Groups for Geometries in Given Diagrams, II: on a Characterization of the Group Aut(HS)

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This paper is a continuation of [12]: we begin to carry out the programme outlined in [12], adapting a more detailed approach.

In this paper a geometry discovered by S. Yoshiara [19] and admitting Aut(HS) as automorphism group is considered. The interest of this geometry also lies in certain ‘irregular’ features, which give it an exceptional position among other c.C₂ geometries. The uniqueness of such a geometry has been proved by R. Weiss and S. Yoshiara [18] using Cayley. Here we give another proof of that result, stressing the theory of coverings of [14, 15].

The paper is organized as follows. In Section 1 a short survey of c.C₂ geometries is given. The geometry we study in this paper is described in Section 2. In Section 3 we give a general outline of the method we use (we could have referred to [12] for that, but we preferred to make this paper as self-contained as possible). The uniqueness proof is given in Sections 4 and 5.

1. INTRODUCTION

A c.C₂ geometry is a residually connected Buekenhout geometry [2] belonging to the following diagram:

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+----+    +----+
|   |    |   |
s  t  c  s  t
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where s and t are finite parameters, the words ‘point’, ‘line’ and ‘plane’ being types. More explicitly, a c.C₂ geometry is a connected 3-partite graph Γ, vertices in the first, second and third class being called ‘points’, ‘lines’ and ‘planes’ respectively, such that:

1. For every point a, the graph induced on the neighborhood Γₐ of a (the residue of a) is the incidence graph of a finite generalized quadrangle of order (s, t), lines and planes of Γ in Γₐ, being ‘points’ and ‘lines’ of that generalized quadrangle.

2. For every plane u, the residue Γᵤ of u (i.e. the graph induced on the neighborhood of u) is the incidence graph of a complete graph Kₜ₊₂ on s + 2 vertices, points and lines of Γ in Γᵤ, being vertices and edges of Kₜ₊₂, respectively.

3. For every line r, the residue Γᵣ of r (namely, the neighborhood of r) is the incidence graph of a generalized digon D₁,n, i.e. a complete 2-partite graph with classes of size 2 and t + 1.

Residues have been defined above: the adjacency relation of the graph is called the incidence relation of the geometry Γ. Edges and 3-cliques of the graph Γ are called flags; 3-cliques are also called chambers. We also assume the following:

4. The automorphism group Aut(Γ) of Γ is flag-transitive (i.e. transitive on the set of chambers of Γ).

Henceforth G will always be a flag-transitive subgroup of Aut(Γ) and G₀ will be the (in fact unique) minimal flag-transitive subgroup of Aut(Γ). A classification of the pairs (Γ, G) as above is known, under some additional conditions ([4, 7, 11, 12, 18]). Indeed, we have the following:

CASE A. Let residues of points of Γ be classical generalized quadrangles of order (s, t) with s ≥ 3 (and t > 1, as ‘classical’ entails ‘thick’). Then one of the following holds ([7, 18]).
Case A.1: \((s, t) = (3, 3)\), \(\text{Aut}(\Gamma) = U_6(2) \cdot 2\), \(G_0 = U_6(2)\).
Case A.2: \((s, t) = (4, 2)\), \(\text{Aut}(\Gamma) = U_4(3) \cdot (2^2)^{122}\), \(G_0 = U_4(3)\).
Case A.3: \((s, t) = (4, 2)\), \(\text{Aut}(\Gamma) = 3 \cdot U_4(3) \cdot (2^2)^{122}\), \(G_0 = 3 \cdot U_4(3)\) (non-split central extension).
Case A.4: \((s, t) = (3, 9)\), \(\text{Aug}(\Gamma) = \text{McL} \cdot 2\), \(G_0 = \text{McL}\).
Case A.5: \((s, t) = (9, 3)\), \(\text{Aut}(\Gamma) = \text{Suz} \cdot 2\), \(G_0 = \text{Suz}\).
Case A.6: \((s, t) = (9, 3)\), \(\text{Aut}(\Gamma) = G_0 = \text{HS} \cdot 2\).

All cases listed above do actually occur. We also note that the geometry of Case A.3 is a triple cover of the one of Case A.2. Hence Case A.2 gives us the only non-simply-connected example in this list. The last example (Case A.6) is quite exceptional from another point of view: the action induced by \(\text{Aut}(\Gamma)\) on the residue \(\Gamma_a\) of a point \(a\) does not contain the classical group naturally associated with the classical generalized quadrangle \(\Gamma\). This is precisely the example we will study in this paper.

Examples A.1 and A.2 (and A.3) are related to the exceptional isomorphism \(S_4(3) = U_4(2)\) (see [4]); both A.1 and A.2 (but not A.3) are the first members of infinite families related with \(U_n(2)\) and \(O^*_n(3)\) respectively, where \(\epsilon = +, -\) or \(\emptyset\) according to whether \(n = 0, 2\) or \(\pm 1 \mod 4\) (see [9]).

**CASE B.** Let \(s = 2\). We obtain affine polar spaces of order 2 and their quotients, in this case ([4, 7]; also see [6]): 11 examples exist, 4 with \(t = 1\), 3 with \(t = 2\) and 4 with \(t = 4\). More details can be found in [12].

**CASE C.** Let \(t = 1\) and \(s \geq 3\) and assume that the action induced by \(\text{Aut}(\Gamma)\) on the residues of planes contains \(A_{s+2}\). A classification for this case is given in [11], modulo a family of strange ‘flat’ examples that are not so easy to classify.

The case in which the 2-transitive action induced by \(\text{Aut}(\Gamma)\) on the \(s + 2\) points of a plane is smaller than \(A_{s+2}\) has not yet been studied.

**CASE D.** Let \(s = 1\). Here we just have \(C_3\) geometries with thin lines, classified in [13].

**CASE E.** Residues of points are flag-transitive thick non-classical generalized quadrangles. Nothing is known about this.

However, even where a classification is known (in particular, in Case A), something should still be done. Indeed, the classification theorem has been obtained by glueing together different pieces where quite different methods had been used. For instance, the earliest and most conspicuous contribution to the classification of Cases A and B is by Buekenhout and Hubaut [3], but their method is quite different from the one used by Weiss and Yoshiara in [18], who essentially finished the job in Case A. So, it would be better to have a unified proof, if possible, in order to prepare a method that might also be useful in other similar contexts. An outline of this programme has been given in [12], and this paper is just a continuation of [12]. S. Yoshiara is also independently working on a programme similar to this.

