Bernoulli numbers. However, series (2) is not convergent (see [1, p. 167]). In addition, Lanczos obtained an efficient method for numerical calculation of the Gamma function (see [2]). In this paper we obtained the following novel convergent asymptotic series.

Theorem 1. *Let $v \geqslant 0$ be a real number and s be a complex number such that $Re(s) \geqslant 1$, where $Re(s)$ denotes the real part of s. Then, we obtained*

$$\ln s! = \tfrac{1}{2}\ln 2\pi + (s+\tfrac{1}{2})\ln s - s + \gamma(s), \tag{3}$$

where

$$\gamma(s) = \sum_{k=1}^{\infty}\frac{a_k(v)}{(s+v)(s+v+1)\cdots(s+v+k-1)},$$

$$a_k(v) = \frac{1}{2k}\int_0^1 (1-2t)(v-t)(v+1-t)\cdots(v+k-1-t)\,\mathrm{d}t.$$

Set $v=1$ in (3), we obtain the Binet's formula (see [7, p. 253])

$$\ln(s-1)! = \frac{1}{2}\ln 2\pi + \left(s-\frac{1}{2}\right)\ln s - s + \sum_{k=1}^{\infty}\frac{a_k(1)}{(s+1)(s+2)\cdots(s+k)}. \tag{4}$$

Furthermore, we generalize equality (3) to all complex numbers s excluding negative integers.

Corollary 1. *For all complex numbers s but not negative integer, let $m \geqslant -[Re(s)]$ be a non-negative integer, where $[x]$ is defined to be the greatest integer not exceeding x. Then, we have*

$$\ln s! = \frac{1}{2}\ln 2\pi + \left(s+m+\frac{1}{2}\right)\ln(s+m+1) - \sum_{p=1}^{m}\ln(s+p)$$
$$- s - m - 1 + \gamma(s+m+1). \tag{5}$$

For the polynomials $a_k(v)'s$ introduced in Theorem 1, there holds the following results.

Theorem 2. *For any $v \geqslant 0$, $a_1(v) = \frac{1}{12}$, and $a_2(v) = v/12$. If $v \geqslant v_0 \approx 0.146094$, the positive root of $240v^3 + 600v^2 + 270v - 53 = 0$, then $a_k(v) > 0$ for $k \geqslant 3$. If $v = 0$, $a_k(v) < 0$ for $k \geqslant 3$. If $0 < v < v_0$, then there exists an integer K, such that $a_k(v) > 0$ for $k \geqslant K$.*

In this paper, for any complex number s with $Re(s) \geqslant 1$, we obtain the following generalized Stirling formula in a new approach (see [7, p. 253])

$$s! = \sqrt{2\pi s}\left(\frac{s}{e}\right)^s e^{\gamma(s)} = \sqrt{2\pi}\left(\frac{s}{e}\right)^s e^{\theta(s)/12s}, \tag{6}$$

where $\theta(s)$ is an analytic function satisfying $0 < \theta(s) < 1$ for real numbers $s \geqslant 1$.

In addition, (4) shows that $\gamma(s)$ is a strictly decreasing function for $s \geqslant 1$. We proved that $\theta(s)$ is also monotone. More precisely, we have

Theorem 3. *$\theta(s)$ is a strictly increasing function for real numbers $s \geqslant 1$.*

Let n be a counting number. Herbert Robbins [5] obtained the following inequality:

$$\frac{1}{12n+1} < \gamma(n) < \frac{1}{12n}. \tag{7}$$

Maria [3] and Nanjundiah [4] improved the above inequality to get

$$\frac{1}{12n+3/2(2n+1)} < \gamma(n) < \frac{1}{12n}, \tag{8}$$

and

$$\frac{1}{12n} - \frac{1}{360n^3} < \gamma(n) < \frac{1}{12n}, \tag{9}$$

respectively. In fact, (9) is stronger than (8) since

$$\frac{1}{12n+3/2(2n+1)} < \frac{1}{12n} - \frac{1}{360n^3}.$$

In this paper, we improve the above results and obtain the following estimate,

$$\frac{1}{12s} - \frac{1}{360s^3} < \gamma(s) < \frac{1}{12s} - \frac{1}{360s(s+1)(s+2)} \tag{10}$$

and more precise estimates. To obtain our results, we need the following asymptote formula.

Theorem 4.

$$\gamma(s) = \frac{1}{12s} + \sum_{q=1}^{\infty}(-1)^q b_q \sum_{p=0}^{\infty} \frac{1}{(s+p)^{q+1}(s+p+1)^{q+1}} \tag{11}$$

and

$$\left| \ln \Gamma(s+1) - \ln\left(\sqrt{2\pi s}\left(\frac{s}{e}\right)^s\right) - \frac{1}{12s} - S_k(s) \right|$$
$$< b_{k+1} \sum_{p=0}^{\infty} \frac{1}{(Re(s)+p)^{k+2}(Re(s)+p+1)^{k+2}}, \tag{12}$$

where

$$b_q = \frac{1}{2}\int_0^{1/2} (1-2t)^2 t^q (1-t)^q \, \mathrm{d}t = \frac{(q!)^2}{4(2q+3)(2q+1)!}. \tag{13}$$

Furthermore, for real numbers $s \geqslant 1$, it holds

$$\frac{1}{12s} + S_{k+1}(s) - \delta_k(s) < \gamma(s) < \frac{1}{12s} + S_{k+1}(s) \quad \text{if } k \geqslant 1 \text{ is odd},$$
$$\frac{1}{12s} + S_{k+1}(s) < \gamma(s) < \frac{1}{12s} + S_{k+1}(s) + \delta_k(s) \quad \text{if } k \geqslant 2 \text{ is even}, \tag{14}$$

where

$$S_k(s) = \sum_{q=1}^{k} (-1)^q b_q \sum_{p=0}^{\infty} \frac{1}{(s+p)^{q+1}(s+p+1)^{q+1}},$$

$$\delta_k(s) = b_{k+1} \sum_{p=0}^{\infty} \frac{1}{4(s+p)^{k+2}(s+p+1)^{k+2}(s+p+\frac{1}{2})^2}.$$

We also generalize Theorem 4 to the following theorem.

