



The pion form factor in improved lattice QCD

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Abstract

We calculate the electromagnetic form factor of the pion in lattice gauge theory. The non-perturbatively improved Sheikoleslami–Wohlerlattice action is used together with the $\mathcal{O}(a)$ improved current. The form factor is compared to results for other choices for the current and features of the structure of the pion deduced from the ‘Bethe–Salpeter wave function’ are discussed.

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1. Introduction

The pion as the simplest particle with only two valence quarks has been the subject of many studies. Global features of the pions—their charge and spin—are easily incorporated in model calculations. The form factor, which directly reflects the internal structure of this elementary particle, is clearly an important challenge. Many earlier calculations are based on ad hoc models that model QCD or sum over selected subsets of Feynman diagrams. However, the most reliable approach, in particular when addressing non-perturbative features as the electromagnetic form factor at intermediate momentum transfers, is the use of lattice QCD. The first lattice results were obtained by Martinelli and Sachrajda [1], which was followed

by a more detailed study by Draper et al. [2], who showed that the form factor obtained through lattice QCD with the Wilson action could be described by a simple monopole form as suggested by vector meson dominance [3]. Below, we extend these early studies in two ways. We use an improved lattice action and an $\mathcal{O}(a)$ improved electromagnetic current operator. Furthermore, we also extend the calculations to lower pion masses than achieved before. Several features of the internal structure of the pion have been obtained previously [4–8] by calculating the ‘Bethe–Salpeter wave function’, which can be used to estimate the relative separation of the quark–antiquark pair in the pion. We also use this approach and compare its predictions to the results of our direct calculation of the pion form factor.

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2. The method

In comparison to earlier lattice calculations of pion properties, the major difference of our approach is the systematic reduction of the discretisation error in the calculation of the matrix elements. We use the non-perturbatively $\mathcal{O}(a)$ improved [9] clover action [10] and the corresponding $\mathcal{O}(a)$ improved current [11–13].

Using this action, we proceed analogous to [2] and calculate the two- and three-point correlation functions for the pion. Projecting onto definite pion three-momentum, the two-point function is

$$G_2(t, \mathbf{p}) = \sum_{\mathbf{x}} \langle \phi(t, \mathbf{x}) \phi^\dagger(0, \mathbf{0}) \rangle e^{i\mathbf{p}\cdot\mathbf{x}}, \quad (1)$$

where ϕ is the operator projecting on a state with the pion quantum numbers. Below, we will consider a π^+ meson, consisting of a u and \bar{d} quark. Neglecting all spin, colour and flavour indices, this operator is given by

$$\phi(x) = \bar{\psi}(x) \gamma_5 \psi(x). \quad (2)$$

In the three point function, which yields the desired form factor, we project onto specific initial and final pion three momenta, \mathbf{p}_i and \mathbf{p}_f

$$G_3(t_f, t; \mathbf{p}_f, \mathbf{p}_i) = \sum_{\mathbf{x}_f, \mathbf{x}} \langle \phi(x_f) j_4(x) \phi^\dagger(0) \rangle \times e^{-i\mathbf{p}_f \cdot (\mathbf{x}_f - \mathbf{x}) - i\mathbf{p}_i \cdot \mathbf{x}}. \quad (3)$$

This function involves the electromagnetic current operator j_μ ; since here we use only the component $\mu = 4$, we do not include a μ -index in the definition of the three point function.

It is well known that the local current,

$$j_\mu^L(x) = \bar{\psi}(x) \gamma_\mu \psi(x), \quad (4)$$

is not conserved on the lattice. The conserved Noether current that belongs to our action,

$$j_\mu^C = \kappa (\bar{\psi}(x) (1 - \gamma_\mu) U_\mu(x) \psi(x + \hat{\mu}) - \bar{\psi}(x + \hat{\mu}) (1 + \gamma_\mu) U_\mu^\dagger(x) \psi(x)), \quad (5)$$

is identical to the conserved current for the Wilson action and still contains corrections of $\mathcal{O}(a)$ at $Q^2 \neq 0$; we use $Q^2 = -q^2$, where q is the four momentum transfer to the pion.