In the rest of this section we state some notation that we will use later.

The **collinearity graph** of a c.\(C_2\) geometry \(\Gamma\) is the graph \(\mathcal{G}(\Gamma)\) having the points and the lines of \(\Gamma\) as vertices and edges, respectively. Note that multiple edges are allowed in \(\mathcal{G}(\Gamma)\). The diameter \(d\) of \(\mathcal{G}(\Gamma)\) is the **diameter** of \(\Gamma\). We say that \(\Gamma\) is an extended generalized quadrangle [4] if \(\mathcal{G}(\Gamma)\) has no multiple edges (namely, if the Intersection Property holds in \(\Gamma\); see [4, 7]).
For the rest of this section we assume $\Gamma$ to be an extended generalized quadrangle. The symbol $\sim$ will denote the adjacency relation in $\mathcal{G}(\Gamma)$ (collinearity in $\Gamma$) and, given a point $p$, $p\sim$ will be the adjacency of $p$ in $\mathcal{G}(\Gamma)$. As the Intersection Property holds in $\Gamma$, the planes of $\Gamma$ are uniquely determined by their sets of points. So, we can write $p \in u$ to mean that the point $p$ is incident with the plane $u$. Given a point $p$ and a plane $u$ such that $p \notin u$ and $p \sim \cap u \neq \emptyset$, let $\alpha(p, u) = |p \sim \cap u|$. The number $\alpha(p, u)$ is even \cite{4}. We set:

$$\alpha_1 = \min(\alpha(p, u) \, | \, p \notin u, \, p \sim \cap u \neq \emptyset),$$

$$\alpha_2 = \max(\alpha(p, u) \, | \, p \notin u, \, p \sim \cap u \neq \emptyset).$$

Given two points $p, q$ of $\Gamma$, let $d(p, q)$ be the distance of $p$ and $q$ in $\mathcal{G}(\Gamma)$. When $d(p, q) = 2$, the set $p \sim \cap q \sim$ is the local subspace of $p$ and $q$. We set $\mu(p, q) = |p \sim \cap q \sim|$ and

$$\mu_1 = \min(\mu(p, q) \, | \, d(p, q) = 2),$$

$$\mu_2 = \max(\mu(p, q) \, | \, d(p, q) = 2).$$

Given a flag-transitive subgroup $G \leqslant \text{Aut}(\mathcal{G}(\Gamma))$ and a chamber $(a, r, u)$ of $\Gamma$, the parabolic subgroups $G_a, G_r, G_u$ are the stabilizers of $a, r, u$ respectively in $G$, the kernels $K_a, K_r, K_u$ are the elementwise stabilizers of the residues $\Gamma_a, \Gamma_r, \Gamma_u$ respectively, and $\tilde{G}_x = G_x/K_x$ is the action of $G_x$ in $\Gamma_x(x = a, r, u)$.

2. THE YOSHIARA GEOMETRY

In this section we describe the geometry for $\text{Aut}(\text{HS})$ discovered by Yoshiara \cite{19} and provide some information on it, taken from \cite{18}.

Let $\mathcal{H}$ be the Higman–Sims graph with 100 vertices and 1100 edges.

Yoshiara \cite{Yo} defines another graph, $\mathcal{Y}$ say, taking the edges of $\mathcal{H}$ as vertices and stating that 2 edges $e = \{x_1, x_2\}, e' = \{x_1', x_2'\}$ of $\mathcal{H}$ are adjacent as vertices of $\mathcal{Y}$ when they are disjoint and $\{x_i, x_i'\}$ is a non-edge of $\mathcal{H}$ for $i, j = 1, 2$. The graph $\mathcal{Y}$ has two families of maximal cliques, one family consisting of cliques of size 10 and the other of cliques of size 11. Take the maximal cliques of size 11 as planes and the edges and vertices of $\mathcal{Y}$ as lines and points, respectively. An extended generalized quadrangle $\Gamma_Y$ is defined in this way, admitting $\text{Aut}(\text{HS})$ ($= \text{Aut}(\mathcal{H})$) as a full automorphism group. The group $G = \text{Aut}(\Gamma_Y) = \text{Aut}(\text{HS})$ is flag-transitive. However, the cliques of $\mathcal{Y}$ of size 11 are shared in two classes, corresponding to the 2 conjugacy classes of $M_{11}$ in $\text{HS}$. The group $\text{Aut}(\text{HS})$ interchanges those 2 conjugacy classes. Hence it interchanges those 2 families of cliques of size 11. But $\text{HS}$ does not do so, of course. Hence $\text{HS}$ is no longer flag-transitive on $\Gamma_Y$.

The geometry $\Gamma_Y$ has parameters $(s, t) = (9, 3)$ and the residues of the points of $\Gamma_Y$ are isomorphic to the Hermitian variety $H_3(3^2)$. Given a chamber $(a, r, u)$ of $\Gamma_Y$ and $G = \text{Aut}(\Gamma_Y)$ ($= \text{Aut}(\text{HS})$), we have:

$$G_a = L_3(4) \cdot 2^2,$$

$$G_r = ((3^2: Q_8 \cdot 2) \times 2) \cdot 2,$$

$$G_u = M_{11},$$

$$K_a = 1,$$

$$K_r = 3^2: Q_8,$$

$$K_u = 1,$$

$$\tilde{G}_a = G_a,$$

$$\tilde{G}_r = 2^2 \cdot 2,$$

$$\tilde{G}_u = G_u.$$

We have 2 possible values for $\alpha(p, u)$, namely $\alpha_1 = 2$ and $\alpha_2 = 6$, and 3 possible values for $\mu(p, q)$, namely 24 ($= \mu_1$), 60 and 120 ($= \mu_2$). The diameter $d$ of $\Gamma_Y$ is 2.

The group $G$ has rank 5 on the set of points of $\Gamma_Y$ with suborbits of size 1, 280, 42, 105 and 672 and sub-sub-orbits (orbits of stabilizers of two collinear points) of size 1, 1, 36, 72, 18, 9 and 144. The sub-sub-orbits of size 36 and 72 are inside the suborbit of size 280. The sub-sub-orbits of size 18, 9 and 144 are inside the suborbits of size 42, 105 and 672, respectively.
There are 11 200 planes, forming 2 orbits for \( HS \cong G \) of size 5600 each.

