The $D$-property of some Lindelöf spaces and related conclusions

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Abstract

It is shown that the space $C_p(\tau_\omega)$ is a $D$-space for any ordinal number $\tau$, where $\tau_\omega = \{ \alpha \leq \tau : \text{cf}(\alpha) \leq \omega \}$. This conclusion gives a positive answer to R.Z. Buzyakova’s question. We also prove that another special example of Lindelöf space is a $D$-space. We discuss the $D$-property of spaces with point-countable weak bases. We prove that if a space $X$ has a point-countable weak base, then $X$ is a $D$-space. By this conclusion and one of T. Hoshina’s conclusion, we have that if $X$ is a countably compact space with a point-countable weak base, then $X$ is a compact metrizable space. In the last part, we show that if a space $X$ is a finite union of $\theta$-refinable spaces, then $X$ is a $\alpha D$-space.

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1. Introduction

The notion of $D$-space was introduced by van Douwen [1]. A neighborhood assignment for a space $X$ is a function $\phi$ from $X$ to the topology of the space $X$, such that $x \in \phi(x)$ for any $x \in X$. A space $X$ is called a $D$-space, if for any neighborhood assignment $\phi$ for $X$ there exists a closed discrete subset $D$ of $X$, such that $X = \bigcup \{ \phi(d) : d \in D \}$. One of the central questions on $D$-spaces posed by van Douwen is whether every Lindelöf space is a $D$-space. We obtain two results related to this problem. Namely, we show that certain known Lindelöf spaces have the $D$-property.

In [2], R.Z. Buzyakova gave an example that answers negatively Reznichenko’s question whether $C_p(X)$ is a $D$-space if $X$ is a countably compact space. To get the conclusion, she firstly discussed the properties of the space $C_p(\tau_\omega)$, where $\tau_\omega = \{ \alpha \leq \tau : \text{cf}(\alpha) \leq \omega \}$, and $\tau$ is an ordinal number. She proved that $C_p(\tau_\omega)$ is a Lindelöf space for any $\tau$. So she raised the following problem: Is $C_p(\tau_\omega)$ a $D$-space for $\tau \geq \omega_2$? In this paper, we show that $C_p(\tau_\omega)$ is a $D$-space for any $\tau$.

In [3], K. Alster constructed an example of Lindelöf space $X$ with countable pseudocharacter which does not admit a continuous one-to-one mapping onto a first countable Hausdorff space. In this paper, we prove that Alster’s example is also a $D$-space.
In [4], it is proved that every space with a point-countable base is a $D$-space. In this note, we prove that $X$ is also a $D$-space, if $X$ has a point-countable weak base. Some results on weak bases can be found in [5–7]. A compact space with a point-countable weak base is metrizable (cf. [6]). We show that if $X$ is a countably compact space with a point-countable weak base, then $X$ is a compact metrizable space.

In [8], it is proved that if $X = \bigcup\{X_i: i \leq n\}, n \in \mathbb{N}$, and $X_i$ is a subparacompact space for each $i \leq n$, then $X$ is a $\alpha D$-space. We know that every subparacompact space is a $\theta$-refinable space. We prove that $X$ is also a $\alpha D$-space, if $X$ is a finite union of $\theta$-refinable spaces.

If $U$ is a standard open set of $C_p(X)$, we say that $U$ depends on a finite set $\{x_1, \ldots, x_n\} \subset X$ if there exist $B_1, \ldots, B_n$ open in $R$ such that $U = \{f: f \in C_p(X), f(x_i) \in B_i \text{ for } i \leq n\}$. Let $N$ denote the set of all natural numbers. Let $Q$ denote the set of all rational numbers of the unit interval. The symbols $\omega$ and $\omega_1$ stand for the first infinite and the first uncountable ordinal numbers, respectively. In notation and terminology we will follow [9–11]. All the spaces in this paper are $T_1$-spaces.

2. Main results

Definition 1. [2] Let $A \subset \tau_\omega$, we say that $B$ is an $\omega$-support of $A$ if $B$ is countable and the following conditions are satisfied:

1. $0 \notin B$;
2. $A \subset B$;
3. If $b \in B$ is non-isolated in $\tau_\omega$, then $b$ is an accumulation point for $B$.

Lemma 2. [2] If $A \subset \tau_\omega$ is countable, then there exists an $\omega$-support $B$ of $A$.

Definition 3. [2] Let $A \subset \tau_\omega$ be countable and an $\omega$-support of itself. Let $f \in C_p(\tau_\omega)$. Define $C_{f,A}$ as follows:

$$C_{f,A}(x) = f(a_x), \quad a_x = \sup\{a \in A: a \leq x\}.$$ 

It is proved in [2] that if $f \in C_p(\tau_\omega)$ and $A \subset \tau_\omega$ is an $\omega$-support of itself then $C_{f,A}(x) \in C_p(\tau_\omega)$.