Theorem 5. *Let s be any complex number except for non-positive integers and let $M \geqslant -Re(s)$ be a non-negative integer. Then, there holds*

$$\gamma(s) = \left(s + M + \frac{1}{2}\right) \ln(s+M+1) - \left(s + \frac{1}{2}\right) \ln s$$
$$- \sum_{p=1}^{M} \ln(s+p) - M - 1 + \gamma(s+M+1) \tag{15}$$

and

$$|\ln \Gamma(s+1) - \Gamma_M(s)| < b_{k+1} \sum_{p=0}^{\infty} \frac{1}{(Re(s)+M+p+1)^{k+2}(Re(s)+M+p+2)^{k+2}}, \tag{16}$$

where

$$\gamma(s+M+1) = \frac{1}{12(s+M+p+1)} + \sum_{q=1}^{\infty} (-1)^q b_q$$
$$\times \sum_{p=0}^{\infty} \frac{1}{(s+M+1+p)^{q+1}(s+M+p+2)^{q+1}},$$

$$\Gamma_M(s) = \frac{1}{2} \ln 2\pi + \left(s + M + \frac{1}{2}\right) \ln(s+M+1) - \sum_{p=1}^{M} \ln(s+p)$$
$$- s - M - 1 + S_k(s+M+1).$$

From Theorem 4, we obtain the following more precise inequalities.

Theorem 6. *For all real numbers $s \geqslant 1$, the following inequality holds*

$$\frac{1}{12s} - \frac{1}{360s^3} + \frac{1 - \dfrac{3}{40(s+\frac{1}{2})^2}}{1 + \dfrac{1}{s(s+1)} + \dfrac{1}{5s^2(s+1)^2}} \frac{1}{1260s^5} < \gamma(s) < \frac{1}{12s} - \frac{1}{360s^3} + \frac{1}{1260s^5}, \tag{17}$$

$$\gamma(s) > \frac{1}{12s} - \frac{1}{360s^3} + \frac{1}{1260s^5} - \left(1 + \frac{4}{21s(s+1)}\right)\frac{1}{1680s^7},$$

$$\gamma(s) < \frac{1}{12s} - \frac{1}{360s^3} + \frac{1}{1260s^5} - \frac{1 - \dfrac{1}{84(s+\frac{1}{2})^2}}{1 + \dfrac{2}{s(s+1)} + \dfrac{1}{s^2(s+1)^2} + \dfrac{1}{7s^3(s+1)^3}} \frac{1}{1680s^7}, \tag{18}$$

$$\gamma(s) > \frac{1}{12s} - \frac{1}{360s^3} + \frac{1}{1260s^5} - \frac{1}{1680s^7} + \frac{1}{1 + \dfrac{10}{3s(s+1)} + \dfrac{3}{s^2(s+1)^2} + \dfrac{1}{s^3(s+1)^3} + \dfrac{1}{9s^4(s+1)^4}} \frac{1}{1188s^9},$$

$$\gamma(s) < \frac{1}{12s} - \frac{1}{360s^3} + \frac{1}{1260s^5} - \frac{1}{1680s^7} + \left(1 + \frac{11}{20s(s+1)} + \frac{11}{140s^2(s+1)^2}\right)\frac{1}{1188s^9}, \tag{19}$$

and

$$\gamma(s) > p(s) - \left(1 + \frac{15951}{15202s(s+1)} + \frac{2730}{7601s^2(s+1)^2} + \frac{910}{22803s^3(s+1)^3}\right)\frac{691}{360360s^{11}},$$

$$\gamma(s) < p(s) - \frac{1}{1 + \dfrac{5}{s(s+1)} + \dfrac{7}{s^2(s+1)^2} + \dfrac{4}{s^3(s+1)^3} + \dfrac{1}{s^4(s+1)^4} + \dfrac{1}{11s^4(s+1)^4}} \times \frac{691}{360360s^{11}}, \tag{20}$$

where $p(s) = \dfrac{1}{12s} - \dfrac{1}{360s^3} + \dfrac{1}{1260s^5} - \dfrac{1}{1680s^7} + \dfrac{1}{1188s^9}$.

2. The proof of Theorem 1

To prove equality (3) is equivalent to prove

$$\gamma(s) = \sum_{k=1}^{\infty} \frac{a_k(v)}{(s+v)(s+v+1)\cdots(s+v+k-1)}, \quad Re(s) \geqslant 1. \tag{21}$$

The following equality is crucial in the proof of (21).

$$\gamma(s) = \frac{1}{2}\sum_{p=0}^{\infty}\int_0^1 \frac{1-2t}{t+s+p}\,\mathrm{d}t, \tag{22}$$

where s is any complex number but not real number less than 1. Equality (22) can be proved from Euler's formula (see [7, p. 237])

$$s! = \lim_{n\to\infty}\frac{n!n^s}{(s+1)(s+2)\cdots(s+n)}. \tag{23}$$

and the Stirling's formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\gamma_n}, \quad 0 < \gamma_n < \frac{1}{12n}, \tag{24}$$

where n is a counting number.

Next, we prove (22). In fact, for counting number n, there holds

$$\begin{aligned}
&\frac{1}{2}\ln 2\pi + \left(s+\frac{1}{2}\right)\ln s - s + \frac{1}{2}\sum_{p=0}^{n}\int_0^1 \frac{1-2t}{t+s+p}\,\mathrm{d}t\\
&\quad = \ln\left(\sqrt{2\pi s}\left(\frac{s}{e}\right)^s\right) + \sum_{p=0}^{n}\ln\frac{(1+s+p)^{(1/2)+s+p}}{e(s+p)^{(1/2)+s+p}}\\
&\quad = \ln\left(\sqrt{2\pi s}\left(\frac{s}{e}\right)^s\right) + \ln\frac{(1+s+n)^{(1/2)+s+n}}{e^{n+1}s^{(1/2)+s}(s+1)(s+2)\cdots(s+n)}\\
&\quad = \ln\left(\frac{\sqrt{2\pi}(1+s+n)^{(1/2)+s+n}}{e^{s+n+1}n!n^s}\,\frac{n!n^s}{(s+1)(s+2)\cdots(s+n)}\right).
\end{aligned}$$

From (24), we can prove

$$\lim_{n\to\infty}\frac{\sqrt{2\pi}(1+s+n)^{(1/2)+s+n}}{e^{s+n+1}n!n^s} = 1.$$

Therefore, it yields

$$\begin{aligned}
&\frac{1}{2}\ln 2\pi + \left(s+\frac{1}{2}\right)\ln s - s + \frac{1}{2}\sum_{p=0}^{\infty}\int_0^1 \frac{1-2t}{t+s+p}\,\mathrm{d}t\\
&\quad = \ln \lim_{n\to\infty}\frac{n!n^s}{(s+1)(s+2)\cdots(s+n)} = \ln s!.
\end{aligned} \tag{25}$$

Eq. (22) can be obtained by comparing (3) and (25).