Some of the previous information is summarized in Figure 1. We finish this section by describing another geometry for \( HS \) discovered by Buekenhout [1], starting from the graph \( \mathcal{K} \). Let us denote this geometry by \( \Gamma_B \). The geometry \( \Gamma_B \) has the following diagram:

![Diagram of \( \Gamma_B \)](image)

and we have \( \text{Aut}(\Gamma_B) = 2 \times \text{Aut}(HS) \). The factor of 2 comes from a diagram automorphism of \( \Gamma_B \). The minimal flag-transitive automorphism group of \( \Gamma_B \) is \( HS \). The geometry \( \Gamma_B \) can be constructed as follows. If \( V \) is the set of vertices of \( \mathcal{K} \), take \( V^* = V \times \{ +, - \} \) as a new set of vertices and define a new graph \( \mathcal{K}^* \) on \( V^* \) assuming that 2 elements \((x, \varepsilon)\) and \((y, \eta)\) of \( V^* \) \((\varepsilon, \eta = + \text{ or } -)\) are adjacent in \( \mathcal{K}^* \) if \( \varepsilon \neq \eta \) and \((x, y)\) is an edge of \( \mathcal{K} \). Take the elements of \( V^* \) as points, giving them the type \(+\) or \(-\) according to whether they have the form \((x, +)\) or \((x, -)\). Take as lines the pairs \{(x, \varepsilon), (y, \varepsilon)\}, where \((x, y)\) is a non-edge in \( \mathcal{K} \), and give them the type \(+\) or \(-\) according to whether \( \varepsilon = + \text{ or } -\). Define incidence as follows: 2 points are incident when they are adjacent as vertices of \( \mathcal{K}^* \); a point \( p \) and a line \( r = \{a, b\} \) are incident if either \( p \in r \) or \( \{p, a\}, \{p, b\} \) are edges of \( \mathcal{K}^* \); two lines \( r = \{a_1, a_2\}, r' = \{a'_1, a'_2\} \) are incident if \( \{a_i, a'_j\} \) is an edge of \( \mathcal{K}^* \) for \( i, j = 1, 2 \).

Clearly, \( \Gamma_B \) admits a diagram automorphism \( \delta \) interchanging the signs \(+\) and \(-\).

Another geometry can be built from \( \Gamma_B \) as follows: take the \( +\)-point flags of \( \Gamma_B \) as ‘points’, the \( +\)-line flags as ‘lines’, and the \( +\)-plane flags as ‘planes’, and assume that a ‘point’ and a ‘line’, a ‘point’ and a ‘plane’, and a ‘line’ and a ‘plane’ are incident if they were incident as flags of \( \Gamma_B \). Let \( \Gamma'_B \) be the geometry obtained in this way. Then \( \text{Aut}(\Gamma'_B) = \text{Aut}(\Gamma_B) \) and \( \Gamma'_B \) belongs to the following Tits diagram (so it is a GAB, in the meaning of [8]):

![Diagram of \( \Gamma'_B \)](image)
Characterization of the group $\text{Aut}(HS)$

$\text{Aut}(HS)$ is the minimal flag-transitive automorphism group of $\Gamma_B'$, and the diagram automorphism $\delta$ considered above for $\Gamma_B$ is now a type-preserving automorphism of $\Gamma_B'$ and defines a quotient $\Gamma_B'/\langle \delta \rangle$. $\text{Aut}(HS)$ is the only flag-transitive automorphism group of $\Gamma_B'/\langle \delta \rangle$ and the 'points' of $\Gamma_B'/\langle \delta \rangle$ are clearly the same as the points of $\Gamma_B$.

We do not know if some interesting relation between $\Gamma_B$ and $\Gamma_B'$ is hidden behind these curiosities.

NOTE. The construction given in [1] for $\Gamma_B$ looks a little different from the one we have described above, but the reader may easily check that they are actually equivalent.

3. OUTLINE OF THE METHOD AND STATEMENT OF THE THEOREM

A local description over a set of types $I = \{1, 2, 3\}$ is a triplet $\tau = ((\Gamma_i, G_i) \mid i = 1, 2, 3)$, where $\Gamma_i$ is a geometry over the set of types $I \setminus \{i\}$ and $G_i$ is a flag-transitive group of type-preserving automorphisms of $\Gamma_i$ (for $i = 1, 2, 3$).

Given a geometry $\Gamma$ over the set of types $I$ and a flag-transitive group $G$ of type-preserving automorphisms of $\Gamma$, we say that the pair $(\Gamma, G)$ belongs to the local description $\tau$ if, given any chamber $(a_1, a_2, a_3)$ of $\Gamma$, where $a_i$ has type $i$ ($i = 1, 2, 3$), we have $(\Gamma_i, G_i) \cong (\Gamma_a, G_a)$ for $i = 1, 2, 3$, where $G_a$ is the action induced on the residue $\Gamma_a$ of $a_i$ by the stabilizer $G_a$ of $a_i$ in $G$ and $(\Gamma_i, G_i) \cong (\Gamma_a, G_a)$ means that there is a type-preserving isomorphism of $\Gamma_i$ onto $\Gamma_a$ inducing an isomorphism of $G_i$ onto $G_a$.

Assume that we know an example $(\Gamma^*, G^*)$ belonging to the local description $\tau$. We set out to prove that it is the only possible example for $\tau$. Also assume that we have been able to prove the following:

(I) The geometry $\Gamma^*$ is simply connected.

By [10, pp. 234–236] or [17], (I) is equivalent to the following. Let $(a_1^*, a_2^*, a_3^*)$ be a chamber of $\Gamma^*$ where $a_i$ has type $i$ ($i = 1, 2, 3$) and let $G_i^*$ be the stabilizer of $a_i^*$ in $G^*$. Then:

(II) The group $G^*$ is the amalgamated product of the subgroups $G_1^*, G_2^*, G_3^*$ with amalgamation of $G_i^* \cap G_j^*$ ($i, j = 1, 2, 3$).

Now let $(\Gamma, G)$ be any example for $\tau$ and, given a chamber $(a_1, a_2, a_3)$, let $G_i$ be the stabilizer of $a_i$ in $G$. Assume that we are able to prove the following:

(III) There are isomorphisms $\alpha_i: G_i^* \rightarrow G_i$ such that, if $\alpha_{ij}$ is the restriction of $\alpha_i$ to $G_i^* \cap G_j^*$, then $\alpha_{ij}$ is an isomorphism of $G_i^* \cap G_j^*$ onto $G_i \cap G_j$ and $\alpha_{ij} = \alpha_{ji}$ ($i, j = 1, 2, 3$).