Lemma 4. [2] Let $A \subset \tau_\omega$ be countable and an $\omega$-support of itself and $B$ be a base of $R$. Let $U \subset C_p(\tau_\omega)$ be open and contain $C_{f,A}$. Then there exist sequences $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ and $\{B_1, \ldots, B_n\}$ with the following properties:

1. $a_i \in A$;
2. $b_i \in A$ for $i < n$ and $b_n = \tau$;
3. $B_i \in B$;
4. $C_{f,A} \in \{g \in C_p(\tau_\omega): g([a_i, b_i]) \subset B_i$ if $a_i \neq b_i$ and $g(a_i) \in B_i$ if $a_i = b_i\} \subset U$.

Theorem 5. $C_p(\tau_\omega)$ is a $D$-space for any $\tau$.

Proof. Let $B$ be a countable base of $R$. Let $\phi$ be any neighborhood assignment for $C_p(\tau_\omega)$, such that for each $d \in C_p(\tau_\omega)$, there exist $B_1, \ldots, B_{n_d} \in B$, and a finite set $\{x_1, \ldots, x_{n_d}\} \subset X$, satisfying $\phi(d) = \{f: f \in C_p(\tau_\omega), f(x_i) \in B_i, i \leq n_d\}$. We will choose $\phi(d_n)$ inductively, such that $d_n+1 \notin \bigcup\{\phi(d_i): i \leq n\}$. Finally, we will show that $X = \bigcup\{\phi(d_n): n \in \mathbb{N}\}$.

Step 1. Take an arbitrary $d_1 \in C_p(\tau_\omega)$. We know that $\phi(d_1)$ depends on a finite set $X_1$. Let $A_1$ be an $\omega$-support of $X_1$.

We let $S_1$ consist of all pairs $\{(a_1, [a_n, b_n]), [B_1, \ldots, B_n]\}$, where $B_i \in B$ and $a_i \in A_1$ for all $i \leq n$, and $b_i \in A_1$ if $i < n$ and $b_n = \tau$. We let $C_1 = \{U_i: s \in S_1\}$, where $U_s = \{f: f \in C_p(\tau_\omega), f([a_i, b_i]) \subset B_i, i \neq b_i, and f(a_i) \in B_i, a_i = b_i, i \leq n_s\}$, $s = \{(a_1, b_1), \ldots, (a_n, b_n), [B_1, \ldots, B_n]\}$. For every $C \in C_1$ put $C^* = \{d: d \in C, C \notin \phi(d)\}$. Let $C_1^* = \{C^*: C \in C_1\}$. Enumerate $C_1^*$ by prime numbers $p$.

Step $n$. If $\phi(d_1) \cup \cdots \cup \phi(d_{n-1})$ covers $C_p(\tau_\omega)$, we will stop the induction. Otherwise, we take the first $C^* \in \bigcup\{C^*_i: i \leq n-1\}$ such that $C^* \setminus \bigcup\{\phi(d_i): i \leq n-1\} \neq \emptyset$, and choose a point $d_n \in C^* \setminus \bigcup\{\phi(d_i): i \leq n-1\}$. Thus $C \subset \phi(d_n)$, and $d_n \in C$, and there is some $j \leq n-1$ such that $C \in C_j$, and $C = (g: g \in C_p(\tau_\omega), g([a_i, b_i]) \subset B_i, a_i \neq b_i, and g(a_i) \in B_i, a_i = b_i, i \leq n_i)$, where $s = \{(a_1, b_1), \ldots, (a_n, b_n), [B_1, \ldots, B_n]\} \in S_j$. If no such
$C^*$ exists in $\bigcup\{C^*_i \colon i \leq n - 1\}$, just take any $d_n \in C_p(\tau_0)$, such that $d_n \notin \bigcup\{\phi(d_i) \colon i \leq n - 1\}$. The set $\phi(d_n)$ depends on a finite set $X_n$. Let $A_n$ be an $o$-support of $X_n \cup A_{n-1}$, and let $S_n$ be the set of all pairs $\{(a_1, b_1), \ldots, (a_k, b_k)\}$, where $B_i \in B, a_i \in A_n, b_i \in A_n$ for each $i < k, b_k = \tau, k \in N$.

Let $C_n = \{C_i \colon \exists s \in S_n \cup S_{n-1}\}$, where $C_i = \{g \in C_p(\tau_0), g([a_i, b_i]) \subseteq B_i \}, if a_i \neq b_i, and g(a_i) \in B_i, if a_i = b_i, i \leq n_1$, and $s = (\{[a_1, b_1], \ldots, [a_n, b_n]\}, \{B_1, \ldots, B_n\})$. We denote $C^*_n = \{C^* \colon C \in C_n\}$, where $C^*$ is the set: $\{d \colon d \in C, C \in \phi(d)\}$. We enumerate $C^*_n$ by the $n$th prime numbers. And the enumeration on $C^*_i, i \leq n - 1$, is left unchanged.

