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Probability theory for the Brier game ☆

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Abstract

The usual theory of prediction with expert advice does not differentiate between good and bad "experts": its typical results only assert that it is possible to efficiently merge not too extensive pools of experts, no matter how good or how bad they are. On the other hand, it is natural to expect that good experts' predictions will in some way agree with the actual outcomes (e.g., they will be accurate on the average). In this paper we show that, in the case of the Brier prediction game (also known as the square-loss game), the predictions of a good (in some weak and natural sense) expert must satisfy the law of large numbers (both strong and weak) and the law of the iterated logarithm; we also show that two good experts' predictions must be in asymptotic agreement. To help the reader's intuition, we give a Kolmogorov-complexity interpretation of our results. Finally, we briefly discuss possible extensions of our results to more general games; the limit theorems for sequences of events in conventional probability theory correspond to the log-loss game. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

There is a substantial body of work in computational learning theory (starting from Littlestone and Warmuth [12]) devoted to the problem of combining predictions given by a pool of experts. The usual goal is to find a merging strategy for the learner whose losses would be almost as small as the losses of the best expert. It was realized, however, that another interesting goal for the learner is to perform *better* than the best expert. For example, Littlestone et al. [11] (whose work was continued by, among

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others, Cesa-Bianchi et al. [2] and Kivinen and Warmuth [9]) considered the problem of performing almost as well as the best linear combination of the experts. Another example is the work by Herbster and Warmuth [8], where the goal for the learner is to "track" the best expert. In both cases, however, it can be argued that the learner's goal is the old one, performing almost as well as the best expert in the pool, but applied to a bigger pool of experts: in the case of comparing the learner to the best linear combination of the experts, we can consider the pool consisting of all linear combinations of the old experts (see, e.g., [22]); and in the case of tracking the best expert, we can consider a pool containing "composite experts" who replicate the predictions of an original expert and are allowed to occasionally switch to a different expert (see [21]). In this paper we will consider a situation where the goal of performing better than the best expert cannot be reduced to the old goal of performing almost as well as the best expert. It is interesting that, as opposed to most of the previous work, we will obtain non-trivial results even in the case where the pool consists of a single expert.

To motivate our definitions, we first consider the following simple example. In the morning of day t ($t=1,2,\ldots$) an expert predicts whether or not it will rain that day; her predictions γ_t are allowed to take values in the interval $[0,1]$ (with $\gamma_t = 1$ interpreted as "definitely rain", $\gamma_t = 0$ as "definitely no rain", and $\gamma_t \in]0,1[$ as the degree of confidence in the event "rain today"). After the outcome $\omega_t \in \{0,1\}$ of the event "rain" is disclosed by Nature (with $\omega_t = 1$ coding "rain" and $\omega_t = 0$ "no rain"), the expert suffers a loss of $(\omega_t - \gamma_t)^2$ (this is the "Brier loss function" popular in meteorology). Suppose the expert's mean error

$$\frac{1}{T}\sum_{t=1}^{T}(\omega_t - \gamma_t) \tag{1}$$

does not tend to 0 as $T \to \infty$. Do we have any grounds to assert that her performance has been unsatisfactory? By the usual strong law of large numbers we can conclude that γ_t are not the true probabilities (conditional on the past) of $\omega_t = 1$, but the expert can still claim that her performance has been good and that nobody else would have performed better; there is no way to falsify this claim within the framework of the traditional probability theory. This problem can be solved, however, using the techniques of the theory of prediction with expert advice: it turns out that there exists a simple strategy for transforming the expert's predictions γ_t which ensures the following: either

$$\frac{1}{T}\sum_{t=1}^{T}(\omega_t - \gamma_t) \to 0 \quad (T \to \infty)$$

or

$$\sum_{t=1}^{T}(\omega_t - \gamma_t)^2 - \sum_{t=1}^{T}(\omega_t - \gamma_t^{\mathrm{L}})^2 \to \infty \quad (T \to \infty),$$

where γ_t^{L} are the transformed predictions (see Theorem 2 below). In other words, either the expert will be right on the average or she will perform infinitely worse than a simple strategy.

The previous example can be interpreted as a worst-case version of the strong law of large numbers; later on (in Section 4) we also state and prove a worst-case version of the law of the iterated logarithm. One more interesting result is the worst-case "theorem of agreement" (Theorem 8 below). Suppose we now have two experts who give predictions $\gamma_t^{(1)}$ and $\gamma_t^{(2)}$ for rain today. The theorem of agreement implies the existence of a (very simple) strategy for transforming $\gamma_t^{(1)}$ and $\gamma_t^{(2)}$ which ensures that at least one of the following three events eventually happens:

$$\gamma_t^{(1)} - \gamma_t^{(2)} \to 0 \quad (t \to \infty)$$

or

$$\sum_{t=1}^{T} \left(\omega_t - \gamma_t^{(1)}\right)^2 - \sum_{t=1}^{T} \left(\omega_t - \gamma_t^{\mathrm{L}}\right)^2 \to \infty \quad (T \to \infty)$$

or

$$\sum_{t=1}^{T} \left(\omega_t - \gamma_t^{(2)}\right)^2 - \sum_{t=1}^{T} \left(\omega_t - \gamma_t^{\mathrm{L}}\right)^2 \to \infty \quad (T \to \infty),$$

where γ_t^{L} are the transformed predictions. In other words, either the experts' predictions will be in agreement or at least one of the experts will perform infinitely worse than that strategy.

It is not immediately clear how all these results are related to algorithmic learning. Actually, the transformation strategy used in the proof of the theorem of agreement (see Section 8.8) is so simple that it hardly deserves to be called a learning strategy. However, in the case of the worst-case strong law of large numbers (and even more so in the case of the worst-case law of the iterated logarithm) non-trivial learning is needed. The idea of the proof (given in Section 8.2) is as follows. If the expert is not correct on the average, we could try to "recalibrate" her predictions by adding a constant ε (positive or negative). The obvious problem is that we do not know a priori which value of ε to choose. Using a learning algorithm which we call the Aggregating Algorithm [18, 20] allows us to home in on the right ε. For the law of the iterated logarithm the idea is similar (though more difficult to implement).

In Section 7, we restate these results (strong law of large numbers, law of the iterated logarithm, and theorem of agreement) in terms of a natural modification of Kolmogorov complexity.

This paper occasionally uses some basic notions of the theory of martingales. Excellent reviews are by Shiryayev [15] and Williams [24]; however, no knowledge of this theory is required for understanding the main ideas of this paper.