From now on, we prove (21). For any counting number k, real number $v \geqslant 0$ and complex number s satisfying $Re(s) \geqslant 1$, it is easy to show that

$$\frac{1}{t+s+p} = \frac{1}{v+s+p} + \sum_{q=1}^{k} \frac{(v-t)(v-t+1)\cdots(v-t+q-1)}{(v+s+p)(v+s+p+1)\cdots(v+s+p+q)} + r_k,$$

where $r_k = \frac{(v-t)(v-t+1)\cdots(v-t+k)}{(v+s+p)(v+s+p+1)\cdots(v+s+p+k)(t+s+p)}$. For $Re(s) \geqslant 1$, since $\lim_{k\to\infty} r_k = 0$, it yields

$$\frac{1}{t+s+p} = \frac{1}{v+s+p} + \sum_{q=1}^{\infty} \frac{(v-t)(v-t+1)\cdots(v-t+q-1)}{(v+s+p)(v+s+p+1)\cdots(v+s+p+q)}. \tag{26}$$

From

$$\left|\frac{(v-t)(v-t+1)\cdots(v-t+q-1)}{(v+s+p)(v+s+p+1)\cdots(v+s+p+q)}\right| \leqslant \frac{\max\{v, 1\}}{(v+Re(s)+p+q-1)(v+Re(s)+p+q)}$$

for $Re(s) \geqslant 1$ and $q \geqslant 1$, and equalities (22) and (26), we have

$$\begin{aligned}
\gamma(s) &= \frac{1}{2}\sum_{p=0}^{\infty}\int_0^1 \frac{1-2t}{t+s+p}\,\mathrm{d}t \\
&= \frac{1}{2}\sum_{p=0}^{\infty}\int_0^1 \left(\frac{1-2t}{v+s+p} + \sum_{q=1}^{\infty}\frac{(1-2t)(v-t)(v-t+1)\cdots(v-t+q-1)}{(v+s+p)(v+s+p+1)\cdots(v+s+p+q)}\right)\mathrm{d}t \\
&= \frac{1}{2}\sum_{p=0}^{\infty}\int_0^1 \sum_{q=1}^{\infty}\frac{(1-2t)(v-t)(v-t+1)\cdots(v-t+q-1)}{(v+s+p)(v+s+p+1)\cdots(v+s+p+q)}\,\mathrm{d}t \\
&= \frac{1}{2}\sum_{q=1}^{\infty}\sum_{p=0}^{\infty}\int_0^1 \frac{(1-2t)(v-t)(v-t+1)\cdots(v-t+q-1)}{(v+s+p)(v+s+p+1)\cdots(v+s+p+q)}\,\mathrm{d}t \\
&= \frac{1}{2}\sum_{q=1}^{\infty}\frac{1}{q}\int_0^1 (1-2t)(v-t)(v-t+1)\cdots(v-t+q-1)\,\mathrm{d}t \\
&\quad \times \sum_{p=0}^{\infty}\left(\frac{1}{(v+s+p)(v+s+p+1)\cdots(v+s+p+q-1)}\right. \\
&\quad \left. - \frac{1}{(v+s+p+1)(v+s+p+2)\cdots(v+s+p+q)}\right) \\
&= \sum_{q=1}^{\infty}\frac{a_q(v)}{(s+v)(s+v+1)\cdots(s+v+q-1)}, \quad Re(s) \geqslant 1.
\end{aligned} \tag{27}$$

The proof of (3) is completed.

3. The proof of Corollary 1

Corollary 1 can be obtained from Theorem 1 and the following recursive formula. For any fixed non-negative integer $m \geqslant -[Re(s)]$ and any complex number s but not nonnegative integer, we have

$$s! = \frac{(s+m+1)!}{(s+1)(s+2)\cdots(s+m+1)}. \tag{28}$$

4. The proof of Theorem 2

It is easy to show that for any $v \geqslant 0$, $a_1(v) = \frac{1}{12}$, $a_2(v) = v/12$. If $v \geqslant \frac{1}{2}$, $0 \leqslant t \leqslant \frac{1}{2}$ and $1 \leqslant m \leqslant k-1$, then $v + m - t \geqslant v + m - 1 + t$ and

$$\begin{aligned} 2ka_k(v) &= \int_0^1 (1-2t)(v-t)(v+1-t)\cdots(v+k-1-t)\,\mathrm{d}t \\ &= \int_0^{1/2} (1-2t)(v-t)(v+1-t)\cdots(v+k-1-t)\,\mathrm{d}t \\ &\quad - \int_0^{1/2} (1-2t)(v+t-1)(v+t)\cdots(v+k-2+t)\,\mathrm{d}t \\ &\geqslant \int_0^{1/2} (1-2t)(v-t)(v+t)\cdots(v+k-2+t)\,\mathrm{d}t \\ &\quad - \int_0^{1/2} (1-2t)(v+t-1)(v+t)\cdots(v+k-2+t)\,\mathrm{d}t \\ &\geqslant \int_0^{1/2} (1-2t)^2(v+t)\cdots(v+k-2+t)\,\mathrm{d}t > 0. \end{aligned} \tag{29}$$

If $v_0 \leqslant v < \frac{1}{2}$, where $v_0 \approx 0.146094$ is the positive root of $240v^3 + 600v^2 + 270v - 53 = 0$, then

$$\begin{aligned} 2ka_k(v) &= \int_0^1 (1-2t)(v-t)(v+1-t)\cdots(v+k-1-t)\,\mathrm{d}t \\ &> \int_0^{1/2} (1-2t)(v-t)(v+1-t)\cdots(v+k-1-t)\,\mathrm{d}t \\ &= \int_0^{v} (1-2t)(v-t)(v+1-t)\cdots(v+k-1-t)\,\mathrm{d}t \\ &\quad + \int_v^{1/2} (1-2t)(v-t)(v+1-t)\cdots(v+k-1-t)\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&> \int_0^v (1-2t)(v-t)(v+1-t)(k-1)!\,\mathrm{d}t \\
&\quad + \int_v^{1/2} (1-2t)(v-t)(v+1-t)(k-1)!\,\mathrm{d}t \\
&= (k-1)! \int_0^{1/2} (1-2t)(v-t)(v+1-t)\,\mathrm{d}t \\
&= \frac{(k-1)!}{960}(240v^3+600v^2+270v-53) \geqslant 0. \qquad (30)
\end{aligned}
$$