Let us show that $\bigcup\{\phi(d_i) \colon n \in N\}$ covers $C_p(\tau_0)$. Take any $f \in C_p(\tau_0)$, let $A = \bigcup\{A_n \colon n \in N\}$. Thus $A$ is an $o$-support of itself. So $C_{f,A} \in C_p(\tau_0)$ (cf. [2]). Thus $C_{f,A} \in \phi(C_{f,A})$. By the properties (1)–(4) of Lemma 4, there exist sequences $\{(a_1, b_1), \ldots, (a_n, b_n)\}$, and $\{B_1, \ldots, B_n\}$ with the following properties: (1) $a_i \in A$; (2) $b_i \in A$ for $i < n$ and $b_n = \tau$; (3) $B_i \in B$; (4) $C_{f,A} \in \{g \in C_p(\tau_0), g([a_i, b_i]) \subseteq B_i \}, if a_i \neq b_i$ and $g(a_i) \in B_i$ if $a_i = b_i, i \leq n\} \subseteq \phi(C_{f,A})$. So there is some $m$ such that $s \in S_m \cup S_{m-1}$, and $s = (\{[a_1, b_1], \ldots, [a_n, b_n]\}, \{B_1, \ldots, B_n\})$. If we let $C = \{g \in C_p(\tau_0), g([a_i, b_i]) \subseteq B_i \}, if a_i \neq b_i and g(a_i) \in B_i, if a_i = b_i, i \leq n\}$, then $C \subseteq C_m$. Thus $C_{f,A} \in C \subseteq \phi(C_{f,A})$. Then we have $C_{f,A} \subseteq C^* and C^* \subseteq C^*_m$. So by induction, we may know that $C^*$ will be covered by $\bigcup\{\phi(d_k) \colon k \leq j\}$ for some $j$. Thus $C_{f,A} \subseteq \phi(d_k)$ for some $k \leq j$. Since $\phi(d_k)$ is standard and it depends on a finite set $X_k = \{s_m \colon m \leq n_k\}$. We assume that $\phi(d_k) \subseteq \{a(\lambda) \colon \lambda < \beta\}$ and $sup\{a(\lambda) \colon \lambda < \alpha\}$ are all rational numbers.

(4) If $\alpha < \beta < \omega_1, a \in A_{\alpha+1}$, $r \in Q and a(\alpha) < r$, then there exists $b \in A_{\beta+1}$ such that $P_{\alpha+1}(b) = a and b(\beta) = r$.

For each $1 \leq \alpha < \omega_1$, and $a \in A_\alpha$, define $x_\alpha \in Q^{\omega_1}$ by letting $x_\alpha(\beta) = a(\beta)$ if $\beta < \alpha and x_\alpha(\beta) = sup\{a(\lambda) \colon \lambda < \alpha\}$ otherwise. Let $X = \bigcup\{X_\alpha \colon 1 \leq \alpha < \omega_1\}$, where $X_\alpha = \{x_\alpha \colon a \in A_\alpha\}$. The topology of $X$ is inherited from $Q^{\omega_1}$.

Conclusion 6. Let $X = \bigcup\{X_\alpha \colon 1 \leq \alpha < \omega_1\}$, where $X_\alpha = \{x_\alpha \colon a \in A_\alpha\}$, and the topology of $X$ is inherited from $Q^{\omega_1}$. Then the space $X$ is a $D$-space.

Proof. Let $\phi$ be any neighborhood assignment of $X$. We may assume that $\phi(x) = \bigcap\{P_{\alpha_i}^{-1}(B_i) \colon i \leq n_k\} \cap X$, where $B_i \in B$ for each $i \leq n$, and $B$ is a countable base of $Q$. We may assume that every member of $B$ is the intersection of some open interval of $R$ with $Q$. We let $S(x) = max\{\alpha_i \colon i \leq n\}$. For any $x \in X$, there exist some $\alpha and a \in A_\alpha such that $x = x_\alpha$.

Since $x \in \phi(x), we can find some open set $V_x^2 in Q^{\omega_1}$, such that $x \in V_x^2 \cap X \subseteq \phi(x)$. We may assume that there are natural numbers $n$ and $m (m \geq 2)$, such that $V_x^2 = \bigcap_{\lambda \leq \alpha_1} V_\lambda$, where $V_\lambda = B_\alpha$ if $\lambda = \alpha_1, \alpha_1 < \alpha$ for each $i \leq n$, and $V_\lambda = B_\gamma$ if $\lambda = \gamma, \gamma_1 \geq \gamma$ for each $\gamma < m$ (we also assume $\gamma_1 \neq \gamma_{j+1}$ for each $j \leq m - 1$), otherwise $V_\lambda = Q$. We know that $x = x_\alpha, a \in A_\alpha, a$ is a strictly increasing sequence, and $x(\gamma) = x(\alpha)$ for any $\gamma \geq \alpha$. So $x(\alpha) \in B_\beta_j$ for each $\gamma < m$. We let $B = \bigcap\{B_\beta_j \colon j \leq m\}$, and max\{y_j \colon j \leq m\} = $\beta(x, B) > \alpha$. So $x(\alpha) \in B, and x(\beta(x, B)) \in B$.