The conserved and improved vector current j_μ^I is of the form [11–13]

$$j_\mu^I = Z_V \{ j_\mu^L(x) + a c_V \partial_\nu T_{\mu\nu} \}, \quad (6)$$

with

$$T_{\mu\nu} = \bar{\psi}(x) i \sigma_{\mu\nu} \psi(x), \\ Z_V = Z_V^0 (1 + a b_V m_q). \quad (7)$$

The bare-quark mass is obtained from:

$$a m_q = \frac{1}{2} \left(\frac{1}{\kappa} - \frac{1}{\kappa_c} \right), \quad (8)$$

where κ_c is the kappa value in the chiral limit and a the lattice spacing. For our simulation we use $\kappa_c = 0.13525$ [14]. The constants in j_μ^I are determined such that the matrix element of the current operator receives no correction to $\mathcal{O}(a)$.

3. Details of the calculation

Our calculations were carried out in the quenched approximation on an $N_\sigma^3 \times N_\tau = 24^3 \times 32$ lattice and based on a set of 100 configurations for the link variables at $\beta = 6$ and $c_{SW} = 1.769$ [9]. After an initial thermalisation of 2500 sweeps, we obtained configurations at intervals of 500 sweeps. Each sweep consists of a pseudo-heatbath step with FHKP updating in the $SU(2)$ subgroups, followed by four over-relaxation steps. In contrast to the Dirichlet conditions in [2], we impose anti-periodic boundary conditions on the quarks and periodic boundary conditions on the gluons. Three values of the hopping parameter κ were used

$$\kappa_1 = 0.1323, \quad \kappa_2 = 0.1338, \quad \text{and} \quad \kappa_3 = 0.1343, \quad (9)$$

corresponding to pion masses¹ of 968, 671 and 541 MeV, respectively. For the improved current, we use the parameters Z_V^0 , b_V and c_V as determined by Bhattacharya et al. [16].

Conservation of the total charge generated at the source at $t = 0$ provides a test [2] for our calculation, relating the $\mu = 4$ component of the three-point

¹ For definiteness, we have taken the lattice spacing $a = 0.105$ fm from [15].

function for $\mathbf{q} = 0$ to the two-point function. For our periodic boundary conditions, it reads

$$G_3(t_f, t; \mathbf{p}, \mathbf{p}) - G_3(t_f, t'; \mathbf{p}, \mathbf{p}) = G_2(t_f, \mathbf{p}), \quad (10)$$

where $t_f < t' < N_\tau$. We find that all configurations we use each satisfy this condition to at least 1 ppm.

For the results discussed below, we chose the pion three-momenta such that $|\mathbf{p}_i|^2 = |\mathbf{p}_f|^2 = 2$ in units of the minimal momentum $\frac{2\pi}{aN_\sigma}$ for our lattice. This guarantees for the elastic pion form factor that $E_f - E_i = q_0 = 0$; it greatly simplifies the kinematic factors appearing in the three-point function. Different values for the three momentum transfer \mathbf{q} were obtained by varying the relative orientation of the initial and final pion momenta.

In order to improve the projection onto the ground state, we smeared the pion operator at the sink in G_2 and G_3 by the method proposed in [7]. We found that a quark–antiquark distance $R = 3$ works best. The quark–antiquark pair was connected by APE smeared gluon links at smearing level 4 and relative weight 2 between straight links and staples.

To extract the desired information from our numerical results, we assume the two-point function of the pion to have the form

$$G_2(t, \mathbf{p}) = \sum_{n=0}^1 \sqrt{Z_R^n(\mathbf{p})Z_0^n(\mathbf{p})} e^{-E_{\mathbf{p}}^n \frac{N_\tau}{2}} \times \cosh \left\{ E_{\mathbf{p}}^n \left(\frac{N_\tau}{2} - t \right) \right\}, \quad (11)$$

including the contribution of the ground state ($n = 0$) and a first excited one ($n = 1$). The Z_R^n denote the matrix elements,

$$Z_R^n(\mathbf{p}) \equiv |\langle \Omega | \phi_R | n, \mathbf{p} \rangle|^2, \quad (12)$$

and $E_{\mathbf{p}}^0, E_{\mathbf{p}}^1$ are the energies of ground and excited state, respectively; the subscript R indicates the operator smearing.

