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Using Symbolic Computation to Exactly Solve for the Bogoyavlenskii's Generalized Breaking Soliton Equation

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Abstract—In this paper, the symbolic computation systems help us find some exact solutions for the Bogoyavlenskii's generalized breaking soliton equation, which are the sums of the arbitrary functions of time and parameterized solitary waves.

Keywords—Breaking soliton equations, Symbolic computation, Computational technique, Mathematical methods in physics, Exact solutions.

The symbolic-computation systems could be useful tools in the investigation of nonlinear evolution equations. A class of those, called the breaking soliton equations, have become an interesting subject within half a decade, as seen, e.g., in [1–6].

In this paper, we consider a symbolic-computation-based method [7,8] in the solution study of the Bogoyavlenskii's generalized breaking soliton equation [1,2], i.e.,

$$(u_{xt} - 4u_x u_{xy} - 2u_y u_{xx} + u_{xxxy})_x = -\alpha^2 u_{yyy}, \tag{1}$$

where the Bogoyavlenskii parameter α^2 is real. What is in the parentheses is in fact a typical $(2+1)$-dimensional breaking soliton equation [2,3,5,6,9,10].

Let us begin with a transformation

$$u(x,y,t) = \partial_x^m \partial_y^n \partial_t^k\, w[z(x,y,t)] + F(t), \tag{2}$$

where $F(t)$ and $z(x,y,t)$ are a couple of functions. The integers, k, m, and n, are determined via the leading-order analysis as $m = 1$ and $k = n = 0$. In the following analysis, when substituting Ansatz (2) into equation (1), we will choose an ordinary differential equation for the third function, $w(z)$, so as to impose conditions upon $z(x,y,t)$. The substitution yields

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

$$
\begin{aligned}
&\alpha^2 w^{(4)} z_y^3 z_x + 3\alpha^2 w''' z_y z_{yy} z_x + \alpha^2 w'' z_{yyy} z_x + w^{(4)} z_t z_x^3 - 6\left(w'''\right)^2 z_y z_x^5 - 6w'' w^{(4)} z_y z_x^5 \\
&+ w^{(6)} z_x^5 z_y + 3w''' z_x^2 z_{xt} + 3\alpha^2 w''' z_y^2 z_{xy} + 3\alpha^2 w'' z_{yy} z_{xy} - 24 w'' w''' z_x^4 z_{xy} - 2w' w^{(4)} z_x^4 z_{xy} \\
&+ 5w^{(5)} z_x^4 z_{xy} + 3\alpha^2 w'' z_y z_{xyy} + \alpha^2 w' z_{xyyy} + 3w''' z_t z_x z_{xx} - 48 w'' w''' z_x^3 z_y z_{xx} - 4w' w^{(4)} z_x^3 z_y z_{xx} \\
&+ 10 w^{(5)} z_y z_x^3 z_{xx} + 3w'' z_{xt} z_{xx} - 48\left(w''\right)^2 z_x^2 z_{xy} z_{xx} - 24 w' w''' z_x^2 z_{xy} z_{xx} + 30 w^{(4)} z_x^2 z_{xy} z_{xx} \\
&- 24\left(w''\right)^2 z_y z_x z_{xx}^2 - 12 w' w''' z_y z_x z_{xx}^2 + 15 w^{(4)} z_y z_x z_{xx}^2 - 18 w' w'' z_{xy} z_{xx}^2 + 15 w''' z_{xy} z_{xx}^2 \\
&+ 3w'' z_x z_{xxt} - 12\left(w''\right)^2 z_x^3 z_{xxy} - 6w' w''' z_x^3 z_{xxy} + 10 w^{(4)} z_x^3 z_{xxy} - 30 w' w'' z_x z_{xx} z_{xxy} \\
&+ 30 w''' z_x z_{xx} z_{xxy} + w'' z_t z_{xxx} - 12\left(w''\right)^2 z_y z_x^2 z_{xxx} - 6 w' w''' z_y z_x^2 z_{xxx} + 10 w^{(4)} z_x^2 z_y z_{xxx} \\
&- 20 w' w'' z_x z_{xy} z_{xxx} + 20 w''' z_x z_{xy} z_{xxx} - 10 w' w'' z_y z_{xx} z_{xxx} + 10 w''' z_y z_{xx} z_{xxx} - 6\left(w'\right)^2 z_{xxy} z_{xxx} \\
&+ 10 w'' z_{xxy} z_{xxx} + w' z_{xxxt} - 4 w' w'' z_x^2 z_{xxxy} + 10 w''' z_x^2 z_{xxxy} - 4\left(w'\right)^2 z_{xx} z_{xxxy} + 10 w'' z_{xx} z_{xxxy} \\
&- 2 w' w'' z_x z_y z_{xxxx} + 5 w''' z_x z_y z_{xxxx} - 2\left(w'\right)^2 z_{xy} z_{xxxx} + 5 w'' z_{xy} z_{xxxx} + 5 w'' z_x z_{xxxxy} \\
&+ w'' z_y z_{xxxxx} + w' z_{xxxxxy} = 0,
\end{aligned} \tag{3}
$$

where $w' = \frac{dw}{dz}$. The leading-order analysis requires that the terms with the highest power of the differential coefficients of $z(x,y,t)$, or $z_x^5 z_y$, vanish, i.e.,

$$-6\left(w'''\right)^2 - 6w'' w^{(4)} + w^{(6)} = 0, \tag{4}$$

which in turn has a special form of solution

$$w[(z(x,y,t)] = -2 \cdot \ln[z(x,y,t)], \tag{5}$$

as the expression for $w(z)$, since every term in equation (4) has the same sum of the degrees of the derivatives, i.e., 6.

For $z(x,y,t)$, we assume that the w' terms in the remainder of equation (3) vanish, i.e.,

$$w'\left(\alpha^2 z_{xyyy} + z_{xxxt} + z_{xxxxxy}\right) = 0, \tag{6}$$

a particular solution of which is found as

$$z\left(x,y,t;\alpha^2\right) = 1 + \exp\left(A\cdot x + B\cdot y - \frac{B^3\alpha^2 + A^4 B}{A^2}\cdot t + C\right), \tag{7}$$

where $A \neq 0$, B, and C are arbitrary constants. Therefore, we construct a trial solution

$$
\begin{aligned}
u\left(x,y,t;\alpha^2\right) &= w' z_x \\
&= F(t) - A\left\{\tanh\left[\frac{1}{2}\left(A\cdot x + B\cdot y - \frac{B^3\alpha^2 + A^4 B}{A^2}\cdot t + C\right)\right] + 1\right\} \\
&= D(t) - A\cdot\tanh\left[\frac{1}{2}\left(A\cdot x + B\cdot y - \frac{B^3\alpha^2 + A^4 B}{A^2}\cdot t + C\right)\right].
\end{aligned} \tag{8}
$$

Using *Mathematica*, we are able to verify that expression (8) is indeed a class of solutions for equation (1). The class turns out to be the sums of the arbitrary functions of time, $D(t) = F(t) - A$ and the Bogoyavlenskii-parameterized (α^2) solitary waves.

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