2. Main definitions

Let Ω (the *outcome space*) and Γ (the *prediction space*) be two sets, $\lambda : \Omega \times \Gamma \to [0, \infty]$ be a non-negative function (the *loss function*), and n be a positive integer (the *number of experts*). These elements define the following perfect-information game (called "game $(\Omega, \Gamma, \lambda)$ with n experts") between Learner, Nature, and a pool of experts which consists of Expert 1, …, Expert n:

for $t = 1, 2, \ldots$
  for $i = 1, \ldots, n$
    Expert i chooses prediction $\gamma_t^{(i)} \in \Gamma$
  end for
  Learner chooses his own prediction $\gamma_t^{\mathrm{L}} \in \Gamma$
  Nature chooses outcome $\omega_t \in \Omega$
end for

(cf., e.g., [20]). Typically, Learner is considered to be playing against all other players. At every trial t Expert i, $i = 1, \ldots, n$, and Learner suffer loss $\lambda(\omega_t, \gamma_t^{(i)})$ and $\lambda(\omega_t, \gamma_t^{\mathrm{L}})$, respectively. Our notation for the total loss suffered by Expert i and Learner over the first T trials will be

$$\mathrm{Loss}_T(\text{Expert } i) = \sum_{t=1}^{T} \lambda(\omega_t, \gamma_t^{(i)})$$

and

$$\mathrm{Loss}_T(\text{Learner}) = \sum_{t=1}^{T} \lambda(\omega_t, \gamma_t^{\mathrm{L}}),$$

respectively. A *play* of the game is a sequence of players' moves:

$$(\gamma_1^{(1)}, \ldots, \gamma_1^{(n)}, \gamma_1^{\mathrm{L}}, \omega_1, \gamma_2^{(1)}, \ldots, \gamma_2^{(n)}, \gamma_2^{\mathrm{L}}, \omega_2, \ldots). \tag{2}$$

To state our results, we need an explication of the notion of a good expert. We want as weak a definition as possible (to make our results as strong as possible), so we will use "adequate" as a technical term.

Definition 1. Expert i, $i = 1, \ldots, n$, is *adequate* for a play (2) of the game $(\Omega, \Gamma, \lambda)$ with n experts if

$$\liminf_{T \to \infty} (\mathrm{Loss}_T(\text{Expert } i) - \mathrm{Loss}_T(\text{Learner})) < \infty$$

(notice that both $\mathrm{Loss}_T(\text{Expert } i)$ and $\mathrm{Loss}_T(\text{Learner})$ implicitly depend on (2)).

In other words, Expert i is not adequate (or is *inadequate*) if her performance is infinitely worse than that of Learner:

$$\mathrm{Loss}_T(\text{Expert } i) - \mathrm{Loss}_T(\text{Learner}) \to \infty \quad (T \to \infty).$$

We will use the expression "good expert" informally, meaning that a good expert will be at least adequate for the actual play. For the interpretation of our results, it is helpful to assume that Experts and Nature are oblivious, in the sense that their moves do not depend on Learner's moves.

In this paper, we will be mainly interested in the *Brier game* $(\Omega, \Gamma, \lambda)$, for which

$$\Omega = \Gamma = [0,1], \quad \lambda(\omega, \gamma) = (\omega - \gamma)^2$$

(since we allow outcomes $0 < \omega < 1$, this definition goes slightly beyond our meteorological interpretation in Section 1). For the purpose of comparison, we will also consider the *log-loss game* $(\Omega, \Gamma, \lambda)$, for which

$$\Omega = \{0,1\}, \quad \Gamma = [0,1], \quad \lambda(\omega, \gamma) = \begin{cases} -\ln \gamma & \text{if } \omega = 1, \\ -\ln(1-\gamma) & \text{otherwise.} \end{cases}$$

3. Strong law of large numbers

In this section and the next one we will consider games with only one expert, $n = 1$, calling Expert 1 just Expert and writing γ_t instead of $\gamma_t^{(1)}$. We say that Learner *can guarantee* a property Π of play (2) if he has a strategy which guarantees that, whatever moves Nature and Expert(s) choose, the resulting play will satisfy Π. (We will also use the phrase "Expert and Nature can guarantee" some property, in the analogous sense.) If Expert is good at predicting Nature's moves ω_t, we expect that she will be right on the average:

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} (\omega_t - \gamma_t) = 0. \tag{3}$$

In other words, we expect that a strong law of large numbers will hold. The following result substantiates this expectation.

Theorem 2. *Learner can guarantee the following implication in the Brier game*:

$$\text{Expert is adequate} \Rightarrow \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} (\omega_t - \gamma_t) = 0. \tag{4}$$

To put it differently, there exists Learner's strategy which guarantees that (3) holds whenever Expert is adequate. Equivalently, Expert whose predictions fail to be correct on the average will be greatly outperformed by Learner.

Remark 3. Actually the statement of the theorem implies the following seemingly stronger assertion: for any $\varepsilon > 0$, Learner has a strategy which guarantees that, for every T,

$$\text{Loss}_T(\text{Learner}) \leqslant \text{Loss}_T(\text{Expert}) + \varepsilon$$

and that, whenever (3) is violated

$$\lim_{T\to\infty} (\text{Loss}_T(\text{Expert}) - \text{Loss}_T(\text{Learner})) = \infty.$$

Similar remarks can be made about most of the theorems in this paper.

Next, we will describe the connection of the usual strong law of large numbers for sequences of events with the log-loss game. The former is closely related to (but slightly weaker than) the following theorem.

Theorem 4. *Learner can guarantee* (4) *in the log-loss game.*

(Notice how this theorem is different from the classical variants of the strong law of large numbers such as Borel's theorem: the classical variants are concerned with almost certain events whereas our result is about the worst-case scenario.)

To deduce Borel's strong law of large numbers from Theorem 4, let P be a probability distribution in $\{0,1\}^\infty$ (with the usual σ-algebra). Consider the following strategy $\mathscr{E}$ for Expert in the log-loss game: after observing a sequence $\omega_1 \ldots \omega_T$ of Nature's moves, Expert predicts with the conditional probability

$$\gamma_{T+1} = \frac{P(\omega_1 \ldots \omega_T 1)}{P(\omega_1 \ldots \omega_T)}$$

(with the uncertainty $\frac{0}{0}$ resolved to, say, $\frac{1}{2}$). Theorem 4 asserts that there exists Learner's strategy $\mathscr{L}$ that guarantees (4). For every finite sequence $\omega_1 \ldots \omega_T$ of Nature's moves put

$$S(\omega_1 \ldots \omega_T) = \exp(\text{Loss}_T(\text{Expert}) - \text{Loss}_T(\text{Learner})),$$

where it is assumed that Nature's moves are $\omega_1 \ldots \omega_T$, Expert plays $\mathscr{E}$, and Learner plays $\mathscr{L}$. Then S is a non-negative martingale with respect to P which tends to infinity outside event (3). Since non-negative martingales are convergent with probability 1 (by Doob's theorem), (3) is P-almost certain (which is the martingale generalization of Borel's strong law).

4. Law of the iterated logarithm

Theorem 2 is an assertion about the convergence of $(1/T)\sum_{t=1}^{T}(\omega_t - \gamma_t)$ to 0. The following theorem gives an estimate of the speed of this convergence.

Theorem 5. *Learner can guarantee the following implication in the Brier game*:

$$\text{Expert is adequate} \Rightarrow \limsup_{T\to\infty} \left| \frac{\sum_{t=1}^{T}(\omega_t - \gamma_t)}{\sqrt{T \ln\ln T}} \right| \leqslant \frac{1}{\sqrt{2}}.$$

The speed of convergence given by this theorem is actually optimal: the constant $\frac{1}{\sqrt{2}}$ cannot be improved.