If $v=0$, then $m-t \geqslant m-1+t$ for $0 \leqslant t \leqslant \frac{1}{2}$ and $2 \leqslant m \leqslant k-1$, and

$$
\begin{aligned}
-2ka_k(0) &= \int_0^1 (1-2t)t(1-t)(2-t)\cdots(k-1-t)\,\mathrm{d}t \\
&= \int_0^{1/2} (1-2t)t(1-t)(2-t)\cdots(k-1-t)\,\mathrm{d}t \\
&\quad + \int_0^{1/2} (2t-1)(1-t)t(1+t)\cdots(k-2+t)\,\mathrm{d}t \\
&> \int_0^{1/2} (1-2t)t(1-t)(1+t)\cdots(k-2+t)\,\mathrm{d}t \\
&\quad + \int_0^{1/2} (2t-1)(1-t)t(1+t)\cdots(k-2+t)\,\mathrm{d}t = 0. \qquad (31)
\end{aligned}
$$

If $0 < v < v_0 \approx 0.146094$, then similarly to (30), we have

$$
\begin{aligned}
2ka_k(v) &> \int_0^{1/2} (1-2t)(v-t)(v+1-t)\cdots(v+k-1-t)\,\mathrm{d}t \\
&= \int_0^v (1-2t)(v-t)(v+1-t)\cdots(v+k-1-t)\,\mathrm{d}t \\
&\quad + \int_v^{1/2} (1-2t)(v-t)(v+1-t)\cdots(v+k-1-t)\,\mathrm{d}t \\
&> \int_0^{v/2} (1-2t)(v-t)(v+1-t)\cdots(v+k-1-t)\,\mathrm{d}t \\
&\quad + \int_v^{1/2} (1-2t)(v-t)(v+1-t)\cdots(v+k-1-t)\,\mathrm{d}t \\
&> \left(1+\frac{v}{2}\right)\cdots\left(k-1+\frac{v}{2}\right) \int_0^{v/2} (1-2t)(v-t)\,\mathrm{d}t \\
&\quad + (k-1)! \int_v^{1/2} (1-2t)(v-t)\,\mathrm{d}t \to \infty, \quad \text{when } k \to \infty, \qquad (32)
\end{aligned}
$$

since

$$\ln \frac{(1+v/2)\cdots(k-1+v/2)}{(k-1)!} = \sum_{p=1}^{k-1} \ln\left(1+\frac{v}{2p}\right) > \sum_{p=1}^{k-1}\left(\frac{v}{2p}-\frac{v^2}{8p^2}\right) \to \infty, \quad \text{when } k \to \infty.$$

Inequality (32) shows that $a_k(v) > 0$ for $v > 0$ and sufficient large counting number k. Thus Theorem 2 is proved.

Note: Theorem 2 can be further improved with more detailed calculations.

5. The proof of Theorem 3

Theorem 3 can be derived directly from Theorem 2. In fact, let $v = 0$ in (21), there holds

$$\theta(s) = 12s\gamma(s) = 1 + \sum_{k=3}^{\infty} \frac{12a_k(0)}{(s+1)(s+2)\cdots(s+k-1)} \tag{33}$$

which is strictly increasing since $a_k(0) < 0$ for $k \geqslant 3$. Let $v = 1$ in (21), there holds

$$\gamma(s) = \sum_{k=1}^{\infty} \frac{a_k(1)}{(s+1)(s+2)\cdots(s+k)} \tag{34}$$

which is strictly decreasing since $a_k(1) > 0$ for $k \geqslant 1$. Equality (6) can be easily obtained from (33) and (34).

6. The proof of (10)

By Theorem 2, and $a_3(0) = -\frac{1}{360}$, we obtain the right hand side inequality of (14),

$$\gamma(s) = \frac{1}{12s} + \sum_{k=3}^{\infty} \frac{a_k(0)}{s(s+1)(s+2)\cdots(s+k-1)} < \frac{1}{12s} - \frac{1}{360s(s+1)(s+2)}. \tag{35}$$

For the left-hand side inequality, we could refer to Theorem 4. But here we give an alternative proof which also generalizes the result in [4] from counting numbers to all real numbers $s \geqslant 1$. From (22), we have

$$
\begin{aligned}
\gamma(s)-\gamma(s+1) &= \frac{1}{2}\sum_{p=0}^{\infty}\int_0^1 \frac{1-2t}{t+s+p}\,\mathrm{d}t - \frac{1}{2}\sum_{p=0}^{\infty}\int_0^1 \frac{1-2t}{t+s+1+p}\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^1 \frac{1-2t}{t+s}\,\mathrm{d}t = \frac{1}{2}\int_0^1 \left(\frac{1+2s}{t+s}-2\right)\mathrm{d}t = \left(s+\frac{1}{2}\right)\ln\frac{s+1}{s} - 1 \\
&= \frac{1}{3(2s+1)^2} + \frac{1}{5(2s+1)^4} + \frac{1}{7(2s+1)^6} + \cdots \\
&> \frac{1}{3(2s+1)^2}\left(\frac{1}{1-\dfrac{1}{3(2s+1)^2}}\right) + \left(\frac{1}{5}-\frac{1}{3^2}\right)\frac{1}{(2s+1)^4} \\
&= \frac{1}{12s(s+1)}\left(1+\frac{1}{6s(s+1)}\right)^{-1} + \frac{1}{180s^2(s+1)^2}\left(1+\frac{1}{4s(s+1)}\right)^{-2} \\
&> \frac{1}{12s(s+1)}\left(1-\frac{1}{6s(s+1)}\right) + \frac{1}{180s^2(s+1)^2}\left(1-\frac{1}{2s(s+1)}\right) \\
&= \frac{1}{12s(s+1)} - \frac{3s(s+1)+1}{360s^3(s+1)^3} = \frac{1}{12s} - \frac{1}{12(s+1)} - \frac{1}{360s^3} + \frac{1}{360(s+1)^3},
\end{aligned}
$$

i.e.,

$$
\gamma(s) - \frac{1}{12s} + \frac{1}{360s^3} > \gamma(s+1) - \frac{1}{12(s+1)} + \frac{1}{360(s+1)^3}.
$$