We let $V_x^1 = \bigcap_{\lambda, < \alpha_1} V_\lambda$, where $V_\lambda = B_\alpha$ if $\lambda = \alpha_1, \alpha_1 < \alpha$ for each $i \leq n$, and $V_\lambda = B if \lambda = \gamma, \gamma_1 < \gamma$ for each $\gamma < m, otherwise V_\lambda = Q$. So $x \in V_x^1 \cap Q^{\omega_1} \subseteq V_x^2 \cap Q^{\omega_1}$.

Now we let $x_\lambda = \bigcap_{\lambda, < \alpha_1} V_\lambda$, where $V_\lambda = B_\alpha if \lambda = \alpha_1, \alpha_1 < \alpha$ for each $i \leq n$, and $V_\lambda = B, if \lambda = \gamma, otherwise V_\lambda = Q$. We may easily see that $V_x^1 \cap x \subseteq V_x \cap X$. Next we show that $V_x \cap X \subseteq V_x^1 \cap X$. For any $y \in V_x \cap X, we have that $y(\beta(x, B)) \in B, and $y(\alpha) \in B$. So $y(\alpha) \leq y(\gamma) \leq y(\beta(x, B))$ for any $\gamma$ satisfying...
\[ \alpha \leq \gamma \leq \beta(x, B) \] by the construction of \( X \). So \( \gamma(y) \in B \) following from that \( B \) is intersection of some interval of \( R \) with \( Q \). Thus \( \gamma(y_j) \in B \subset B_{y_j} \) following from \( \gamma_j \geq \alpha \) for each \( j \leq m \). So \( y \in V_1 \cap X \). Hence we have \( V_1 \cap X \subset V_1 \cap X \). So \( x \in V_1 \cap X \subset \phi(x) \).

Thus for any \( x \in X \), we assume \( x \in X_{\alpha} \). Then there is some open set \( V_\alpha \subset Q^{\omega_1} \), such that \( x \in V_\alpha \cap X \subseteq \phi(x) \).

We may assume that \( V_\alpha = \bigcap_{\lambda < \alpha} V_\lambda \), where \( V_\lambda = B \) if \( \lambda = \omega \) or for some ordinal number \( \beta(x, B) > \alpha \), and there is some \( n \), such that \( V_\lambda = B_n \) for \( \lambda = \omega \) and \( \alpha_i < \alpha \) for each \( i \leq n \), otherwise \( V_\lambda = Q \). We denote \( A(x) = \{ B \subset B : x(a) \in B, \text{ and there exists } \beta(x, B) > \alpha, \text{ such that } x \in F(x, B, \beta(x, B)) \text{ and } F(x, B, \beta(x, B)) \subset \phi(x) \} \), where \( F(x, B, \beta(x, B)) = \bigcap_{\lambda < \omega} F_\lambda \cap X \cap \phi(x) \), and \( F_\lambda = \{ \lambda \} \), if \( \lambda < \alpha \), and \( F_\lambda = B \) if \( \lambda = \alpha \) or \( \lambda = \beta(x, B) \), otherwise \( F_\lambda = Q \).

From above proof, we know that \( A(x) \neq \emptyset \) for each \( x \in X \). We define \( A_1 \subset A \) by \( F_\alpha \subset \phi(x) \), such that some \( x \in F_\alpha \subset \phi(x) \). Then \( x \in X_{\alpha} \), where \( F(x, B, \beta(x, B)) = \{ y : y \in F(x, B, \beta(x, B)) \subset \phi(y) \} \). Since \( |X| \leq \omega, |A(x)| \leq \omega \) for any \( x \in X_1 \), we have that \( |A_1| \leq \omega \). Enumerate \( A_1 \) by prime numbers \( p \).

Step 1. Firstly, we choose the first member of \( A_1 \), and denote it by \( f_1 \). Let \( x_1 \in F_1 \), we have that \( f_1 \subset \phi(x) \). Let \( \alpha_1 \) be the first ordinal such that \( x_1 \in X_{\alpha} \).

Secondly, we choose the first member of \( A_1 \), and denote it by \( f_2 \), such that \( f_2 \subset \phi(x) \). We let \( \alpha_2 = \min \{ \gamma : x_2 \in X_\gamma \} \). Then \( \beta_1 = sup(\{ \beta(x, B) : x \in X_1 \cup \{ \alpha_1 \} \cup \{ S(x_1) \} \} \), then \( \beta_1 \) is countable. We let \( A_2 = \{ F(x, B, \beta(x, B)) : B \subset A(x) \cap X_{\alpha_1}, \alpha < \beta_1 \} \). Enumerate \( A_2 \) by the squares \( \omega \) of prime numbers.