The three-point function is parametrised as

$$G_3(t_f, t; \mathbf{p}_f, \mathbf{p}_i) = F(Q^2) \sqrt{Z_R^0(\mathbf{p}_f)Z_0^0(\mathbf{p}_i)} e^{-E_{\mathbf{p}_f}^0(t_f-t) - E_{\mathbf{p}_i}^0 t} + \left\{ \sqrt{Z_R^1(\mathbf{p}_f)Z_0^0(\mathbf{p}_i)} (1, \mathbf{p}_f | j_\mu(0) | 0, \mathbf{p}_f) \times e^{-E_{\mathbf{p}_f}^1(t_f-t) - E_{\mathbf{p}_i}^0 t} + (1 \leftrightarrow 0) \right\}. \quad (13)$$

Effects involving, for example, the production of pion pairs, as well as ‘wrap around effects’ due to the propagation of states beyond t_f are exponentially suppressed ($< \mathcal{O}(e^{-5})$); similarly, an elastic contribution from the excited state was estimated to be of the order of 1% or less. All these effects are not reflected in our chosen parametrisation.

All parameters in the 2- and 3-point functions—energies E , Z -factors and the form factor $F(Q^2)$ —were fit simultaneously to the data from all configurations. For the three-point function, we chose $t_f = 11$ and let the current insertion time t vary from 0 to 10. For maximum spatial symmetry, all values corresponding to the same value $|\mathbf{p}|$ in the two-point function and all $\mathbf{p}_{i,f}$ yielding the same \mathbf{q} in the three-point function were combined for the fit. The value for the parameters and their error in these simultaneous fits was obtained through a single elimination jackknife procedure. Since we satisfy Eq. (10) to high accuracy, we show $F(0) = 1$ in the results below instead of using the result from a fit at $Q^2 = 0$, which would be less accurate in this case.

4. Results

Our method to extract the pion form factor is non-perturbatively improved in two respects: we use an improved action and an improved current operator. We can get an impression of the importance of the latter effect by comparing the conserved Noether current corresponding to the improved action with the improved current (which is also conserved). The results² are shown in Fig. 1 for the lightest of our three quark masses. The form factor from the improved current is systematically lower than the one from the conserved current. The difference grows with Q^2 and reaches about 25% at the largest momentum transfer considered here.

The structure of the improved current can be further understood by comparing the improved current to the renormalised local current,

$$j_\mu^{L,R} \equiv Z_V \bar{\psi} \gamma_\mu \psi, \quad (14)$$

² In all our results we only show the statistical errors.