Theorem 6. *Expert and Nature can guarantee the conjunction*

$$\text{Expert is adequate} \;\&\; \liminf_{T\to\infty} \frac{\sum_{t=1}^{T}(\omega_t - \gamma_t)}{\sqrt{T \ln\ln T}} = -\frac{1}{\sqrt{2}},$$
$$\&\; \limsup_{T\to\infty} \frac{\sum_{t=1}^{T}(\omega_t - \gamma_t)}{\sqrt{T \ln\ln T}} = \frac{1}{\sqrt{2}}$$

in the Brier game.

The proof is given in Section 8; the statement of the theorem, however, is made plausible by the following simple observation: when $\gamma_t = \frac{1}{2}$, for all t, and Nature is "obedient" in that it generates only 0's and 1's independently with equal probabilities, then the usual law of the iterated logarithm shows that

$$\liminf_{T\to\infty} \frac{\sum_{t=1}^{T}(\omega_t - \gamma_t)}{\sqrt{T \ln\ln T}} = -\frac{1}{\sqrt{2}}, \qquad \limsup_{T\to\infty} \frac{\sum_{t=1}^{T}(\omega_t - \gamma_t)}{\sqrt{T \ln\ln T}} = \frac{1}{\sqrt{2}}$$

almost surely.

For the log-loss game Learner can guarantee an equality; more accurately,

Theorem 7. *Learner can guarantee the following implication in the log-loss game*:

$$(\text{Expert is adequate and } D_T \to \infty \quad \text{as } T \to \infty)$$
$$\Rightarrow -\sqrt{2} = \liminf_{T\to\infty} \frac{\sum_{t=1}^{T}(\omega_t - \gamma_t)}{\sqrt{D_T \ln\ln D_T}} \leqslant \limsup_{T\to\infty} \frac{\sum_{t=1}^{T}(\omega_t - \gamma_t)}{\sqrt{D_T \ln\ln D_T}} = \sqrt{2},$$

where $D_T = \sum_{t=1}^{T} \gamma_t(1-\gamma_t)$.

Of course, Theorem 5 ceases to be true if "$\leqslant$" is replaced by "$=$" (it suffices to consider the following strategies for Expert and Nature: both always choose $\frac{1}{2}$).

Again Theorem 7 easily implies the usual martingale law of the iterated logarithm for sequences of events.

5. Theorem of agreement

In this section we consider the Brier game with two experts. If both experts are good at predicting Nature's moves ω_t, it is natural to expect that the agreement property

$$\lim_{t\to\infty} (\gamma_t^{(1)} - \gamma_t^{(2)}) = 0 \tag{5}$$

will hold (see, e.g., [16, 4]). Our first task will be to show that the following elaboration of (5) holds for good experts:

$$\sum_{t=1}^{\infty} \left(\gamma_t^{(1)} - \gamma_t^{(2)}\right)^2 < \infty. \tag{6}$$

Theorem 8. *Learner can guarantee the following implication in the Brier game*:

$$\text{Experts 1 and 2 are adequate} \Rightarrow \sum_{t=1}^{\infty} \left(\gamma_t^{(1)} - \gamma_t^{(2)}\right)^2 < \infty.$$

In other words, if the experts' predictions fail to agree in the sense of (6), at least one of them will be greatly outperformed by Learner.

For the log-loss game the methods of Vovk [17] allow us to prove

Theorem 9. *Learner can guarantee the following implication in the log-loss game*:

$$\text{Experts 1 and 2 are adequate} \Rightarrow \sum_{t=1}^{\infty} \mathscr{H}\left(\gamma_t^{(1)}, \gamma_t^{(2)}\right) < \infty,$$

where $\mathscr{H}$ is the Hellinger distance:

$$\mathscr{H}(p,q) = (\sqrt{p} - \sqrt{q})^2 + (\sqrt{1-p} - \sqrt{1-q})^2.$$

For the log-loss game the conclusion that the sum of the Hellinger distance between good experts' predictions converges is in some sense the strongest possible (see [17]); it turns out (Theorem 23 below) that condition (6) is the strongest possible (in the same sense) in the Brier game.

6. Weak law of large numbers

All results we have stated so far are asymptotic: they assert something about *infinite* outcome sequences. It has been argued, however, that such results are not interesting because we never observe infinite sequences. In this section we state a simple result about finite outcome sequences, considering games that last only $T < \infty$ trials, where T is a positive integer constant. (This is the simplest variant of the weak law of large numbers; a natural generalization would be to replace the fixed number T of trials by a stopping time.) For such finite-horizon games, a *play* is a sequence of players' moves during the trials $1, \ldots, T$:

$$\left(\gamma_1^{(1)}, \ldots, \gamma_1^{(n)}, \gamma_1^{\mathrm{L}}, \omega_1, \ldots, \gamma_T^{(1)}, \ldots, \gamma_T^{(n)}, \gamma_T^{\mathrm{L}}, \omega_T\right). \tag{7}$$

Definition 10. Let C be a positive constant and T be a positive integer constant. Expert i, $i = 1, \ldots, n$, is *C-adequate* for a play (7) of the game $(\Omega, \Gamma, \lambda)$ of duration T if

$$\mathrm{Loss}_T(\text{Expert } i) - \mathrm{Loss}_T(\text{Learner}) \leqslant C.$$

Now we can state a variant of the weak law of large numbers for the Brier game.

Theorem 11. *Let C be a positive constant and T be integer. Learner can guarantee the following implication in the Brier game of duration T with one expert*:

$$\text{Expert is } C\text{-adequate} \Rightarrow \left| \frac{1}{T} \sum_{t=1}^{T} (\omega_t - \gamma_t) \right| \leqslant \sqrt{\frac{C + \ln 2/2}{T}}.$$

For comparison, we also state an elaboration of the usual weak law of large numbers for sequences of events (see, e.g., [1, Theorem 1.1] and its elaboration, Eq. (1.18)).

Theorem 12. *Let C be a positive constant and T be integer. Learner can guarantee the following implication in the log-loss game of duration T with one expert*:

$$\text{Expert is } C\text{-adequate} \Rightarrow \left| \frac{1}{T} \sum_{t=1}^{T} (\omega_t - \gamma_t) \right| \leqslant \frac{1}{2} \sqrt{\frac{e^C}{T}}.$$

7. Kolmogorov-complexity interpretation

In this section we will generalize Levin's modification of Kolmogorov complexity and Martin-Löf's notion of randomness to a wide class of prediction games including the Brier game. Within our framework, Levin's and Martin-Löf's definitions describe complexity and randomness in the log-loss game. For the details of the theory of Kolmogorov complexity, the reader can consult Li and Vitanyi [10].

We will assume that the sets Ω and Γ are equipped with some computability structure that allows one to speak of, say, computable functions on $\Omega \times \Gamma$ (in our examples this somewhat vague assumption will be obviously satisfied). The loss function λ is assumed to be computable.