Step 2. We choose the first member \( F_{n+1} \) of \( A_n \), and denote it by \( f_{n+1} \), for some \( n \), such that \( f_{n+1} \subset \phi(x) \). Thus we choose a point \( x_{n+1} \in F_{n+1} \) such that \( x_{n+1} \cap (\phi(x)) \subset \phi(x) \). Assume \( x_{n+1} \) is equipped with \( (n+1) \)st powers of \( f_{n+1} \).

Let us prove that \( X = \bigcup \{ \phi(x_i) : i \leq n \} \). Let \( \beta = sup(\{ \beta_n : n \leq n \} \). Take an arbitrary \( x \in X \). Assume \( x \in \bigcup \{ X_\gamma : \gamma \leq \beta \} \). Then there exists \( \alpha_1 < \beta \) such that \( x \in X_{\alpha_1} \). Since \( A(x) \neq \emptyset \), and there exists \( F(x, B, \beta(x, B)) \subset \phi(x) \). Thus \( x \in F(x, B, \beta(x, B)) \). By induction, we may know that \( F^n(x, B, \beta(x, B)) \) will be covered by \( \bigcup \{ \phi(y) : i \leq n \} \). Thus there is some \( m \leq n \) such that \( x \in (\phi(x)) \). Now assume that \( x \in \bigcup \{ X_\gamma : \gamma \leq \beta \} \). Let \( x' \in X_\beta \), such that \( \beta = sup(\{ \gamma(x) : \gamma \leq \beta \}) \). Then \( x \in X_{\gamma(x)} \).

By [4], we know that every space with a point-countable base is a \( D \)-space. The conclusion can be generalized to the case of weak bases. Let us review the definition of weak bases (cf. [5]). A collection \( B = \bigcup \{ B_x : x \in X \} \) is called a weak base of \( X \), if for any \( x \in X \) the following conditions hold: (1) For each \( x \in X \), \( B_x \) is closed under finite intersections and \( x \in \bigcap B_x \); (2) A subset \( U \subset X \) of \( X \) is open iff for any \( x \in U \), there is some \( B \in B_x \), such that \( x \in B \subset U \).

**Theorem 7.** If \( X \) has a point-countable weak base, then \( X \) is a \( D \)-space.
Proof. Let $B = \bigcup \{B_x : x \in X\}$ be a point-countable weak base of $X$, and let $\phi$ be any neighborhood assignment of $X$. We may assume $X = \{x_\alpha : \alpha < \gamma\}$, where $\gamma$ is a cardinal number. Suppose for each $\alpha < \beta$, we have chosen a discrete set $D_\alpha$ satisfying:

1. $x_\alpha \in \{\phi(d) : d \in \bigcup \{D_i : i \leq \alpha\}\}$;
2. $\bigcup \{D_\eta : \eta < \alpha\}$ is a closed discrete set of $X$;
3. $D_\alpha \cap \bigcup \{\phi(d) : d \in \bigcup \{D_i : i < \alpha\}\} = \emptyset$;
4. For any $x \in X \setminus \bigcup \{\phi(d) : d \in \bigcup \{D_\eta : \eta \leq \alpha\}\}$ and for any $B \in B_x$, if $x \in B \subset \phi(x)$, then $B \cap D_\eta = \emptyset$, for any $\eta \leq \alpha$.

Before we construct $D_\beta$ let us show that $D'_\beta = \bigcup \{D_\alpha : \alpha < \beta\}$ is a closed discrete set of $X$.

By condition (2), we have known that for any $\alpha < \beta$, $D'_\alpha = \bigcup \{D_\eta : \eta < \alpha\}$ is a closed discrete set of $X$. We now show that $D'_\beta = \bigcup \{D_\alpha : \alpha < \beta\}$ is a closed discrete set of $X$.