Then $G$ is a homomorphyic image of $G^*$ by (II). If $G^*$ is simple or $G_0 \leq G^* \leq \text{Aut}(G_0)$ for some simple group $G_0$ such that $\text{Aut}(G_0)/G_0$ is small enough, then $G^* = G$ and it follows that $(\Gamma^*, G^*)$ is the only possible example for $\tau$.

The next theorem selects the local description we have to consider in this paper.

THEOREM 1 [7, 18]. Let $\Gamma$ be a $c.C_2$ geometry such that residues of points of $\Gamma$ are classical generalized quadrangles of order $(s, t)$ with $s \geq 3$ (and $t > 1$) and let $G \leq \text{Aut}(\Gamma)$ be a flag-transitive automorphism group of $\Gamma$ such that the action $\tilde{G}_a$ in $\Gamma_a$ of the stabilizer $G_a$ of a point $a$ of $\Gamma$ does not contain the classical simple group naturally associated with $\Gamma_a$. Then $\Gamma$ is an extended generalized quadrangle and $(\Gamma, G)$ belongs to the following local description:

$\tau_\gamma = ((H_3(3^2), L_3(4) \cdot 2^2), (D_{1,3}, 2^2 \cdot 2), (K_{11}, M_{11}))$. 
Clearly, if $\Gamma_y$ is the Yoshiara geometry described in Section 2, then $(\Gamma_y, \text{Aut}(HS))$ belongs to the local description $\tau_y$ above. We aim to prove the following:

**Theorem 2. (Main theorem).** The pair $(\Gamma_y, \text{Aut}(HS))$ is the only possible example for $\tau_y$.

As we have said before, Theorem 2 has been proved in [18] exploiting Cayley in the final part of the proof. We will prove that theorem avoiding computers and proving the claims (I) and (III) above.

## 4. The Simple Connectedness of the Yoshiara Geometry

Let $\Gamma$ be an extended generalized quadrangle and let $\mathcal{B}(\Gamma)$ be the collinearity graph of $\Gamma$. We define a 2-dimensional simplicial complex $\mathcal{K}(\Gamma)$ as follows: the graph $\mathcal{B}(\Gamma)$ is the 1-skeleton of $\mathcal{K}(\Gamma)$ and the set of faces of $\mathcal{K}(\Gamma)$ is the set of all 3-subsets of planes of $\Gamma$.

A triplet $\{a, b, c\}$ of distinct pairwise collinear points of $\Gamma$ is a **full triangle** if it is a face of $\mathcal{K}(\Gamma)$; otherwise, $\{a, b, c\}$ is said to be an **empty triangle**. Clearly, $\mathcal{K}(\Gamma)$ is simply connected iff every simple closed path of $\mathcal{B}(\Gamma)$ splits in full triangles. The following is a specialization of Lemma 2 of [15].

**Lemma 1** [15]. An extended generalized quadrangle $\Gamma$ is simply connected iff $\mathcal{K}(\Gamma)$ is simply connected.

Let $\Gamma_y$ be the Yoshiara geometry described in Section 2. We will prove the following:

**Lemma 2.** The geometry $\Gamma_y$ is simply connected.

**Proof.** By way of contradiction, assume that $\Gamma_y$ is not simply connected. Let $\tilde{f}: \tilde{\Gamma} \rightarrow \Gamma_y$ be the universal covering of $\Gamma_y$. The chamber system $\tilde{\Gamma}$ is indeed a geometry [10, Lemma 1]. Let $\tilde{Z}$ be the group of deck transformations of $\tilde{f}$. As $\tilde{f}$ is universal, $\tilde{\Gamma}$ has a flag-transitive automorphism group $\tilde{G} = \tilde{Z} \cdot \text{Aut}(HS)$, and the pair $(\tilde{\Gamma}, \tilde{G})$ has the same local description, the same parabolics and the same kernels as $(\Gamma_y, \text{Aut}(HS))$ [14, 16]. We have $\tilde{Z} \neq 1$, as we have assumed that $\Gamma_y$ is not simply connected.

Our first goal is to obtain some more information on $\tilde{\Gamma}$ and $\Gamma_y$. In view of that, it will be convenient to generalize our problem a little. So, let $\Gamma$ be a $c.C_2$ geometry with a flag-transitive automorphism group $G$ such that $(\Gamma, G)$ has the same local description, the same parabolics and the same kernels as $(\Gamma_y, \text{Aut}(HS))$, and let $f: \Gamma \rightarrow \Gamma_y$ be a covering such that $G = Z \cdot \text{Aut}(HS)$, where $Z$ is the group of deck transformations of $f$. In particular, we might have $\Gamma = \tilde{\Gamma}$ and $f = \tilde{f}$ or $\Gamma = \Gamma_y$ and $f$ equal to the identity mapping.

Clearly, $\Gamma$ is an extended generalized quadrangle, just like $\Gamma_y$.

Let $a, b$ be distinct collinear points of $\Gamma$ and let $G_a, G_b$ be the stabilizers in $G$ of $a$ and $b$, respectively. Then $G_b$ and $G_{ab} = G_a \cap G_b$ behave in $\Gamma_b$ like their analogues in $\Gamma_y$. In particular, $G_{ab}$ has 6 orbits in $\Gamma_b$ of size 1, 36, 72, 9, 18 and 144, respectively. Of course $\{a\}$ is the orbit of size 1. The orbit of size 36 is the set of all points $c$ collinear with $b$ and adjacent to $a$ inside $\Gamma_b$ (i.e. $\{a, b, c\}$ is a full triangle). Let $X_{ba}^+$ be this set and let $X_{ba}^-$ be the orbit of size 72. Let $X_{ba}^i$ be the orbit of size $i$ ($i = 9, 18$ or 144).