Let $\mathcal{S}$ be a prediction strategy, i.e., a function that maps every finite sequence $\omega_1 \ldots \omega_T$ of outcomes into a prediction $\mathcal{S}(\omega_1 \ldots \omega_T) \in \Gamma$. In this section, our notation for the total loss incurred over the first T trials by Learner who follows $\mathcal{S}$ will be $\mathrm{Loss}_{\mathcal{S}}(\omega_1 \ldots \omega_T)$, where $\omega_1 \ldots \omega_T$ are the realized outcomes. The function $\mathrm{Loss}_{\mathcal{S}}(\sigma)$ of a finite sequence $\sigma \in \Omega^*$ is called the *loss process of* $\mathcal{S}$; a real-valued function on Ω^* is a *loss process* if it coincides with $\mathrm{Loss}_{\mathcal{S}}$ for some prediction strategy $\mathcal{S}$. Notice that a function $M : \Omega^* \to [0, \infty]$ is a loss process if and only if

$$\forall \sigma \in \Omega^* \, \exists \gamma \in \Gamma \, \forall \omega \in \Omega : M(\sigma * \omega) = M(\sigma) + \lambda(\omega, \gamma),$$

where $\sigma * \omega$ is σ extended by adding one more element ω on the right.

We are interested in the loss processes corresponding to computable prediction strategies; for the usual games (such as the log-loss or Brier game) these are exactly the computable loss processes. It would be ideal for our purpose of defining algorithmic notions of complexity and randomness if the class of computable loss processes contained a smallest (say, to within an additive constant) element. Unfortunately, for the interesting loss functions such a smallest element does not exist: given a computable prediction strategy $\mathscr{S}$, it is easy to construct a computable prediction strategy that greatly outperforms $\mathscr{S}$ on at least one outcome sequence. Levin, developing ideas of Kolmogorov, suggested (for the special case of the log-loss game) the following solution to the problem of non-existence of a smallest computable loss process.

Definition 13. A function $M : \Omega^* \to [0, \infty]$ is a *loss superprocess* if

$$\forall \sigma \in \Omega^* \, \exists \gamma \in \Gamma \, \forall \omega \in \Omega : M(\sigma * \omega) \geqslant M(\sigma) + \lambda(\omega, \gamma). \tag{8}$$

It is *upper semicomputable* if there exists a computable sequence of computable functions $M_i : \Omega^* \to [0, \infty]$ such that, for every $\sigma \in \Omega^*$, $M(\sigma) = \inf_i M_i(\sigma)$.

The notion of a loss superprocess is closely related to the notions of supermartingale and submartingale (especially when the conventional foundations of probability theory are replaced by the game-theoretic foundations; cf. [14, Chapter 12, 5, 19]. We have chosen the prefix "super" because both supermartingales and loss superprocesses describe situations where Learner (or the bettor) can suffer "unfair" extra losses.

Definition 14. The game $(\Omega, \Gamma, \lambda)$ is a *Levin game* if there exists a smallest, to within an additive constant, upper semicomputable loss superprocess M. (In other words, M is an upper semicomputable loss superprocess such that for any other upper semicomputable loss superprocess N there exists a constant C such that $\sup_{\sigma \in \Omega^*}(M(\sigma) - N(\sigma)) < \infty$.) Such M is called a *λ-universal loss superprocess*.

Levin proved the existence of a log-loss universal loss superprocess and used it to give an alternative definition of randomness in the sense of Martin-Löf. (The log-loss universal loss superprocess is the logarithm of Levin's a priori semimeasure.) It is shown in [23] that a wide class of games, including the log-loss game and Brier game, are Levin games. (The idea is to apply the Aggregating Algorithm, see Theorem 24 below, to a universal computable sequence of upper semicomputable loss superprocesses.) We fix a universal loss superprocess $\mathscr{K}^{(\Omega,\Gamma,\lambda)}$ in every Levin game $(\Omega, \Gamma, \lambda)$; for the Brier game we will use the shorthand $\mathscr{K}^{\mathrm{Brier}}$. We will drop the superscript if the game is clear from the context.

The intuition behind the universal loss superprocess $\mathscr{K}^{(\Omega,\Gamma,\lambda)}$ is that

$$\mathscr{K}^{(\Omega,\Gamma,\lambda)}(\omega_1, \ldots, \omega_T) \tag{9}$$

is an intrinsic measure of difficulty of prediction of a sequence $\omega_1, \ldots, \omega_T$: the loss of no computable prediction strategy is much less than (9), but the latter can be attained "in the limit". The universal loss superprocess when applied to $(\omega_1, \ldots, \omega_T) \in \Omega^*$ eventually learns all regularities in $(\omega_1, \ldots, \omega_T)$ relevant to the on-line prediction of ω_1, then $\omega_2, \ldots$, finally ω_T, but this process of learning never ends: the upper semicomputability (but not computability) of $\mathscr{K}^{(\Omega,\Gamma,\lambda)}$ means that there always remains possibility of discovering new regularities in $\omega_1 \dots \omega_T$ which will decrease $\mathscr{K}^{(\Omega,\Gamma,\lambda)}$'s estimate of the loss attainable on $\omega_1 \dots \omega_T$.

Equipped with this generalization of Kolmogorov complexity, we can now define the notion of randomness.

Definition 15. An outcome sequence $\omega_1\omega_2\dots$ is (Ω,Γ,λ)-*random* w.r.t. a computable prediction strategy $\mathscr{S}$ if

$$\sup_T \left(\mathrm{Loss}_{\mathscr{S}}(\omega_1 \dots \omega_T) - \mathscr{K}^{(\Omega,\Gamma,\lambda)}(\omega_1 \dots \omega_T)\right) < \infty. \tag{10}$$

The following proposition (proven in [23]) shows that the strong limit theorems (strong law of large numbers, law of the iterated logarithm, and theorem of agreement; cf. Theorems 19, 20, and 22 below) that we prove in this paper are stronger than the usual limit theorems for sequences of events.

Proposition 16 (Vovk [23]). *If a binary sequence is log-loss random w.r.t. a computable prediction strategy, then it is Brier random w.r.t. that strategy.*

On the other hand, it is easy to see that the log-loss randomness is different from the Brier randomness: e.g., the sequence $(0,0,\dots)$ is Brier random w.r.t. the prediction strategy $\gamma_t = 1/t$, $\forall t$ (since $\sum_t t^{-2}$ is convergent) but not log-loss random (since $\sum_t t^{-1}$ is divergent).

Our definition of randomness is not the only possible one. For example, one could replace $\sup_T$ by $\limsup_{T\to\infty}$ in (10); it is clear, however, that this modified definition will be equivalent to the original definition. The following definition also looks natural.

Definition 17. An outcome sequence $\omega_1\omega_2\dots$ is *weakly* (Ω,Γ,λ)-*random* w.r.t. a computable prediction strategy $\mathscr{S}$ if

$$\liminf_T \left(\mathrm{Loss}_{\mathscr{S}}(\omega_1 \dots \omega_T) - \mathscr{K}^{(\Omega,\Gamma,\lambda)}(\omega_1 \dots \omega_T)\right) < \infty. \tag{11}$$

Again it turns out that this leads to the same notion, which demonstrates that our definition is fairly robust.