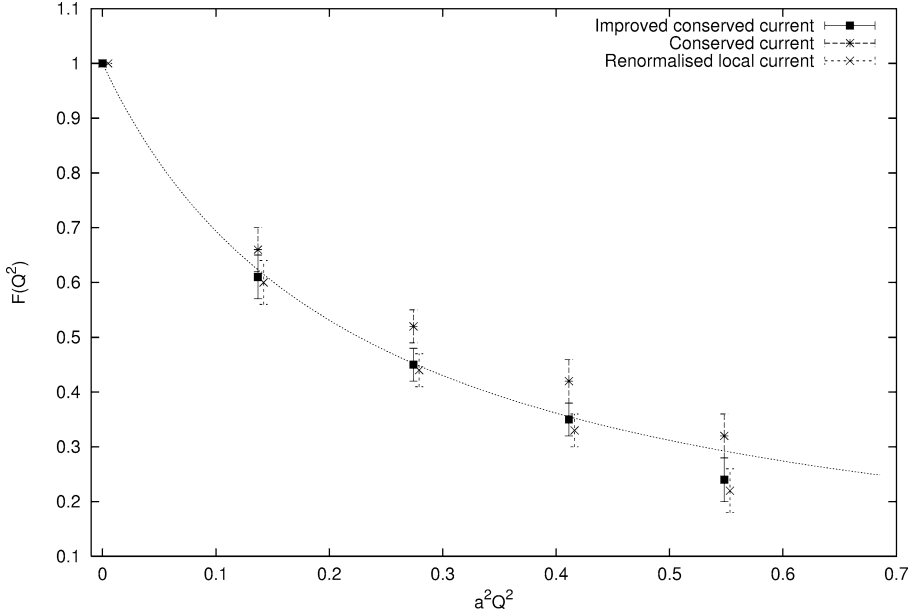


Fig. 1. Form factors for the three currents with lowest quark mass. Solid line: monopole form with m_ρ taken from literature (see text). Data for the local current shifted horizontally for clarity.

which is not conserved. For our kinematics with $q_0 = 0$, we can also extract a form factor from j_4^L . It is also shown in Fig. 1 and can be seen to lie very close to the improved current. This means that the contribution of the term proportional to c_V in j_4^L is very small. Closer inspection shows that while the matrix element of the tensor term can become almost comparable to that of the γ_4 term, the overall tensor contribution is reduced by the small coefficient c_V . Similar statements also hold for our two additional κ -values.

It is worth mentioning that with the Z_V , c_V and b_V values taken from [16], and performing a fit at $Q^2 = 0$ we obtained $F^I/F^C = 1$ to better than 1% with a statistical error of about 5%.

In the previous study [2] of the pion form factor, where the Wilson action was used, the results were compared to a monopole form factor

$$F(Q^2) = \left\{ 1 + \frac{Q^2}{m_\rho^2} \right\}^{-1}, \quad (15)$$

a form suggested by vector meson dominance. We also show in Fig. 1 a monopole form factor using the value for the ρ -mass obtained by interpolating the results from [14] which uses the same action as we do. This monopole form factor describes our results for the im-

proved current at all but the highest Q^2 very well. As in [2], we observe that the conserved current lies consistently above the monopole form factor. A similar behaviour was found also for our other two κ -values.

In Fig. 2 we show our results for improved form factors for all three values for κ . The corresponding quark- and pion-masses are given in Table 1. The form factors systematically decrease with decreasing pion mass. The form factors for the two lightest pions, with $m_\pi = 541$ and 671 MeV, come very close together. As can be seen, the statistical error of the extracted form factors grows as the quark mass decreases. Nevertheless, we are still able to obtain conclusive results for the smallest quark mass. The corresponding pion mass of 541 MeV is substantially lower than in the previous work, where $m_\pi \sim 1$ GeV. Given that we have only three data sets, we have not attempted to extrapolate our improved form factor to the physical pion mass.

We also fitted our results for the improved form factors to a monopole form factor. In doing so, we omitted the highest momentum data point and extracted in each case a vector meson mass, m_V , shown in Table 1. They are close to the values for m_ρ taken from interpolations to literature data [14].