Clearly, one of the following holds:

(i) all triangles of $\mathcal{B}(\Gamma)$ are full (namely, $\Gamma$ is a $BH$-extension, in the meaning of [4]);
(ii) we have $X_{ba}^+ \cup X_{ba}^- = a^- \cap b^-$. 
We need to draw some consequences from (ii). So, assume (ii). For every \( i = 1, 2, 3, 4 \) or 5, let \( Y_{ba}^i \) be the set of planes \( v \) on \( b \) not containing \( a \) and such that \( \alpha(a, v) = 2i \) (see Section 1). For every point \( c \in X_{ba}^+ \), let \( Y_c^i \) be the set of planes on \( c \) in \( Y_{ba}^i \). Let \( n_i, x_i \) be the sizes of \( Y^i \) and \( Y_c^i \), respectively (clearly, the size of \( Y_c^i \) does not depend on the choice of \( c \in X_{ba}^+ \)). Given \( d \in X_{ba}^i \) (\( i = 9, 18 \) or 144), let \( Y_d^i \) be the set of planes on \( d \) in \( Y_{ba}^i \) and let \( x_{ij} \) be the size of \( Y_d^i \) (clearly, \( x_{ij} \) does not depend on the choice of \( d \) in \( X_{ba}^i \)). It is easily seen that the following must hold:

\[
\begin{align*}
  n_i &= 36 x_i \quad (i = 1, 2, 3, 4, 5) \\
  9x_i + 18x_{i,18} + 144x_{i,144} &= (11 - 2i)n_i \quad (i = 1, 2, 3, 4, 5), \\
  x_2 + 2x_3 + 3x_4 + 4x_5 &= 4, \\
  x_1 + x_2 + x_3 + x_4 + x_5 &= 3.
\end{align*}
\]

There are just two 5-tuples \((x_1, x_2, x_3, x_4, x_5)\) satisfying these equations, namely:

(a) \((1, 0, 2, 0, 0)\);
(b) \((0, 2, 1, 0, 0)\).

The values of \( n_i \) are given by \( n_i = 36 x_i \). In each of the cases (a) and (b) above, 3 distinct possibilities arise for the 9-tuple \((x_{ij}, i = 1, 2, 3, 4, 5, j = 9, 18, 144)\).

Case (a). We only give the possibilities for \((x_{i,9}, x_{i,18}, x_{i,144})\):

(a.1) \((4, 0, 2)\);
(a.2) \((2, 2, 2)\);
(a.3) \((0, 4, 2)\).

Clearly, \( x_{3,j} = 4 - x_{1,j} \) and \( x_{i,j} = 0 \) for \( j = 9, 18, 144 \) and \( i \neq 1, 3 \).

Case (b). We only give the possibilities for \((x_{3,9}, x_{3,18}, x_{3,144})\):

(b.1) \((0, 2, 1)\);
(b.2) \((2, 1, 1)\);
(b.3) \((4, 0, 1)\).

Clearly, \( x_{2,j} = 4 - x_{3,j} \) and \( x_{ij} = 0 \) for \( j = 9, 18, 144 \) and \( i \neq 2, 3 \). Case (a.1) indeed occurs in the Yoshiara geometry \( Y_r \). We will now prove that the remaining cases cannot occur.

The geometry \( \Gamma \) has the same parabolic subgroups as \( Y_r \): whence, if \( v \) is a plane on \( b \), the stabilizer \( G_{bv} \) of \( v \) and \( b \) is \( M_{10} \) in its natural action on the 10 points of \( v \) other than \( b \). Then, if \( c, d \) are 2 further points of \( v \), the stabilizer \( G_{bcd} \) of \( b, c, d \) has order 8, and the 8 points of \( v - \{b, c, d\} \) form an orbit of \( G_{bcd} \).

Let (a.2) or (a.3) occur and let \( c \in X_{ba}^{18} \). The group \( G_{abc} \) has order 16 and has a subgroup \( H \) of order 8 fixing a plane \( v \in Y_{ba}^3 \) on \( c \). Let \( d \) be the (unique) point of \( v \) adjacent to \( a \) inside \( I_s \); namely, there is a plane containing all of \( a, b, d \). The group \( H \) also fixes \( d \). Then \( H = G_{bcd} \), stabilizer of 2 points in \( M_{10} \). But \( (a \cap v) - \{b, c, d\} \) and \( v - (a \cup \{c\}) \) are distinct 4-subsets of \( v \) fixed by \( H \). This does not fit with the fact that the stabilizer in \( M_{10} \) of 2 points \( c, d \) is transitive with the remaining 8 points: whence neither of (a.2) or (a.3) can occur.

Let any of (b.1), (b.2), (b.3) occur. Take a point \( c \in X_{ba}^+ \) and let \( v \) be the (unique) plane in \( Y_c^3 \) on \( c \). The group \( G_{abc} \) fixes \( v \). As it fixes \( b \) and \( c \) in \( v \) and has order 8, it fixes a third point \( d \) on \( v \): whence \( G_{abc} (= G_{bcd}) \) is the stabilizer of 2 points \( c, d \) in \( M_{10} \) and the remaining 8 points of \( v - \{b, c, d\} \) form an orbit of \( G_{abc} \) in \( v \). But \( G_{abc} \) fixes the 4-sets \( (a \cap v) - \{b, c\} \) and \( v - (a \cup \{c\}) \). We have a contradiction: none of (b.1), (b.2), (b.3) can occur. So, we have established the following:

Statement 1. (a.1) is the only possibility in case (ii).
The following can also be obtained by easy computations:

**Statement 2.** Let (ii) (hence (a.1)) occur. If \( c \in X_{ba} \), then the 4 planes of \( \Gamma_b \) on \( c \) belong to \( Y_{ba} \). Every plane in \( Y_{ba}^1 \) has 1 point in \( X_{ba}^0 \) and 8 points in \( X_{ba}^{144} \). Every plane in \( X_{ba}^2 \) has 1 point in \( X_{ba}^{18} \), 4 points in \( X_{ba}^{144} \) and 4 points in \( X_{ba} \).

Of course, the previous discussion applies to both \( \Gamma_y \) and \( \hat{\Gamma} \), as any of \( \Gamma = \Gamma_y \) or \( \Gamma = \hat{\Gamma} \) was allowed. In particular, as (i) does not hold in \( \Gamma_y \), (ii), (a.1) and Statement 2 apply to \( \Gamma_y \).

From now on, we assume that \( \Gamma = \Gamma_y \).

**Statement 3.** Every simple closed path of \( \mathcal{G}(\Gamma) \) splits into triangles.

Indeed, every simple closed path of \( \mathcal{G}(\Gamma) \) splits into triangles and/or quadrangles, because \( \mathcal{G}(\Gamma) \) has diameter \( d = 2 \).

Let \( (a, b, c, d, e) \) be a pentagon in \( \mathcal{G}(\Gamma) \). We first prove that \( (a, b, c, d, e) \) splits into triangles and/or quadrangles.