Proposition 18 (Vovk [23]). *In the log-loss game and Brier game, weak randomness is equivalent to randomness.*

Therefore, if a sequence $\omega_1\omega_2\dots$ is not random w.r.t. $\mathscr{S}$, then the universal loss superprocess *beats* $\mathscr{S}$ on it, in the sense that

$$\mathrm{Loss}_{\mathscr{S}}(\omega_1 \dots \omega_T) - \mathscr{K}(\omega_1 \dots \omega_T) \to \infty \quad (T \to \infty).$$

Using the algorithmic notion of randomness, we can restate some of our results in a more intuitive way. (Theorem 19 below corresponds to Theorem 2, Theorem 20 corresponds to Theorem 5, Theorem 21 corresponds to Theorem 6, and Theorem 22 corresponds to Theorem 8.)

Theorem 19 (Strong law of large numbers). *If* $\omega_1\omega_2\ldots$ *is Brier random w.r.t. a computable prediction strategy* $\mathscr{S}$, *then property* (3) *holds, where* $\gamma_t = \mathscr{S}(\omega_1\ldots\omega_{t-1})$ *is* $\mathscr{S}$*'s prediction at trial t.*

In other words, a computable prediction strategy whose predictions fail to be correct on the average will be greatly outperformed by the universal loss superprocess.

Theorem 20 (Law of the iterated logarithm). *If a sequence* $\omega_1\omega_2\ldots\in[0,1]^\infty$ *is Brier random w.r.t. a computable prediction strategy* $\mathscr{S}$,

$$\limsup_{T\to\infty}\left|\frac{\sum_{t=1}^{T}(\omega_t-\gamma_t)}{\sqrt{T\ln\ln T}}\right| \leqslant \frac{1}{\sqrt{2}}.$$

Theorem 21. *There exists a computable prediction strategy* $\mathscr{S}$ *and a sequence* $\omega_1\omega_2\ldots\in[0,1]^\infty$ *Brier random w.r.t.* $\mathscr{S}$ *such that*

$$\liminf_{T\to\infty}\frac{\sum_{t=1}^{T}(\omega_t-\gamma_t)}{\sqrt{T\ln\ln T}} = -\frac{1}{\sqrt{2}},\qquad \limsup_{T\to\infty}\frac{\sum_{t=1}^{T}(\omega_t-\gamma_t)}{\sqrt{T\ln\ln T}} = \frac{1}{\sqrt{2}}. \tag{12}$$

Theorem 22 (Theorem of agreement). *If the outcome sequence is Brier random w.r.t. two computable prediction strategies, the predictions output by these strategies satisfy* (6).

The next theorem complements Theorem 22.

Theorem 23 (Criterion of randomness). *Suppose the outcome sequence is Brier random w.r.t. one of two computable prediction strategies. It is random w.r.t. the other if and only if the predictions output by the two strategies satisfy* (6).

(The analogous result for the log-loss game, proven in [17], is closely connected with a theorem of Kabanov, Liptser, and Shiryayev in conventional probability theory.)

8. Some proofs

8.1. Aggregating Algorithm

Our main technical tool is the "Aggregating Algorithm" [18, 20]. Its role in "Brier probability theory" is analogous to the role of Kolmogorov's axiom of σ-additivity in the usual probability theory. (For example, in the proof of Theorem 2 in the next subsection we will represent the complement of event (3) as the union of a countable family of "null" events; the Aggregating Algorithm will allow us to conclude that the whole union is also "null".)

We will not describe the Aggregating Algorithm here (the reader can consult Vovk [20] or Haussler et al. [7] for details) and will only state a theorem (different parts of which were proven in [6, 18, 7]) describing its properties for the two games we are interested in. In this theorem the pool of experts is *infinite* and consists of Expert 1, Expert 2, and so on.

Theorem 24 (DeSantis et al. [6], Vovk [18] and Haussler et al. [7]). *Let $p_1, p_2, \dots$ be a sequence of non-negative numbers summing to* 1 (*the weights of the experts*). *The Aggregating Algorithm* (*with suitable parameters*) *defines Learner's strategy in the Brier* (*resp. log-loss*) *game which guarantees that, whatever Experts' and Nature's moves, the following inequality will hold at every trial T and for every Expert $i = 1, 2, \dots$:*

$$\mathrm{Loss}_T(\mathrm{Learner}) \leqslant \mathrm{Loss}_T(\mathrm{Expert}\, i) + a \ln \frac{1}{p_i}, \tag{13}$$

where $a = \frac{1}{2}$ (resp. $a = 1$). (When $p_i = 0$, $\ln 1/p_i = \infty$, and so (13) *becomes trivially true.*)

(Of course, this theorem can be applied to a finite pool of experts as well: any finite pool of experts can be modeled as an infinite pool in which all but finitely many experts are given zero weights.)

The games for which inequality (13) can be guaranteed for some finite a are called *perfectly mixable* (some interesting games, such as the absolute loss game, also require a coefficient $c > 1$ before $\mathrm{Loss}_T(\mathrm{Expert}\, i)$); it can be shown that perfectly mixable games which satisfy some conditions of computability are Levin games.

8.2. Proof of Theorem 2

An *event* is a property of the sequence of Expert's and Nature's moves $\gamma_1\omega_1\gamma_2\omega_2\dots$. We will say that an event E (such as (3)) is *full* if Learner has a strategy $\mathscr{L}$ that guarantees

$$E \text{ or } \lim_{T\to\infty}(\mathrm{Loss}_T(\mathrm{Expert}) - \mathrm{Loss}_T(\mathscr{L})) = \infty,$$

where $\mathrm{Loss}_T(\mathscr{L})$ is defined in the natural way; the complement of a full event is called *null*; we will say that $\mathscr{L}$ *forces* the event E. Therefore, our goal is to prove that (3) is full. First we will simplify our task. Let us say that an event E is *weakly full* (and its complement is *weakly null*) if Learner has a strategy $\mathscr{L}$ that *weakly forces* E, i.e., guarantees

$$E \text{ or } \sup_T(\mathrm{Loss}_T(\mathrm{Expert}) - \mathrm{Loss}_T(\mathscr{L})) = \infty.$$

Lemma 25. *An event E is weakly full if and only if it is full.*

Proof. Let E be weakly forced by Learner's strategy $\mathscr{L}$. For every $C>0$ let $\mathscr{L}^{(C)}$ be the following strategy for Learner:
- before $\mathrm{Loss}_T(\mathrm{Expert}) - \mathrm{Loss}_T(\mathscr{L})$ reaches C, act in accordance with $\mathscr{L}$;
- after that, repeat Expert's predictions.

Mixing the strategies $\mathscr{L}^{(2^k)}$, $k=1,2,\dots$, with the Aggregating Algorithm with weights 2^{-k}, we obtain a strategy that forces E. □

So we only need to prove that (3) is weakly full. The next lemma will simplify our task even further.

Lemma 26. *If events $E_1, E_2, \dots$ are full (resp. weakly full), then their intersection $\bigcap_{k=1}^{\infty} E_k$ is also full (resp. weakly full).*

Proof. It suffices to apply the Aggregating Algorithm to Learner's strategies (weakly) forcing $E_1, E_2, \dots$. □

The next lemma, despite its triviality, is very important.