Firstly, $\beta$ is a limit ordinal number. If $x \in X \setminus \bigcup \{\phi(d) : d \in \bigcup \{D_\alpha : \alpha < \beta\}\}$, then there is some $B \in B_x$, such that $x \in B \subset \phi(x)$. So $B \cap D_\alpha = \emptyset$ for any $\alpha < \beta$ by (4). Thus $B \cap D'_\beta = \emptyset$. If $x \in \bigcup \{\phi(d) : d \in \bigcup \{\phi(d) : d \in \bigcup \{D_\eta : \eta < \alpha\}\}\} \setminus D'_\beta$, we let $x_\alpha$ be the minimal ordinal, such that $x \in \bigcup \{\phi(d) : d \in D_{x_\alpha}\} \setminus \bigcup \{\phi(d) : d \in \bigcup \{D_\eta : \eta < \alpha\}\}$. Let $V_x = (((\bigcup \{\phi(d) : d \in D_{x_\alpha}\}) \setminus \bigcup \{\phi(d) : d \in \bigcup \{D_\eta : \eta < \alpha\}\}) \cap V'_x$, where $V'_x$ is an open set, such that $x \in V'_x$ and $|V'_x \cap D_{x_\alpha}| < 1$. Let $x \in V_x$ and $V_x$ is an open set. Then $V_x \cap D_\beta = \emptyset$. Thus there exists some $B \in B_x$, such that $x \in B \subset V_x$. So $B \cap D'_\beta = \emptyset$. Thus for any $x \notin D'_\beta$, there is some $B \in B_x$, such that $B \cap D'_\beta = \emptyset$. So $D'_\beta$ is a closed set. We may easily see that $D'_\beta$ is also a discrete set of $X$ following from the fact that $D_\alpha \cap \bigcup \{\phi(d) : d \in \bigcup \{D_i : i < \alpha\}\} = \emptyset$.

Secondly, $\beta = \alpha + 1$ for some $\alpha$. So $D'_\beta = \bigcup \{D_\eta : \eta < \alpha\} = \bigcup \{D_\eta : \eta \leq \alpha\} = \bigcup \{D_\eta : \eta < \alpha\} \cup \bigcup \{D_\eta : \eta = \alpha\}$. By assumption, we know that $D'_\alpha = \bigcup \{D_\eta : \eta < \alpha\}$ is a discrete closed set of $X$, so is $D'_\beta$.

We let $U_\beta = \bigcup \{\phi(d) : d \in \bigcup \{D_\eta : \eta < \beta\}\}$. Now we will construct $D_\beta$.

If $x_\beta \notin U_\beta$, then $D_\beta = \emptyset$. So we assume $x_\beta \in U_\beta$ and let $B_{x_\beta} = \{B^* : B \in B_{x_\beta}\}$, where $B^* = \{x : x \in B \subset \phi(x)\}$ and $B_{x_\beta} = \{B : x \in B \in B_{x_\beta}\}$. So $B_{x_\beta}$ is a countable family. Enumerate it by prime numbers $p$. Let $y_1 = x_\beta$ and take the first member $B^*_1$ of $B_{x_\beta}$ such that $B^* \cup \phi(y_1) \cup U_\beta = \emptyset$. We choose a point $y_2 \in B^* \setminus (\phi(y_1) \cup U_\beta)$. Then $B \subset \phi(y_2)$. The family $B_{y_2}$ is countable, and we denote $B_{y_2}^* = \{B^* : B \in B_{y_2} \setminus B_{y_2}^*\}$. We enumerate $B_{y_2}^*$ by the squares $p^2$ of prime numbers.

Suppose we have finished $n$ steps. We have $\phi(y_1), \ldots, \phi(y_n)$, and $B_{y_n}^*, i \leq n$. If $\bigcup \{\phi(y_i) : i \leq n\} \cup U_\beta = X$, then stop the induction, and let $D_\beta = \{y_i : i \leq n\}$. So we assume $\bigcup \{\phi(y_i) : i \leq n\} \cup U_\beta = X$, then we take the first member of $\bigcup \{B_{y_n}^* : i \leq n\}$, such that $B^* \setminus \bigcup \{\phi(y_i) : i \leq n\} \cup U_\beta \neq \emptyset$. We choose a point $y_{n+1} \in (B^* \setminus \bigcup \{\phi(y_i) : i \leq n\} \cup U_\beta)$. Then $B \subset B \subset \phi(y_{n+1})$. If $\bigcup \{B_{y_n}^* : i \leq n\}$ is contained in $\bigcup \{\phi(y_i) : i \leq n\} \cup U_\beta$, then we let $\min\{\eta' : x_{\eta'} \notin \phi(y_i) : i \leq n\} \cup U_\beta = \eta$, and denote $x_\eta$ by $y_{n+1}$. And we let $B_{y_{n+1}}^* = \{B^* : B \in B_{y_{n+1}}^* \setminus \bigcup \{B_{y_n}^* : i \leq n\}\}$, and enumerate it by the $(n + 1)$st powers of prime numbers.