Repeatedly using the above method, for any counting number $m \geqslant 2$, we have

$$
\begin{aligned}
\gamma(s) - \frac{1}{12s} + \frac{1}{360s^3} &> \gamma(s+1) - \frac{1}{12(s+1)} + \frac{1}{360(s+1)^3} \\
&\geqslant \gamma(s+m) - \frac{1}{12(s+m)} + \frac{1}{360(s+m)^3}.
\end{aligned}
$$

Inequality (35) yields $\lim_{m\to\infty}\gamma(s+m)=0$ and

$$
\begin{aligned}
\gamma(s) - \frac{1}{12s} + \frac{1}{360s^3} &> \gamma(s+1) - \frac{1}{12(s+1)} + \frac{1}{360(s+1)^3} \\
&\geqslant \lim_{m\to\infty}\left(\gamma(s+m) - \frac{1}{12(s+m)} + \frac{1}{360(s+m)^3}\right) = 0.
\end{aligned}
$$

Therefore, the right-hand side inequality of (10) is proved and finally (10) is proved.

7. The proof of Theorems 4 and 5

The proof of Theorem 5 is very similar to the proof of Theorem 4. So, we will only prove Theorem 4. To prove Theorem 4, (22) plays the key role. In addition, the following two formulas are easy to verify,

$$\int_0^1 \frac{1-2t}{t+s+p}\,\mathrm{d}t = \int_0^{1/2} \frac{(1-2t)^2}{(t+s+p)(1-t+s+p)}\,\mathrm{d}t, \tag{36}$$

$$\frac{1}{(t+s+p)(1-t+s+p)} = \frac{1}{(s+p)(1+s+p)} - \frac{t(1-t)}{(s+p)(1+s+p)(t+s+p)(1-t+s+p)}. \tag{37}$$

Using (37) repeatedly, for any counting number k, there holds

$$\frac{1}{(t+s+p)(1-t+s+p)} = \sum_{q=0}^{k}(-1)^q \frac{t^q(1-t)^q}{(s+p)^{q+1}(1+s+p)^{q+1}} + (-1)^{k+1}R_k(p,t,s), \tag{38}$$

where $R_k(p,t,s) = \frac{t^{k+1}(1-t)^{k+1}}{(s+p)^{k+1}(1+s+p)^{k+1}(t+s+p)(1-t+s+p)}$. Thus,

$$\begin{aligned}\frac{1}{(t+s+p)(1-t+s+p)} &= \lim_{k\to\infty}\left(\sum_{q=0}^{k}(-1)^q \frac{t^q(1-t)^q}{(s+p)^{q+1}(1+s+p)^{q+1}} + (-1)^{k+1}R_k(p,t,s)\right)\\ &= \sum_{q=0}^{\infty}(-1)^q \frac{t^q(1-t)^q}{(s+p)^{q+1}(1+s+p)^{q+1}}.\end{aligned} \tag{39}$$

Combining (22), (36), and (39) yields

$$\begin{aligned}\gamma(s) &= \frac{1}{2}\sum_{p=0}^{\infty}\int_0^1 \frac{1-2t}{t+s+p}\,\mathrm{d}t = \frac{1}{2}\sum_{p=0}^{\infty}\int_0^{1/2} \frac{(1-2t)^2}{(t+s+p)(1-t+s+p)}\,\mathrm{d}t\\ &= \frac{1}{2}\sum_{p=0}^{\infty}\int_0^{1/2}\sum_{q=0}^{\infty}(-1)^q \frac{(1-2t)^2 t^q(1-t)^q}{(s+p)^{q+1}(1+s+p)^{q+1}}\\ &= \sum_{q=0}^{\infty}(-1)^q b_q \sum_{p=0}^{\infty}\frac{1}{(s+p)^{q+1}(s+p+1)^{q+1}}\\ &= \frac{1}{12s} + \sum_{q=1}^{\infty}(-1)^q b_q \sum_{p=0}^{\infty}\frac{1}{(s+p)^{q+1}(s+p+1)^{q+1}}.\end{aligned}$$

Equality (11) is proved. To prove (12) and (14), we need the following formula,

$$\frac{t^{k+1}(1-t)^{k+1}}{(s+p)^{k+1}(1+s+p)^{k+1}(s+p+\frac{1}{2})^2} < R_k(p,t,s) < \frac{t^{k+1}(1-t)^{k+1}}{(s+p)^{k+2}(1+s+p)^{k+2}}, \quad 0<t<\frac{1}{2}. \tag{40}$$

Inequalities (12) and (14) can be derived directly from (38) and (40). Equality (13) is proved as follows.

$$\begin{aligned} b_q &= \frac{1}{2}\int_0^{1/2}(1-2t)^2 t^q(1-t)^q\,\mathrm{d}t = \frac{1}{4}\int_0^1 (1-2t)^2 t^q (1-t)^q\,\mathrm{d}t \\ &= \frac{1}{4}\int_0^1 (1-4t(1-t))t^q(1-t)^q\,\mathrm{d}t = \frac{1}{4}\int_0^1 t^q(1-t)^q\,\mathrm{d}t - \int_0^1 t^{q+1}(1-t)^{q+1}\,\mathrm{d}t \\ &= \frac{1}{4}B(q+1,q+1) - B(q+2,q+2) = \frac{1}{4}\frac{\Gamma^2(q+1)}{\Gamma(2q+2)} - \frac{\Gamma^2(q+2)}{\Gamma(2q+4)} \\ &= \frac{(q!)^2}{4(2q+3)(2q+1)!}. \end{aligned}$$

8. The proof of Theorem 6

To prove Theorem 6, we need to prove the following equality first.

$$(a+1)^{2n-1} - a^{2n-1} = \sum_{k=0}^{n-1} \frac{2n-1}{2k+1} C_{n-1+k}^{2k} a^{n-1-k}(a+1)^{n-1-k}. \tag{41}$$