Lemma 27. *Let $\mathscr{L}$ be Learner's strategy. The event*

$$\sup_T (\mathrm{Loss}_T(\mathrm{Expert}) - \mathrm{Loss}_T(\mathscr{L})) < \infty \tag{14}$$

is weakly full (and, therefore, full).

Proof. By the law of the excluded middle, the following disjunction always holds:

$$\sup_T (\mathrm{Loss}_T(\mathrm{Expert}) - \mathrm{Loss}_T(\mathscr{L})) < \infty,$$

or

$$\sup_T (\mathrm{Loss}_T(\mathrm{Expert}) - \mathrm{Loss}_T(\mathscr{L})) = \infty. \qquad \square$$

Now we can start proving the theorem. Consider Learner's strategy $\gamma_t^{\mathrm{L}} = \gamma_t + \varepsilon$, where ε is a constant (typically small in absolute value). (We are allowing Learner to make predictions outside $[0,1]$; this freedom does not really help Learner since the outcomes are always in $[0,1]$.) Equivalent transformations of (14) give

$$\exists C: \sum_{t=1}^{T} (\omega_t - \gamma_t)^2 \leqslant \sum_{t=1}^{T} (\omega_t - \gamma_t - \varepsilon)^2 + C,$$

$$\exists C: 2\varepsilon \sum_{t=1}^{T} (\omega_t - \gamma_t) \leqslant T\varepsilon^2 + C.$$

For $\varepsilon > 0$, this is equivalent to

$$\exists C : \frac{1}{T}\sum_{t=1}^{T}(\omega_t - \gamma_t) \leqslant \frac{\varepsilon}{2} + \frac{C}{2\varepsilon T}$$

and for $\varepsilon < 0$, this is equivalent to

$$\exists C : \frac{1}{T}\sum_{t=1}^{T}(\omega_t - \gamma_t) \geqslant \frac{\varepsilon}{2} + \frac{C}{2\varepsilon T}.$$

We can see that for any $\varepsilon > 0$ the event

$$-\varepsilon \leqslant \liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T}(\omega_t - \gamma_t) \leqslant \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T}(\omega_t - \gamma_t) \leqslant \varepsilon \tag{15}$$

is weakly full. Application of Lemmas 25 and 26 concludes the proof of the theorem.

8.3. Proof of Theorem 19

To see why Theorem 19 is true, it suffices to notice that, when Learner plays against a computable strategy for Expert using the strategy described in the previous subsection, which guarantees (4), the corresponding loss process is computable and so is bounded below by $\mathscr{K}^{\text{Brier}}$ to within an additive constant. (Strictly speaking, not all details of Learner's strategy are described in the previous subsection, but they are easy to fill out.)

In the same way, proofs of Theorems 20 and 22 follow from the proofs of Theorems 5 and 8 and we do not spell them out.

8.4. Proof of Remark 3

It suffices to merge (using the Aggregating Algorithm) Learner's strategy that always replicates Expert's move with Learner's strategy guaranteeing (4), taking the former with weight close to 1.

8.5. Proof of Theorem 4

We have stated quite a few theorems related to the log-loss game, but we will only prove one of them: the log-loss game is very close to the traditional probability theory and the proofs of limit theorems for this game can usually be deduced from the well-known proofs of the theory of martingales (cf. [19], Chapter 13 of [14], and [5]).

We give a simple proof of Theorem 4 (this proof can be elaborated, along the lines of the proof of Theorem 5 in the next subsection, to obtain a proof of Theorem 7). With every prediction $\gamma \in [0,1]$ we associate the probability measure P_γ in $\{0,1\}$ which assigns the values

$$P_\gamma(0) = 1 - \gamma, \quad P_\gamma(1) = \gamma$$

to the points 0 and 1, respectively. Let $\varepsilon > 0$. Consider Learner's strategy which, at every trial t, outputs the prediction associated with the probability distribution that assigns the weight

$$\frac{P_{\gamma_t}(\omega)e^{\varepsilon(\omega-\gamma_t)}}{P_{\gamma_t}(0)e^{\varepsilon(0-\gamma_t)} + P_{\gamma_t}(1)e^{\varepsilon(1-\gamma_t)}}$$

to $\omega \in \{0,1\}$ (the denominator is just the normalizing constant). Expert's loss over the first T trials is

$$-\sum_{t=1}^{T} \ln P_{\gamma_t}(\omega_t)$$

(for simplicity we assume in this proof that $P_{\gamma_t}(\omega_t)$ is always positive) and Learner's loss is

$$-\sum_{t=1}^{T} \ln \frac{P_{\gamma_t}(\omega_t)e^{\varepsilon(\omega_t-\gamma_t)}}{P_{\gamma_t}(0)e^{\varepsilon(-\gamma_t)} + P_{\gamma_t}(1)e^{\varepsilon(1-\gamma_t)}},$$

so Lemma 27 implies that the event

$$\exists C\, \forall T : -\sum_{t=1}^{T} \ln P_{\gamma_t}(\omega_t) \leqslant -\sum_{t=1}^{T} \ln \frac{P_{\gamma_t}(\omega_t)e^{\varepsilon(\omega_t-\gamma_t)}}{P_{\gamma_t}(0)e^{\varepsilon(-\gamma_t)} + P_{\gamma_t}(1)e^{\varepsilon(1-\gamma_t)}} + C$$

is full. The last inequality can be transformed as follows:

$$\varepsilon \sum_{t=1}^{T} (\omega_t - \gamma_t) \leqslant \sum_{t=1}^{T} \ln (P_{\gamma_t}(0)e^{\varepsilon(-\gamma_t)} + P_{\gamma_t}(1)e^{\varepsilon(1-\gamma_t)}) + C,$$

$$\varepsilon \sum_{t=1}^{T} (\omega_t - \gamma_t) \leqslant \sum_{t=1}^{T} \ln (P_{\gamma_t}(0)(1 + \varepsilon(-\gamma_t) + \varepsilon^2\gamma_t^2) + P_{\gamma_t}(1)(1 + \varepsilon(1-\gamma_t) + \varepsilon^2(1-\gamma_t)^2)) + C$$

(we have assumed that $\varepsilon \leqslant 1$ and used the inequality $e^t \leqslant 1 + t + t^2$, which is true at least for $|t| \leqslant 1$);

$$\varepsilon \sum_{t=1}^{T} (\omega_t - \gamma_t) \leqslant \sum_{t=1}^{T} \ln (1 + \varepsilon^2) + C,$$

$$\varepsilon \sum_{t=1}^{T} (\omega_t - \gamma_t) \leqslant \sum_{t=1}^{T} \varepsilon^2 + C$$

(we have used the inequality $\ln(1+t) \leqslant t$);

$$\frac{1}{T} \sum_{t=1}^{T} (\omega_t - \gamma_t) \leqslant \varepsilon + \frac{C}{\varepsilon T}.$$

This and the inequality obtained in the same way with ε replaced by $-\varepsilon$ imply that the event (15) is full (in the log-loss game). It remains to apply Lemma 26.

8.6. *Proof of Theorem 5*

First we will prove an elaboration of Lemma 27.