In this way, we get $D_\beta = \{y_n : n \in N\}$. We have that $D_\beta \cap U_\beta = \emptyset$. Let us show that $D_\beta$ is closed. Fix any $x \notin \bigcup \{\phi(d) : d \in D_\beta\} \cup U_\beta$. If $B \in B_x$ and $x \in B \subset \phi(x)$, then $x \in B_{y_\eta}^*$. Suppose $B \cap D_\beta = \emptyset$, then there is some $n$ such that $B \in B_{y_n}$, and $x \in B_{y_n}^* \in B_{y_n}^*$. Thus $B^*$ will be covered by $\bigcup \{\phi(d) : d \in D_\beta\}$ by the induction. Contradiction. So $B \cap D_\beta = \emptyset$. For any $x \in \bigcup \{\phi(d) : d \in D_\beta\} \cup U_\beta \setminus D_\beta$, we may easily get a $B \in B_x$, such that $B \cap D_\beta = \emptyset$. Thus $D_\beta$ is a closed set of $X$, and we may know that it is discrete following from its construction. We have shown that $\bigcup \{D_\eta : \eta < \beta\}$ is a closed discrete set of $X$, so the set $\bigcup \{D_\eta : \alpha < \beta\} \cup D_\beta$ is also a closed discrete set. From the above discussion, we know that the conditions (1), (2), (3) and (4) hold.

Let $D = \bigcup \{D_\alpha : \alpha < \gamma\}$, we may easily see that $X = \bigcup \{\phi(d) : d \in D\}$. Next we will prove that $D$ is a closed discrete set of $X$.

For any $x \in X$, let $\eta = \min\{\alpha : \text{such that } x \in \bigcup \{\phi(d) : d \in D_\alpha\}\}$, we denote $V_x = \bigcup \{\phi(d) : d \in D_\eta\} \setminus \bigcup \{D_\delta : \delta < \eta\} \cap V'_x$, where $V'_x$ is an open set of $X$, and $x \in V'_x$, satisfying that $|V'_x \cap D_\eta| \leq 1$. So $V_x$ is an open set of $X$, and $x \in V_x, |V_x \cap | \leq 1$. Thus $D$ is a discrete closed set of $X$. So $X$ is a $D$-space. □
Corollary 8. If $X$ is a countably compact space with a point-countable weak base, then $X$ is a compact metrizable space.

In [8], it is proved if $X = \bigcup\{X_i : i \leq n\}$ and $X_i$ is a subparacompact space for each $i \leq n$, then $X$ is a $\alpha D$-space. A space $X$ is called a $\alpha D$-space, if for any open cover $U$ of $X$, and any closed set $F$ of $X$, there exists a closed discrete set $D_F \subset F$, such that $F \subset \bigcup\{D_f : d \in D_F\}$, where $d \in F(d) \in U$ for each $d \in D_F$ (cf. [8]). Every subparacompact (metacompact) space is a $\theta$-refinable space. So we want to study the $\alpha D$-property of finite union of $\theta$-refinable spaces.

From [9], we know that a space $X$ is called a $\theta$-refinable space, if for any open cover $U$ of $X$ there is an open refinement $V = \bigcup\{V_n : n \in N\}$ such that:

1. For any $n \in N$, $V_n$ is a cover of $X$.
2. For any $x \in X$ there is some $n \in N$ such that $\text{ord}(x, V_n) < \omega$, where $\text{ord}(x, V_n) = \{|V : x \in V \in V_n|\}$.
3. $V = \bigcup\{V_n : n \in N\}$ is called a $\theta$-refinement of $U$.

Lemma 9. [8] Suppose that $X$ is a space, $Y$ is a subspace of $X$ such that every closed subspace $F$ of $X$ contained in $X \setminus Y$ is a $\alpha D$-space, and $V$ is an open covering of $X$. Then, for every locally finite in $Y$ family $F$ of subsets of $Y$ refining $V$, there exist a locally finite in $X$ subset $A$ and a pointer $f : A \to V$ such that $\bigcup F \subset \bigcup f(A)$ (where $f(A) = \{f(d) : d \in A\}$). (In addition, if $V$ is an open set of $X$, and $V \cap \bigcup F = \emptyset$. Then we can get a locally finite (in $X$) set $A$, such that $A \cap V = \emptyset$.)

Theorem 10. $X = \bigcup\{X_i : i \leq n\}$ and every $X_i$ is a $\theta$-refinable space, for each $i \leq n$, then $X$ is a $\alpha D$-space.

Proof. We prove by induction.

If $n = 0$, then $X = \emptyset$, and the statement is obviously true. Assume now that the statement holds for $n = k$ for some $k \in \omega$, and let us show that it is also true for $n = k + 1$.

Let $V$ be an open cover of $X$. For each $i = 1, 2, \ldots, k + 1$, fix a covering $P_i = \bigcup\{\eta_{ij} : j \in \omega\}$ of $X_i$ satisfying the following conditions:

1. $P_i$ refines $V$;
2. $\bigcup \eta_{ij} = X_i$ for each $j \in \omega$;
3. For any $x \in X_i$, there is some $j \in \omega$, such that $\text{ord}(x, \eta_{ij}) < \omega$.

We let $X_{ij}^m = \{x : x \in X_i, \text{ord}(x, \eta_{ij}) \leq m\}, m \in N$. So we have $X = \bigcup\{X_{ij}^m : m \in N, i \leq k + 1, j \in \omega\}$.