Let \( A_a, B_a, C_a \) be the orbits of \( G_a \) of size 105, 42 and 672, respectively. If we consider the order of the stabilizer \( G_{ac} \) of \( a \) and \( c \), we easily see that \( c \in A_a \) iff \( a \in A_c \), \( c \in B_a \) iff \( a \in B_c \) and \( c \in C_a \) iff \( a \in C_c \), where \( A_a, B_a, C_a \) are the analogues of \( A_a, B_a, C_a \).

We say that a plane \( v \) on \( a \) is close or far from \( a \) according to whether \( a^- \cap v \neq \emptyset \) or \( a^- \cap v = \emptyset \).

Let \( c \in A_a \). Then 48 of the planes on \( c \) are close to \( a \) and we have \( \alpha(a, v) = 2 \) for each plane \( v \) on \( c \) close to \( a \). Let \( c \in B_a \). Then 80 of the planes on \( c \) are close to \( a \) and we have \( \alpha(a, v) = 6 \) for each plane \( v \) on \( c \) close to \( a \). Let \( c \in C_a \). Then 80 of the planes on \( c \) are close to \( a \). We have \( \alpha(a, v) = 6 \) for 60 of them and \( \alpha(a, v) = 2 \) for the remaining ones.

Let \( c \in B_a \cup C_a \). Then \( a \in B_c \cup C_c \) and 80 of the planes on \( a \) are close to \( c \). As at least 48 of the planes on \( a \) are close to \( d \), there are at least 16 planes on \( a \) which are close to both \( c \) and \( d \). Then we find points \( c', d' \) such that \( c \sim c' \sim a \sim d' \sim d \) and \( c \sim d' \). The pentagon \( (a, b, c, d, e) \) splits into \( (a, b, c, c'), (c, d, d', c'), (d, e, a, d') \) and \( (a, c', d') \).

The case of \( d \in B_a \cap C_a \) is clearly quite similar to the above.

Let us now assume that \( c, d \in A_a \). Then we have \( |c^- \cap a^-| = |d^- \cap a^-| = 24 \). If either \( a^- \cap c^- \cap d^- \neq \emptyset \) or there are points \( b' \in a^- \cap c^- \) and \( e' \in a^- \cap d^- \) such that \( b' \sim e' \), then we are done. So, we may assume that \( a^- \cap c^- \cap d^- = \emptyset \) and that \( b' \neq e' \) for every \( b' \in a^- \cap c^- \) and \( e' \in a^- \cap d^- \). There are just 4.242 triplets \( (b', a', e') \) such that \( b' \in a^- \cap c^- \), \( e' \in a^- \cap d^- \), \( a' \sim a \) and both \( \{a, a', b'\} \) and \( \{a, a', e'\} \) are full triangles. Each plane on \( a \) meets \( c^- \cap a^- \) in either 0 or 2 points, because \( c \in A_a \). Similarly, each plane on \( a \) meets \( d^- \cap a^- \) in either 0 or 2 points. Then every point \( a' \sim a \) appears in at most 16 triplets \( (b', a', e') \) as above. It appears in exactly 16 of them when each of the planes on \( a \) and \( a' \) meets \( (a^- \cap c^-) \cup (a^- \cap d^-) \), 2 of them meet \( a^- \cap c^- \) and the other 2 meet \( a^- \cap d^- \). Hence there are at least 144 points \( a' \sim a \) appearing in triplets as above. But \( A_a \) contains 105 points. Then \( a' \in B_c \cup C_c \) for some point \( a' \sim a \) collinear with points \( b', e' \) belonging to \( a^- \cap c^- \) and \( a^- \cap d^- \) respectively. Hence the pentagon \( (a, b, c, d, e) \) splits into \( (a, b, c, b'), (a, b', a'), (a, a', e'), (a, e', d, e') \) and \( (a', b', c, d, e') \). On the other hand, the pentagon \( (a', b', c, d, e') \) splits into triangles and quadrangles because \( a' \in B_c \cup C_c \). Hence \( (a, b, c, d, e) \) splits into quadrangles and triangles.

We still have to prove that every quadrangle of \( \mathcal{G}(\Gamma) \) splits into triangles.

Let \( (a, b, c, d) \) be a quadrangle in \( \mathcal{G}(\Gamma) \).

If \( b, d \) belong to the same connected component of the graph induced by \( \mathcal{G}(\Gamma) \) on \( a^- \cap c^- \), then the quadrangle \( (a, b, c, d) \) splits into triangles along some path from \( b \) to
Characterization of the group Aut(HS)
If $\tilde{a}$ is a plane of $\tilde{f}$ on $\tilde{b}$ such that $\tilde{a} \cap \tilde{a}^{-}\neq \emptyset$, then $\alpha(a, \tilde{f}(\tilde{u})) = 6$ and $\alpha(a, \tilde{f}(u))$ obtains two distinct contributions from $\alpha(\tilde{a}, \tilde{u})$ and $\alpha(\tilde{a}', \tilde{u})$. Then $\alpha(\tilde{a}, \tilde{u}) = \alpha(\tilde{a}', \tilde{u}) = 2$, whichever of the cases (i) or (ii) holds in $\tilde{f}$.

Hence $|\tilde{a}^{-} \cap \tilde{b}^{-}| = 36$. Indeed, there are exactly 72 planes $u$ on $b$ not on $a$ and such that $\alpha(a, u) = 6$ and a point $\tilde{d}$ in $\tilde{a}^{-} \cap \tilde{b}^{-}$ cannot be mapped by $\tilde{f}$ onto a point $d$ belonging to a plane $v$ containing both $b$ and $a$. Otherwise, if $\tilde{v} \in \tilde{f}^{-1}(v)$ contains $\tilde{a}$ and $\tilde{d}' \in \tilde{f}^{-1}(d) \cap \tilde{v}$, we have $\tilde{d} = \tilde{d}'$ because $\tilde{a}$ and $\tilde{d}'$ belong to the same fibre of $\tilde{f}$ and they are both collinear with $\tilde{b}$; hence $\tilde{d} \sim \tilde{a}$ and we have the contradiction $d(\tilde{a}, \tilde{a}') = 2$.