Lemma 28. *Let $\mathscr{L}_i$, $i=1,2,\ldots$, be Learner's strategies in the one-expert Brier game and p_i, $i=1,2,\ldots$, be their* weights: $p_i>0$, $\forall i$, $\sum_i p_i=1$. *The event*

$$\sup_{i,T}\left(\mathrm{Loss}_T(\mathrm{Expert})-\mathrm{Loss}_T(\mathscr{L}_i)-\frac{1}{2}\ln\frac{1}{p_i}\right)<\infty$$

is full.

Proof. Let us mix $\mathscr{L}_1,\mathscr{L}_2,\ldots$ into one strategy $\mathscr{L}$ with the Aggregating Algorithm. By Lemma 27, the event

$$\sup_T(\mathrm{Loss}_T(\mathrm{Expert})-\mathrm{Loss}_T(\mathscr{L}))<\infty$$

is full. The statement of the lemma now follows from

$$\mathrm{Loss}_T(\mathscr{L})\leqslant\mathrm{Loss}_T(\mathscr{L}_i)+\frac{1}{2}\ln\frac{1}{p_i}$$

(see Theorem 24). □

Now we prove Theorem 5 by elaborating the proof of Theorem 2. First, we notice that it is enough to prove Theorem 5 with the absolute value sign dropped. Indeed, suppose we have a strategy that guarantees the implication

$$\text{Expert is adequate}\Rightarrow\limsup_{T\to\infty}\frac{\sum_{t=1}^T(\omega_t-\gamma_t)}{\sqrt{T\ln\ln T}}\leqslant\frac{1}{\sqrt{2}}. \tag{16}$$

Feeding this strategy with the values $1-\gamma_t$ in place of γ_t and $1-\omega_t$ in place of ω_t, we obtain a strategy that guarantees the implication

$$\text{Expert is adequate}\Rightarrow\limsup_{T\to\infty}\frac{\sum_{t=1}^T(\gamma_t-\omega_t)}{\sqrt{T\ln\ln T}}\leqslant\frac{1}{\sqrt{2}};$$

combining the last two inequalities, we obtain the formula of Theorem 5.

To prove that Learner can guarantee (16), we let $\mathscr{L}_i$ predict with $\gamma_t+\varepsilon_i$, where ε_i are positive constants to be chosen later. By Lemma 28 the following event is full:

$$\exists C\,\forall i,\ T:\sum_{t=1}^T(\omega_t-\gamma_t)^2\leqslant\sum_{t=1}^T(\omega_t-\gamma_t-\varepsilon_i)^2+\frac{1}{2}\ln\frac{1}{p_i}+C,$$

$$\exists C\,\forall i,\ T:\sum_{t=1}^T(\omega_t-\gamma_t)\leqslant\frac{T\varepsilon_i}{2}+\frac{1}{4\varepsilon_i}\ln\frac{1}{p_i}+\frac{C}{2\varepsilon_i}. \tag{17}$$

Let $\delta > 0$ be a small constant. Taking

$$i = \lfloor \log_{1+\delta}(T+1) \rfloor$$

(intuitively, i is an economic encoding of the approximation $(1+\delta)^i$ to T),

$$p_i = e^{-c} i^{-1-\delta}$$

(we take as large weights as possible; e^{-c} is a normalizing constant), and

$$\varepsilon_i = \sqrt{\frac{\ln i}{2(1+\delta)^i}}$$

(this is close to the value of ε that minimizes the approximation $T\varepsilon/2 + \ln\ln T/4\varepsilon$ to the right-hand side of (17)), we rewrite the inequality in (17) as

$$\sum_{t=1}^{T} (\omega_t - \gamma_t) \leqslant \frac{T}{2} \sqrt{\frac{\ln i}{2(1+\delta)^i}} + \frac{1}{4} \sqrt{\frac{2(1+\delta)^i}{\ln i}} \ln (i^{1+\delta}) + \frac{C}{2} \sqrt{\frac{2(1+\delta)^i}{\ln i}} + c.$$

Since, as $T \to \infty$, $\ln i$ grows as $\ln\ln T$ and $(1+\delta)^i$ grows as T, the right-hand side of the last inequality asymptotically does not exceed

$$2^{-1.5} T(1+\delta) \sqrt{\frac{\ln\ln T}{T}} + 2^{-1.5} \sqrt{\frac{T}{\ln\ln T}} (1+\delta) \ln\ln T = \frac{1+\delta}{\sqrt{2}} \sqrt{T \ln\ln T}.$$

Lemma 26 ensures that we can take $\delta \to 0$, which proves that Learner can guarantee (16).

8.7. Strategies for Learner's adversaries

We will need the following martingale-theoretic result (I learned it from Philip Dawid).

Lemma 29. *Let ξ_T be a supermartingale with uniformly bounded increments. Then*

$$\limsup_{T\to\infty} \xi_T = \infty \Rightarrow \liminf_{T\to\infty} \xi_T = -\infty$$

almost surely. (*In particular*, $\lim_{T\to\infty} \xi_T = \infty$ *with probability* 0.)

Proof. Without loss of generality, assume $\xi_0 = 0$ (if necessary, replace all ξ_T by $\xi_T - \xi_0$). We are required to prove that the event

$$\limsup_{T\to\infty} \xi_T = \infty, \qquad \liminf_{T\to\infty} \xi_T > -\infty$$

is null. It is sufficient to prove, for every constant C, that

$$\limsup_{T\to\infty} \xi_T = \infty, \qquad \forall T : \xi_T > -C \tag{18}$$

is null. Fix C. Let ξ^* be the "stopped" supermartingale $\xi^*_T = \xi_{\min(T,\tau)}$, where $\tau = \min\{t \mid \xi_t \leqslant -C\}$. We are required to prove that the event

$$\limsup_{T\to\infty} \xi^*_T = \infty, \qquad \forall T : \xi^*_T > -C$$

(which is just a different expression for (18)) is null. On this event the positive supermartingale $\xi^*_T + C + C^*$, where C^* is an upper bound on $\sup_{T\geqslant 1} |\xi_T - \xi_{T-1}|$, tends to infinity, so it is indeed null. □

This lemma enables us to easily prove Theorem 6.

Proof of Theorem 6. Let $\mathscr{L}$ be Learner's strategy. Consider Expert and Nature's randomized strategy under which Expert always predicts $\frac{1}{2}$ and Nature chooses 0 or 1 independently with equal probabilities. Then, by the usual law of the iterated logarithm,

$$\liminf_{T\to\infty} \frac{\sum_{t=1}^T (\omega_t - \gamma_t)}{\sqrt{T \ln\ln T}} = -\frac{1}{\sqrt{2}}, \qquad \limsup_{T\to\infty} \frac{\sum_{t=1}^T (\omega_t - \gamma_t)}{\sqrt{T \ln\ln T}} = \frac{1}{\sqrt{2}} \tag{19}$$

with probability 1 and, by Lemma 29,

$$\lim_{T\to\infty} (\mathrm{Loss}_T(\mathrm{Expert}) - \mathrm{Loss}_T(\mathscr{L})) = \infty$$

with probability 0. From Martin's [13] theorem about the determinacy of quasi-Borel games we can deduce that Expert and Nature have a deterministic strategy that guarantees the conjunction of

$$\liminf_{T\to\infty} (\mathrm{Loss}_T(\mathrm{Expert}) - \mathrm{Loss}_T(\mathscr{L})) < \infty$$

and (19). □

Proof of Theorem 21 is based on the same idea.