We know that for each $i, j, m$, $X_{ij}^m$ is a closed set of $X_i$. For each $l < m$, if $V_l$ is an open set of $X$, and $X_l \subset V_l$, then $X_{l+1} \setminus V_l$ is a closed set of $X_{l+1}$, and there exists a discrete family (in $X_l$) $F_{ij}^{l+1}$, which is a refinement of $\eta_{ij}$, and $X_{l+1} \setminus V_l = \bigcup F_{ij}^{l+1}$. Let us show that for each $i, j, m$, we can get a discrete closet set $D_{ij}^m$ of $X$, and a pointer $f_{ij}^l : D_{ij}^m \to V$, such that $X_{ij}^m \subset \bigcup f_{ij}^l(D_{ij}^m)$.

We have $X_{ij}^1 = \bigcup F_{ij}^1$, where $F_{ij}^1$ is a discrete (in $X_i$) family, and it is a refinement of $V$. Then by Lemma 9, we have a discrete (in $X$) subset $D_{ij}^1$, and a pointer $f_{ij}^1 : D_{ij}^1 \to V$, such that $X_{ij}^1 \subset \bigcup f_{ij}^1(D_{ij}^1)$. We have $X_{ij}^2 \setminus \bigcup f_{ij}^1(D_{ij}^1) = \bigcup F_{ij}^2$, and $F_{ij}^2$ is discrete (in $X_i$) family, and it is a refinement of $V$. By Lemma 9, we can get a discrete (in $X$) subset $D_{ij}^2$, and a pointer $f_{ij}^2 : D_{ij}^2 \to V$, such that $X_{ij}^2 \setminus \bigcup f_{ij}^1(D_{ij}^1) \subset \bigcup f_{ij}^2(D_{ij}^2)$, and $(\bigcup f_{ij}^1(D_{ij}^1)) \cap D_{ij}^2 = \emptyset$. So $X_{ij}^2 \subset \bigcup f_{ij}^1(D_{ij}^1) \cup \bigcup f_{ij}^2(D_{ij}^2)$. In this way, we can get discrete (in $X$) subsets $D_{ij}^m, l < m$, and pointers $f_{ij}^l : D_{ij}^m \to V$, such that $X_{ij}^m \subset \bigcup f_{ij}^l(D_{ij}^l)$, $l \leq m$, and $D_{ij}^m \cap (\bigcup f_{ij}^l(D_{ij}^l)) = \emptyset$. We may easily see that every closed subset of $X_{ij}^m$ also has such a property.

We denote $\{X_{ij}^m : m \in N, i \leq k + 1, j \in \omega\}$ by $F_q : q \in N$. For each $q \in N$, and every closed set $M_q \subset F_q$ (closed in $F_q$), we have a discrete set (in $X$) $D_q$, and a pointer $f_q : D_q \to V$, such that $M_q \subset \bigcup f_q(D_q)$.

We let $D_{ij}^m$ be a discrete set of $X$, and have a pointer $f_{ij}^l : D_{ij} \to V$, such $F_{ij} \subset \bigcup f_{ij}(D_{ij})$. Let $q \in N$, suppose for each $m \leq q$, we have discrete sets $D_m$, and a pointer $f_{m} : D_m \to V$, such that $\{f_m : m \leq q\} \subset \bigcup (f_m(D_m)) : m \leq q$, and $D_m \cap \bigcup (f_m(D_m)) : p < m = \emptyset$. Let us construct a discrete set $D_{q+1}$, and a pointer $f_{q+1}$.
Since $F_{q+1} \setminus \bigcup \{ f_m(D_m): m \leq q \} = M_{q+1}$ is a closed set of $F_{q+1}$. The set $F_{q+1} = X_{ij}^m$ for some $i$, $j$, $m$. Every closed set (closed in $X_{ij}^m$ of $X_{ij}$ have the property what we have proved, so we can get a closed discrete set $D_{q+1}$ of $X$, and a pointer $f_{q+1}: D_{q+1} \to \mathcal{V}$, such that $F_{q+1} \setminus \bigcup \{ f_m(D_m): m \leq q \} \subset \bigcup \{ f_{q+1}(D_{q+1}) \}$, and $D_{q+1} \cap \bigcup \{ f_m(D_m): m \leq q \} = \emptyset$. So $\bigcup \{ F_m: m \leq q + 1 \} \subset \bigcup \{ f_m(D_m): m \leq q + 1 \}$.

We may easily see that $D = \bigcup \{ D_q: q \in \mathbb{N} \}$ is a discrete closed set of $X$ following from the fact $D_{q+1} \cap \bigcup \{ f_m(D_m): m \leq q \} = \emptyset$, and $X = \bigcup f(D)$, where $f: D \to \mathcal{V}$ is a pointer and satisfying $f|D_q = f_q$ for each $q \in \mathbb{N}$. So $X$ is a $\alpha D$-space. □

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