Therefore $\tilde{a}^{-} \cap \tilde{b}^{-}$ contributes one half of the orbit $X_{\tilde{b}a}$ of $G_{\tilde{b}a}$ of size 72 and forms one half of the orbit of $G_{\tilde{b}a}$ of size 72. We recall that the fibres of $\tilde{f}$ are the orbits of the group $\tilde{Z}$ of deck transformations of $\tilde{f}$. Those orbits are imprimitivity classes for $\tilde{G} = \tilde{Z} \cdot G$, as $\tilde{Z} \cong \tilde{G}$. Then $G_{\tilde{b}a}$ ($= G_{\tilde{b}a}$) fixes $\tilde{f}^{-1}(a)$. It cannot fix $\tilde{a}^{-} \cap \tilde{b}^{-}$ because that set is just one half of one of the orbits of $G_{\tilde{b}a}$. Then the orbit of $G_{\tilde{b}a}$ containing $\tilde{a}'$ contains at least one more element $\tilde{a}''$. Clearly, we have $\tilde{a}'' \in \tilde{f}^{-1}(a) - \{\tilde{a}, \tilde{a}'\}$, because $G_{\tilde{b}a}$ fixes $\tilde{f}^{-1}(a)$.

Let $U$ be the set of planes on $\tilde{b}$ containing some points of $\tilde{a}^{-}$ and $\tilde{a}'$ and $U'$ be defined in a similar way with respect to $\tilde{a}''$. Counting flags $(\tilde{d}, \tilde{u})$ with $\tilde{u} \in U$ and $\tilde{d} \in \tilde{a} \cap \tilde{a}^{-}$, we see that $|U| = 72$, because $\alpha(a', \tilde{u}) = 2$. Similarly, $|U'| = 72$. Then both $U$ and $U'$ are mapped by $\tilde{f}$ onto the set of 72 planes $u$ of $\Gamma$ containing $b$ but not $a$ and such that $\alpha(a, u) = 6$. This forces $U = U'$. Then $d(\tilde{a}', \tilde{a}'') = 3$ and each of $\tilde{a}, \tilde{a}', \tilde{a}''$ is collinear with 2 points of each of the planes in $U$.

Now let $\tilde{d}'$ be a point of a plane $\tilde{v}$ containing both $\tilde{a}$ and $\tilde{b}$, and let $g$ be an element of $G_{\tilde{b}a}$ mapping $\tilde{b}$ onto $\tilde{d}'$. Assume $\tilde{d}' \neq \tilde{b}$. Two of the 3 planes on $\tilde{b}$ and $\tilde{d}'$ not on $\tilde{a}$ are mapped by $\tilde{f}$ onto planes of $\tilde{f}(U)$, because $\tilde{f}(\tilde{d}')$ belongs to 2 planes of $\tilde{f}(U)$. But 2 of those 3 planes also belong to $g(U)$. Hence some of them belong to $U \cap g(U)$. Let $\tilde{u} \in U \cap g(U)$. Then each of the points $\tilde{a}'$, $\tilde{a}''$, $g(\tilde{a}')$, $g(\tilde{a}'')$ is collinear with 2 points of $\tilde{u}$. We cannot obtain more than 4 points of $\tilde{u}$ in this way and the pair $\{\tilde{a}', \tilde{a}''\}$ already provides 4 such points. Moreover, $g$ fixes $\tilde{f}^{-1}(a)$ as it fixes $\tilde{a}$. Furthermore, distinct points in the same fiber of $\tilde{f}$ have distance $\geq 3$. This forces $\{g(\tilde{a}'), g(\tilde{a}'')\} = \{\tilde{a}', \tilde{a}''\}$. It is now easily seen that $G_{\tilde{a}}$ fixes $\{\tilde{a}', \tilde{a}''\}$.

Therefore, $\tilde{G}_{\tilde{a}a} = L_{3}(4) \cdot 2$, $\tilde{G}_{\tilde{a}}$ permutes $\tilde{a}'$ and $\tilde{a}''$ and $\tilde{f}^{-1}(a)$ consists of disjoint pieces of size 3 such that 2 elements of $\tilde{f}^{-1}(a)$ belong to the same piece iff they have distance 3.

As $\tilde{f}^{-1}(a)$ contains just 2 points at distance 3 from $\tilde{a}$, there are just 2 homotopy classes of triangles based at $a$ and not null-homotopic. Given a plane $\tilde{u} \in U$, let $\{\tilde{b}, \tilde{b}'\}$, $\{\tilde{c}, \tilde{c}'\}$ and $\{\tilde{d}, \tilde{d}'\}$ be the pairs of points of $\tilde{u}$ collinear with $\tilde{a}$, $\tilde{a}'$ and $\tilde{a}''$ respectively. Let $b' = \tilde{f}(\tilde{b}')$, $c' = \tilde{f}(\tilde{c}')$, $d' = \tilde{f}(\tilde{d})$ and $d'' = \tilde{f}(\tilde{d}')$.

None of the triangles $(a, b, c)$, $(a, c', d)$, $(a, d', b')$ is null-homotopic. Then 2 of them must be mutually homotopic by the above, modulo changes in orientation. But we have also seen that we can fix any of $\tilde{a}$, $\tilde{a}'$, $\tilde{a}''$ permuting the other 2. It may be that, when we do so, $(a, b, c)$ is mapped onto $(a, c, d)$ instead of $(a, c', d)$, for instance. But this does not cause any problems. Indeed, $(a, c, d)$ and $(a, c', d)$ differ in the triangle $(a, c, c')$, which is full and hence null-homotopic. Therefore, any 2 of the triangles $(a, b, c)$, $(a, c', d)$, $(a, d', b')$ are homotopic modulo changes in orientation. It follows from this that $T = (a, b, c)$ and $T^{-1} = (a, c, b)$ are representative of the 2 homotopy classes of non-null-homotopic triangles based at $a$.

We already know that every simple closed path of $\mathcal{H}(\Gamma)$ splits into triangles (Statement 3) and it follows from the above that none of the empty triangles is null-homotopic. Therefore the homotopy group of $\mathcal{H}(\Gamma)$ has order 3: namely, $\tilde{Z} = 3$ [14–16].