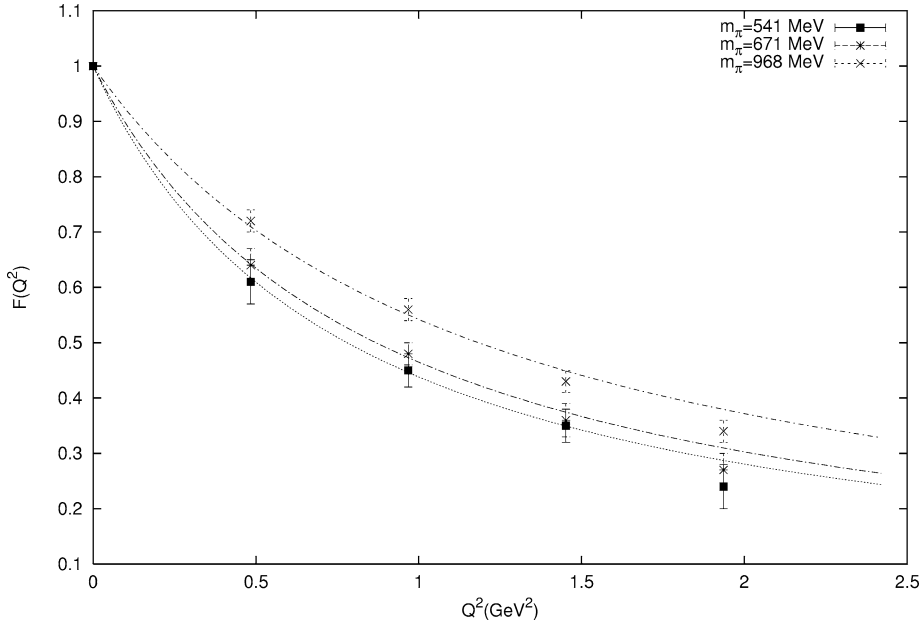


Fig. 2. Form factors as a function of Q^2 for the three pion masses. Curves: monopole fits to the data.

Table 1

Masses and RMS-values for the different kappa values, in lattice units. The ρ -mass has been taken from [14]

| κ | m_q | m_π | m_ρ | m_V | $\langle r^2 \rangle_{BS}^{1/2}$ | $\langle r^2 \rangle_{FF}^{1/2}$ |
|----------|-------|----------|----------|-----------|----------------------------------|----------------------------------|
| 0.13230 | 0.082 | 0.515(2) | 0.625(5) | 0.597(14) | 2.530(2) | 4.23(10) |
| 0.13380 | 0.040 | 0.357(2) | 0.513(5) | 0.496(15) | 2.615(2) | 4.94(15) |
| 0.13430 | 0.026 | 0.288(2) | 0.476(7) | 0.470(19) | 2.629(2) | 5.21(21) |

In examining the two-point Green function for various quark–antiquark distances, we also obtain the ‘Bethe–Salpeter wavefunction’,

$$\Phi_{BS}(R) = \sqrt{\frac{Z_R^0(\mathbf{0})}{Z_0^0(\mathbf{0})}}. \tag{16}$$

Following the procedure in [5,8], we obtain $\langle r^2 \rangle_{BS}^{1/2}$, shown in Table 1. These RMS-radii are compared to the values extracted from the low- Q^2 behaviour of the form factor,

$$\left. \frac{dF(Q^2)}{dQ^2} \right|_{Q^2=0} = -\frac{1}{6} \langle r^2 \rangle_{FF} = -\frac{1}{m_V^2}, \tag{17}$$

where in the last step we have assumed a monopole form and use the fitted parameter m_V . In agreement with the findings of [4–6], we see that the Bethe–

Salpeter predictions are very insensitive to the value of the quark or pion mass. However, it is well known [6] that the information that can be obtained from the Bethe–Salpeter approach as described above is only an approximation. It assumes, in the extraction of $\langle r^2 \rangle$, that the center of mass of the pion is always halfway between the valence quark and antiquark, not allowing for the motion of the gluons. The extraction of $\langle r^2 \rangle$ from the calculated pion form factor does not involve this restriction for the valence (anti)quark motion. As can be seen, the more reliable determination from $F(Q^2)$ leads, as expected, to a larger radius. Moreover, this radius shows a substantial dependence on the mass.

We have presented here the first calculation of the electromagnetic form factor of the pion based on an $\mathcal{O}(a)$ improved action and the concomitant improved vector current. This is seen to lead to significant changes in the prediction for the internal structure of the pion. We observe a decrease of the form factor for decreasing pion mass, which in turn leads to an increase of the RMS-radius. This mass-dependence of the radius is not seen in the Bethe–Salpeter approach. Furthermore, the mass of the pion we reach in our calculations is considerably closer to the physical value than in previous work.

The computational effort involved in taking the improvement into account is small. Since it guarantees elimination of $\mathcal{O}(a)$ discretisation errors to all orders in the coupling constant, use of this method in future work seems logical in pushing the calculations further towards the physical limit.

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