Let

$$p_n = \sum_{k=0}^{n-1} \frac{2n-1}{2k+1} C_{n-1+k}^{2k} a^{n-1-k}(a+1)^{n-1-k}.$$

Then

$$\sum_{n=1}^{\infty} p_n x^n = \frac{x + a(a+1)x^2}{(1-a^2x)(1-(a+1)^2x)}. \tag{42}$$

More precisely, let $q = a(a+1)$, we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} p_n x^n &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{2n-1}{2k+1} C_{n-1+k}^{2k} q^{n-1-k} x^n \\
&= \sum_{k=0}^{\infty} \sum_{n=k+1}^{\infty} \frac{2n-1}{2k+1} C_{n-1+k}^{2k} q^{n-1-k} x^n \\
&= \sum_{k=0}^{\infty} \frac{x^{k+1}}{2k+1} \sum_{n=0}^{\infty} (2n+2k+1) C_{n+2k}^{2k} q^n x^n \\
&= \sum_{k=0}^{\infty} \frac{x^{k+1}}{2k+1} \left(2 \sum_{n=0}^{\infty} (n+2k+1) C_{n+2k}^{2k} q^n x^n - (2k+1) \sum_{n=0}^{\infty} C_{n+2k}^{2k} q^n x^n \right) \\
&= \sum_{k=0}^{\infty} \frac{x^{k+1}}{(2k+1)!} \left(2 \sum_{n=0}^{\infty} (n+2k+1)(n+2k) \cdots (n+1) q^n x^n \right. \\
&\quad \left. -(2k+1) \sum_{n=0}^{\infty} (n+2k) \cdots (n+1) q^n x^n \right) \\
&= \sum_{k=0}^{\infty} \frac{x^{k+1}}{(2k+1)!} \left(2 \frac{\mathrm{d}^{2k+1}}{\mathrm{d}y^{2k+1}} \sum_{n=0}^{\infty} y^{n+2k+1} - (2k+1) \frac{\mathrm{d}^{2k}}{\mathrm{d}y^{2k}} \sum_{n=0}^{\infty} y^{n+2k} \right) \quad (y = qx) \\
&= \sum_{k=0}^{\infty} \frac{x^{k+1}}{(2k+1)!} \left(2 \frac{\mathrm{d}^{2k+1}}{\mathrm{d}y^{2k+1}} \left(\frac{y^{2k+1}}{1-y} \right) - (2k+1) \frac{\mathrm{d}^{2k}}{\mathrm{d}y^{2k}} \left(\frac{y^{2k}}{1-y} \right) \right) \\
&= \sum_{k=0}^{\infty} \frac{x^{k+1}}{(2k+1)!} \left(2 \sum_{p=0}^{2k+1} C_{2k+1}^{p} (y^{2k+1})^{(2k+1-p)} \left(\frac{1}{1-y} \right)^{(p)} \right. \\
&\quad \left. -(2k+1) \sum_{p=0}^{2k} C_{2k}^{p} (y^{2k})^{(2k-p)} \left(\frac{1}{1-y} \right)^{(p)} \right) \\
&= \sum_{k=0}^{\infty} \frac{x^{k+1}}{(2k+1)!} \left(2 \sum_{p=0}^{2k+1} C_{2k+1}^{p} \frac{(2k+1)!}{p!} y^p \frac{p!}{(1-y)^{p+1}} \right. \\
&\quad \left. -(2k+1) \sum_{p=0}^{2k} C_{2k}^{p} \frac{(2k)!}{p!} y^p \frac{p!}{(1-y)^{p+1}} \right) \\
&= \sum_{k=0}^{\infty} \frac{x^{k+1}}{1-y} \left(2 \left(1 + \frac{y}{1-y} \right)^{2k+1} - \left(1 + \frac{y}{1-y} \right)^{2k} \right)
\end{aligned}
$$

$$= 2\sum_{k=0}^{\infty}\frac{x^{k+1}}{(1-y)^{2k+2}} - \frac{x}{1-y}\sum_{k=0}^{\infty}\frac{x^k}{(1-y)^{2k}}$$
$$= 2\frac{\dfrac{x}{(1-y)^2}}{1-\dfrac{x}{(1-y)^2}} - \frac{x}{1-y}\frac{1}{1-\dfrac{x}{(1-y)^2}} = \frac{x+xy}{(1-y)^2-x}$$
$$= \frac{x+a(a+1)x^2}{(1-a^2x)(1-(a+1)^2x)}$$

On the other hand,

$$\sum_{n=1}^{\infty}((a+1)^{2n-1}-a^{2n-1})x^n = x(1+a)\sum_{n=1}^{\infty}(a+1)^{2n-2}x^{n-1} - xa\sum_{n=1}^{\infty}a^{2n-2}x^{n-1}$$
$$= \frac{x(1+a)}{1-x(1+a)^2} - \frac{xa}{1-xa^2} = \frac{x(1+ax+a^2x)}{(1-xa^2)(1-x(1+a)^2)}. \quad (43)$$

Equality (41) is consequently obtained from (42) and (43) and (41) is equivalent to

$$\frac{1}{(2n-1)a^{2n-1}} - \frac{1}{(2n-1)(a+1)^{2n-1}} = \sum_{k=0}^{n-1}\frac{C_{n-1+k}^{2k}}{2k+1}\frac{1}{a^{n+k}(a+1)^{n+k}}. \quad (44)$$

Taking $k=1$ in (14), we have

$$\frac{1}{12s} + S_2(s) - \delta_1(s) < \gamma(s) < \frac{1}{12s} + S_2(s). \quad (45)$$

Set $n=3$ and $a=s+p$ in (44), there holds

$$\frac{1}{(s+p)^5} - \frac{1}{(s+p+1)^5} = \frac{5}{(s+p)^3(s+p+1)^3} + \frac{5}{(s+p)^4(s+p+1)^4}$$
$$+ \frac{1}{(s+p)^5(s+p+1)^5},$$

and from above we obtain

$$\frac{1}{1+\dfrac{1}{s(s+1)}+\dfrac{1}{5s^2(s+1)^2}}\left(\frac{1}{5(s+p)^5} - \frac{1}{5(s+p+1)^5}\right)$$
$$< \frac{1}{(s+p)^3(s+p+1)^3} < \frac{1}{5(s+p)^5} - \frac{1}{5(s+p+1)^5}, \quad p\geqslant 0. \quad (46)$$