Proof of Theorem 21. Let, for all t, $\gamma_t = \frac{1}{2}$. It is well known that, for all log-loss random $\omega_1, \omega_2, \ldots$, (12) holds. It remains to apply Proposition 16. □

8.8. Proof of Theorems 8 and 23

Proof of Theorem 8. Let Learner's predictions be

$$\gamma_t^{\mathrm{L}} = \frac{\gamma_t^{(1)} + \gamma_t^{(2)}}{2}.$$

Using the notation

$$r_t = \gamma_t^{\mathrm{L}} - \omega_t, \quad \varepsilon_t = \frac{\gamma_t^{(1)} - \gamma_t^{(2)}}{2},$$

we can write the cumulative (over the first T trials) mean loss of the two experts minus the cumulative loss of Learner as

$$\sum_{t=1}^{T}\left(\frac{1}{2}((r_t+\varepsilon_t)^2+(r_t-\varepsilon_t)^2)-r_t^2\right)=\sum_{t=1}^{T}\varepsilon_t^2.$$

So, when $\sum_{t=1}^{\infty}\varepsilon_t^2=\infty$, Learner outperforms at least one expert in the weak sense of

$$\sup_T(\mathrm{Loss}_T(\mathrm{Expert})-\mathrm{Loss}_T(\mathrm{Learner}))=\infty;$$

an argument similar to that of Lemma 25 shows that Learner has a strategy that outperforms at least one expert in the strong sense of

$$\lim_{T\to\infty}(\mathrm{Loss}_T(\mathrm{Expert})-\mathrm{Loss}_T(\mathrm{Learner}))=\infty. \quad \square$$

Proof of Theorem 23. We only need to prove (cf. Theorem 22) that if the two strategies agree in the sense of (6), then randomness of the outcome sequence w.r.t. one of them implies the randomness w.r.t. the other. If these two prediction strategies output predictions $\gamma_t^{(1)}$ and δ_t, we consider a third prediction strategy that output predictions $\gamma_t^{(2)}$ satisfying

$$\delta_t=\frac{\gamma_t^{(1)}+\gamma_t^{(2)}}{2}$$

(we allow $\gamma_t^{(2)}$ to take values outside $[0,1]$; it is easy to see that we can do so without invalidating the proof). We are required to prove that if $\gamma_t^{(1)}$ and δ_t (equivalently, $\gamma_t^{(1)}$ and $\gamma_t^{(2)}$) agree in the sense of (6) and the universal loss superprocess beats $\gamma_t^{(1)}$, then the universal loss superprocess beats δ_t. This is an immediate consequence of the following observation: the universal loss superprocess beats $\gamma_t^{(1)}$ and performs at least as well as $\gamma_t^{(2)}$ (to within an additive constant), and so the agreement between $\gamma_t^{(1)}$ and $\gamma_t^{(2)}$ ensures that the universal loss superprocess beats δ_t as well (see the proof of Theorem 8). $\square$

8.9. Strong law of large numbers is violated for the absolute-loss game

In this section we will prove that the strong law of large numbers does not hold for the absolute-loss game, for which $\Omega=\{0,1\}$, $\Gamma=[0,1]$, and $\lambda(\omega,\gamma)=|\omega-\gamma|$ (cf. Section 9 below). Let Nature output 0 or 1 independently with equal probabilities and Expert always predict 1. Then property (3) is violated with probability 1 and the difference $\mathrm{Loss}_T(\mathrm{Expert})-\mathrm{Loss}_T(\mathscr{L})$ (where $\mathscr{L}$ is Learner's strategy) is a martingale and so cannot tend to ∞ with probability 1 (see Lemma 29).

8.10. Proof of Theorem 11

An event E is *C-full* if Learner has a strategy $\mathscr{L}$ that *C-forces* E, in the sense that it guarantees

$$E \text{ or } \mathrm{Loss}_T(\mathrm{Expert})-\mathrm{Loss}_T(\mathrm{Learner})>C.$$

We are required to prove that the event

$$\left|\frac{1}{T}\sum_{t=1}^{T}(\omega_t-\gamma_t)\right| \leqslant \sqrt{\frac{C+\ln 2/2}{T}} \tag{20}$$

is C-full.

Consider the following two strategies for the Learner: at trial t, $\mathscr{L}^+$ recommends the move $\gamma_t^{\mathrm{L}}=\gamma_t+\varepsilon$ and $\mathscr{L}^-$ recommends the move $\gamma_t^{\mathrm{L}}=\gamma_t-\varepsilon$; ε is a positive constant that will be chosen later. We ascribe the same weight $\frac{1}{2}$ to these two strategies and mix them with the Aggregating Algorithm obtaining Learner's strategy $\mathscr{L}$. The event $\mathrm{Loss}_T(\mathrm{Expert})-\mathrm{Loss}_T(\mathscr{L})\leqslant C$ is C-full (cf. Lemma 27); therefore, by Theorem 24, the intersection of the events

$$\mathrm{Loss}_T(\mathrm{Expert})-\mathrm{Loss}_T(\mathscr{L}^+)\leqslant C+\frac{\ln 2}{2} \tag{21}$$

and

$$\mathrm{Loss}_T(\mathrm{Expert})-\mathrm{Loss}_T(\mathscr{L}^-)\leqslant C+\frac{\ln 2}{2} \tag{22}$$

is also C-full. The calculations of Section 8.2 show that (21) implies

$$\frac{1}{T}\sum_{t=1}^{T}(\omega_t-\gamma_t)\leqslant\frac{\varepsilon}{2}+\frac{C+\ln 2/2}{2\varepsilon T}, \tag{23}$$

setting ε to $\sqrt{(C+\ln 2/2)/T}$ (so as to minimize the right-hand side of (23)), we obtain

$$\frac{1}{T}\sum_{t=1}^{T}(\omega_t-\gamma_t)\leqslant\sqrt{\frac{C+\ln 2/2}{T}}. \tag{24}$$

Analogously, (22) implies

$$\frac{1}{T}\sum_{t=1}^{T}(\omega_t-\gamma_t)\geqslant-\sqrt{\frac{C+\ln 2/2}{T}}. \tag{25}$$

Therefore, the intersection of (24) and (25) is C-full. It remains to compare (24) and (25) with (20).

9. Further research

It is plausible that at least some of our results can be proven for other specific perfectly mixable games; two examples of such games are Cover's universal portfolio game [3] and the Kullback–Leibler game [7] (these two games, like all other popular perfectly mixable games, are Levin games). There is also little doubt that some results (such as the strong law of large numbers or the theorem of agreement) can be proven for wide classes of games, and an interesting problem is to find, for each of these results, the weakest possible conditions on a game under which that result still continues to hold.

Even for the games which are not perfectly mixable (such as the absolute loss game), it might be possible to prove some interesting limit theorems. The results for such games, however, must be very different from the usual set including the law of large numbers, the law of the iterated logarithm, and the theorem of agreement (see Section 8.9 above).

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