The rest of Statement 4 follows easily now.
The final contradiction. Keep the assumption $\hat{r} \neq \Gamma_\gamma$. Consider the subgroup $\hat{G} = \hat{Z} \cdot HS = 3 \cdot HS$ of $\hat{G}$. It is a central extension of $HS$. It must split because $HS$ has Schur multiplier 2. Then we have $\hat{G} = 3 \times HS$ and $\hat{Z} = \mathbb{Z}(\hat{G}) = 3$. The group $\hat{G}$ is transitive on the points of $\hat{r}$ because $HS$ is transitive on the 1100 points of $\Gamma_\gamma$. As $\hat{G} = 3 \times HS$, $\hat{G}$ has 2 systems of imprimitivity classes on the 3300 points of $\hat{r}$; namely, a system $\{P, P', P''\}$ of 3 classes of size 1100 and the system of the 1100 orbits of $\hat{Z}$, of size 3. The stabilizer $\hat{G}_a$ of a point $\hat{a} \in P$ is $L_3(4) \cdot 2_1$ (see [5]) and, if $\hat{v}$ is a plane of $\hat{r}$ on $\hat{a}$, the stabilizer $\hat{G}_{\hat{a}\hat{v}}$ of the flag $(\hat{a}, \hat{v})$ is $M_{10}$ and is 3-transitive on the 10 points of $\hat{v} - \{\hat{a}\}$. Furthermore, $HS$ fixes each of $P, P', P''$. It follows from the above that each plane of $\hat{r}$ is contained in one of $P, P', P''$. Then $P, P', P''$ are 3 connected components of $\mathfrak{H}($ $\hat{r})$. But $\hat{r}$ is residually connected [14]: the final contradiction is reached.

Therefore, $\hat{r} = \Gamma_\gamma$: Lemma 2 is proved.

5. PROOF OF THE MAIN THEOREM

By Section 4, $\Gamma_\gamma$ is simply connected. Hence (II) of Section 3 holds for $G^* = \text{Aut}(HS)$ with $G^*_1 = L_3(4) \cdot 2^2$, $G^*_2 = (\langle 3^2 : Q_8 \rangle \cdot 2) \cdot 2$, $G^*_3 = M_{10}$, stabilizers of a point $a$, a line $r$ and a plane $u$, respectively, in a chamber $(a, r, u)$ of $\Gamma_\gamma$.

All we have to prove in order to establish Theorem 2 is that (III) of Section 3 holds for any pair $(\Gamma, G)$ belonging to the local description $\tau_\gamma$ of $G^*(\Gamma_\gamma, \text{Aut}(HS))$.

So, let $(\Gamma, G)$ belong to $\tau_\gamma$, let $(a, r, u)$ be a chamber of $\Gamma(a, r, u)$ being a point, a line and a plane, respectively) and let $G_a, G_r, G_u, K_a, K_r, K_u$ and $G_a, G_r, G_u$ have the meaning stated in Section 1.

LEMMA 3. (Weiss and Yoshiara [18]). We have $K_a = K_r = 1$.

We note that no use of computers is made in [18] to prove the above. Let us set $G_{ar} = G_a \cap G_r$, $G_{au} = G_a \cap G_u$, $G_{ru} = G_r \cap G_u$. Clearly, we have $G_{ar} = (\langle 3^2 : Q_8 \rangle \cdot 2) \cdot 2 = (\langle 3^2 : Q_8(h) \rangle \cdot \langle k \rangle)$, $G_{ru} = 3^2 : Q_8 \cdot 2 = 3^2 : Q_8(g)$ and $G_{ar} \cap G_{ru} = K_a = 3^2 : Q_8$, where the elements $k$ and $g$ are involutions, $h$ has order a power of 2 and $g$ interchanges the 2 points $a, b$ of $r$. Clearly, $G_{ar} = G_{ab}(= G_a \cap G_b)$ and $G_{ru} = 2^2 = \langle h, k \rangle / \langle h^2 \rangle$ acts on the 4 planes on $r$ as the Sylow 2-subgroup of $A_4$ (the equality $G_{ar} = G_{ab}$ holds because $\Gamma'$ is a generalized quadrangle, by Theorem 1). Furthermore, $\langle G_{au}, k \rangle = L_3(4) \cdot 2_1$ and $G_a = \langle G_{aux}, h, k \rangle = L_3(4) \cdot (2_1 333)$. Hence we have $\langle G_{au}, h \rangle = L_3(4) \cdot 2$.

Clearly, $G_r = G_{ar}: \langle g \rangle$ splits over $G_{ar}$. The action of $g$ on $K_r = G_{aru}$ is uniquely determined inside $G_u$.

We want to show that the structure of $G_u$ is uniquely determined by the above. Then (III) will follow. In order to do that, we have to show that just one action is possible for $g$ on $G_{ar}$ extending the (known) action of $g$ on $K_r$.

Let $g_1, g_2$ be 2 such actions consistent with the information we have on $g$. Then $g_2^{-1} g_1$ centralizes $K_r$. Let $\zeta$ be the automorphism of $G_{ar}$ defined by conjugation by $g_2^{-1} g_1$. Clearly, $\zeta$ fixes $k$, stabilizes $H = K_r \cdot \langle h \rangle$ and centralizes $K_r$. Let $S = Q_8(h)$. $S$ is a Sylow 2-subgroup of $H$. Then $\zeta(S) = xSx^{-1}$ for some $x \in O_2(K_r) = 3^2$. As $\zeta$ centralizes $Q_8 = K_r$, we have $Q_8 = xQ_8x^{-1}$. This forces $x = 1$ and $\zeta$ stabilizes $S$.

We also have $\zeta(h) = hf$ for some element $f$ of $Q_8$. Hence $\zeta(y) = yf^i$, where $i = 0$ or 1 according to whether $y \in Q_8$ or $y \in hQ_8 = Q_8h$. It follows from this that $f$ centralizes $Q_8$ (hence $f$ centralizes $S$ too).

For every $y \in S$ and $x \in O_2(K_r)$ we have $\zeta(y)x(\zeta(y))^{-1} = xyy^{-1}$ because $\zeta$ centralizes $0_2(K_r)$; namely, $f(yyy^{-1})f^{-1} = yyy^{-1}$. It follows from this that $f$ centralizes $O_2(K_r)$. Then $f \in Z(K_r)$. But $K_r$ is easily recognizable as $G_{aru}$ inside $G_u$ and there we see that $Z(K_r) = 1$. Then $f = 1$, $\zeta = 1$, $g_1 = g_2$ and $G_u$ is uniquely determined. We are done.
ACKNOWLEDGMENTS

I am deeply indebted to S. Yoshiara for a number of comments and suggestions which have been very useful in writing this paper, and will surely be useful for the job started in [12], if I will accomplish it. Indeed, as mentioned before, Yoshiara is independently carrying out a programme similar to that of [12], perhaps superior to my attempts.

REFERENCES


Received 13 April 1990 and accepted in revised form 16 September 1990

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