With direct calculation, we have $b_1 = \frac{1}{120}$ and $b_2 = \frac{1}{840}$. Therefore, from (46)

$$\begin{aligned}
S_2(s) - \delta_1(s) =& -\frac{1}{120}\sum_{p=0}^{\infty}\frac{1}{(s+p)^2(s+p+1)^2} + \frac{1}{840}\sum_{p=0}^{\infty}\frac{1}{(s+p)^3(s+p+1)^3} \\
&- \frac{1}{840}\sum_{p=0}^{\infty}\frac{1}{4(s+p)^3(s+p+1)^3(s+p+\frac{1}{2})^2} \\
=& -\frac{1}{360}\sum_{p=0}^{\infty}\left(\frac{1}{(s+p)^3} - \frac{1}{(s+p+1)^3} - \frac{1}{(s+p)^3(s+p+1)^3}\right) \\
&+ \frac{1}{840}\sum_{p=0}^{\infty}\frac{1}{(s+p)^3(s+p+1)^3} \\
&- \frac{1}{840}\sum_{p=0}^{\infty}\frac{1}{4(s+p)^3(s+p+1)^3(s+p+\frac{1}{2})^2} \\
=& -\frac{1}{360s^3} + \sum_{p=0}^{\infty}\left(\frac{1}{252(s+p)^3(s+p+1)^3}\right. \\
&\left. - \frac{1}{3360(s+p)^3(s+p+1)^3(s+p+\frac{1}{2})^2}\right) \\
>& -\frac{1}{360s^3} + \frac{1}{252}\left(1 - \frac{3}{40\left(s+\frac{1}{2}\right)^2}\right)\sum_{p=0}^{\infty}\frac{1}{(s+p)^3(s+p+1)^3} \\
>& -\frac{1}{360s^3} + \frac{1}{252}\left(\frac{1 - \dfrac{3}{40\left(s+\frac{1}{2}\right)^2}}{1 + \dfrac{1}{s(s+1)} + \dfrac{1}{5s^2(s+1)^2}}\right) \\
&\times \sum_{p=0}^{\infty}\left(\frac{1}{5(s+p)^5} - \frac{1}{5(s+p+1)^5}\right) \\
=& -\frac{1}{360s^3} + \frac{1 - \dfrac{3}{40\left(s+\frac{1}{2}\right)^2}}{1260 + \dfrac{1260}{s(s+1)} + \dfrac{252}{s^2(s+1)^2}}\frac{1}{s^5}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
S_2(s) &= -\frac{1}{120}\sum_{p=0}^{\infty}\frac{1}{(s+p)^2(s+p+1)^2}+\frac{1}{840}\sum_{p=0}^{\infty}\frac{1}{(s+p)^3(s+p+1)^3}\\
&= -\frac{1}{360}\sum_{p=0}^{\infty}\left(\frac{1}{(s+p)^3}-\frac{1}{(s+p+1)^3}-\frac{1}{(s+p)^3(s+p+1)^3}\right)\\
&\quad+\frac{1}{840}\sum_{p=0}^{\infty}\frac{1}{(s+p)^3(s+p+1)^3}\\
&= -\frac{1}{360s^3}+\sum_{p=0}^{\infty}\frac{1}{252(s+p)^3(s+p+1)^3}\\
&< -\frac{1}{360s^3}+\frac{1}{252}\sum_{p=0}^{\infty}\left(\frac{1}{5(s+p)^5}-\frac{1}{5(s+p+1)^5}\right)\\
&= -\frac{1}{360s^3}+\frac{1}{1260s^5}.
\end{aligned}$$

Inequality (17) is proved. To prove (18), let $n=4$ and $a=s+p$ in (44), we have

$$\begin{aligned}
\frac{1}{(s+p)^7}-\frac{1}{(s+p+1)^7} &= \frac{7}{(s+p)^4(s+p+1)^4}+\frac{14}{(s+p)^5(s+p+1)^5}\\
&\quad+\frac{7}{(s+p)^6(s+p+1)^6}+\frac{1}{(s+p)^7(s+p+1)^7},
\end{aligned}$$

and from above we obtain

$$\begin{aligned}
&\frac{1}{1+\dfrac{2}{s(s+1)}+\dfrac{1}{s^2(s+1)^2}+\dfrac{1}{7s^3(s+1)^3}}\left(\frac{1}{7(s+p)^7}-\frac{1}{7(s+p+1)^7}\right)\\
&\quad<\frac{1}{(s+p)^4(s+p+1)^4}\\
&\quad<\frac{1}{7(s+p)^7}-\frac{1}{7(s+p+1)^7},\quad p\geqslant 0. \qquad (47)
\end{aligned}$$

Taking $k=2$ in (14), there holds

$$\frac{1}{12s}+S_3(s)<\gamma(s)<\frac{1}{12s}+S_3(s)+\delta_2(s).$$

Computing $S_3(s)$ directly yields

$$\begin{aligned}
S_3(s) = & -\frac{1}{120}\sum_{p=0}^{\infty}\frac{1}{(s+p)^2(s+p+1)^2}+\frac{1}{840}\sum_{p=0}^{\infty}\frac{1}{(s+p)^3(s+p+1)^3} \\
& -\frac{1}{5040}\sum_{p=0}^{\infty}\frac{1}{(s+p)^4(s+p+1)^4} \\
= & -\frac{1}{360s^3}+\frac{1}{252}\sum_{p=0}^{\infty}\frac{1}{(s+p)^3(s+p+1)^3}-\frac{1}{5040}\sum_{p=0}^{\infty}\frac{1}{(s+p)^4(s+p+1)^4} \\
= & -\frac{1}{360s^3}+\frac{1}{252}\sum_{p=0}^{\infty}\left(\frac{1}{5(s+p)^5}-\frac{1}{5(s+p+1)^5}-\frac{1}{(s+p)^4(s+p+1)^4}\right. \\
& \left.-\frac{1}{5(s+p)^5(s+p+1)^5}\right) \\
& -\frac{1}{5040}\sum_{p=0}^{\infty}\frac{1}{(s+p)^4(s+p+1)^4} \\
= & -\frac{1}{360s^3}+\frac{1}{1260s^5}-\frac{1}{240}\sum_{p=0}^{\infty}\frac{1}{(s+p)^4(s+p+1)^4} \\
& -\sum_{p=0}^{\infty}\frac{1}{1260(s+p)^5(s+p+1)^5} \\
> & -\frac{1}{360s^3}+\frac{1}{1260s^5}-\left(\frac{1}{240}+\frac{1}{1260s(s+1)}\right)\sum_{p=0}^{\infty}\frac{1}{(s+p)^4(s+p+1)^4} \\
> & -\frac{1}{360s^3}+\frac{1}{1260s^5}-\left(\frac{1}{240}+\frac{1}{1260s(s+1)}\right) \\
& \times\sum_{p=0}^{\infty}\left(\frac{1}{7(s+p)^7}-\frac{1}{7(s+p+1)^7}\right) \quad \text{(from (47))} \\
= & -\frac{1}{360s^3}+\frac{1}{1260s^5}-\left(1+\frac{4}{21s(s+1)}\right)\frac{1}{1680s^7}.
\end{aligned}$$

Similarly,

$$S_3(s)+\delta_2(s) = -\frac{1}{360s^3}+\frac{1}{1260s^5}-\frac{1}{240}\sum_{p=0}^{\infty}\frac{1}{(s+p)^4(s+p+1)^4}$$

$$
\begin{aligned}
&-\sum_{p=0}^{\infty}\frac{1}{1260(s+p)^5(s+p+1)^5}+\delta_2(s)\\
&<-\frac{1}{360s^3}+\frac{1}{1260s^5}-\frac{1}{240}\sum_{p=0}^{\infty}\frac{1}{(s+p)^4(s+p+1)^4}\\
&\quad+\frac{1}{5040}\sum_{p=0}^{\infty}\frac{1}{4(s+p)^4(s+p+1)^4(s+p+\frac{1}{2})^2}\\
&<-\frac{1}{360s^3}+\frac{1}{1260s^5}-\left(\frac{1}{240}-\frac{1}{20160(s+\frac{1}{2})^2}\right)\sum_{p=0}^{\infty}\frac{1}{(s+p)^4(s+p+1)^4}\\
&=-\frac{1}{360s^3}+\frac{1}{1260s^5}-\frac{1-\dfrac{1}{84(s+\frac{1}{2})^2}}{1+\dfrac{2}{s(s+1)}+\dfrac{1}{s^2(s+1)^2}+\dfrac{1}{7s^3(s+1)^3}}\\
&\quad\times\frac{1}{1680s^7}\quad\text{(from (47)).}
\end{aligned}
$$

Inequality (18) is thus proved. (19) can be similarly proved by using the following two equations.

$$
\begin{aligned}
S_4(s)=&-\frac{1}{360s^3}+\frac{1}{1260s^5}-\frac{1}{1680s^7}+\sum_{p=0}^{\infty}\frac{1}{132(s+p)^5(s+p+1)^5}\\
&+\sum_{p=0}^{\infty}\frac{1}{240(s+p)^6(s+p+1)^6}+\sum_{p=0}^{\infty}\frac{1}{1680(s+p)^7(s+p+1)^7}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{(s+p)^9}-\frac{1}{(s+p+1)^9}=&\frac{9}{(s+p)^5(s+p+1)^5}+\frac{30}{(s+p)^6(s+p+1)^6}\\
&+\frac{27}{(s+p)^7(s+p+1)^7}+\frac{9}{(s+p)^8(s+p+1)^8}\\
&+\frac{1}{(s+p)^9(s+p+1)^9}
\end{aligned}
$$

which is obtained by setting $n=5$ and $a=s+p$ in (44). And (20) can be proved by using the following two equations,

$$S_5(s) = -\frac{1}{360s^3} + \frac{1}{1260s^5} - \frac{1}{1680s^7} + \frac{1}{1188s^9} - \sum_{p=0}^{\infty} \frac{691}{32760(s+p)^6(s+p+1)^6} - \sum_{p=0}^{\infty} \frac{409}{18480(s+p)^7(s+p+1)^7} - \sum_{p=0}^{\infty} \frac{1}{132(s+p)^8(s+p+1)^8} - \sum_{p=0}^{\infty} \frac{1}{1188(s+p)^9(s+p+1)^9}$$

and

$$\frac{1}{(s+p)^{11}} - \frac{1}{(s+p+1)^{11}} = \frac{11}{(s+p)^6(s+p+1)^6} + \frac{55}{(s+p)^7(s+p+1)^7} + \frac{77}{(s+p)^8(s+p+1)^8} + \frac{44}{(s+p)^9(s+p+1)^9} + \frac{11}{(s+p)^{10}(s+p+1)^{10}} + \frac{1}{(s+p)^{11}(s+p+1)^{11}}$$

which is obtained by setting $n=6$ and $a=s+p$ in (44).

Appendix and a proposed problem

We list a few $a_k(v)$ defined in (3) and obtain by using "DERIVE".

$$a_1(v) = \frac{1}{12},$$
$$a_2(v) = \frac{v}{12},$$
$$a_3(v) = \frac{30v^2+30v-1}{360},$$
$$a_4(v) = \frac{10v^3+30v^2+19v-1}{120},$$
$$a_5(v) = \frac{70^4+420v^3+756v^2+378v-25}{840},$$
$$a_6(v) = \frac{42v^5+420v^4+1456v^3+2016v^2+863v-66}{504},$$

$$a_7(v) = \frac{420v^6 + 6300v^5 + 35490v^4 + 92400v^3 + 108000v^2 + 41250v - 3499}{5040},$$

$$a_8(v) = \frac{60v^7 + 1260v^6 + 10458v^5 + 43470v^4 + 93960v^3 + 97200v^2 + 33953v - 3117}{720},$$

$$a_9(v) = \frac{990v^8 + 27720v^7 + 317856v^6 + 1920996v^5 + 6542580v^4 + 12363120v^3 + 11642400v^2 + 3780546v - 369689}{11880},$$

$$a_{10}(v) = \frac{110v^9 + 3960v^8 + 59928v^7 + 495264v^6 + 2427480v^5 + 7154400v^4 + 12190464v^3 + 10644480v^2 + 3250433v - 334844}{1320}.$$

It is easy to prove that $a_k(v)$ is a polynomial of degree $k-1$ in v. We propose for the future research that, for $k \geqslant 3$, the coefficients of $a_k(v)$ are always positive except the constant terms.

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Further Reading

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