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Tail Estimation of the Stable Index α

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Abstract—A refined tail-estimation procedure for measuring the index of stability of stable Paretian or α-stable distributions is proposed. The estimator is more suitable for α-stable laws than the widely used estimator proposed in [1].

Keywords—Stable distributions, Tail estimation, Hill estimator, Pickands estimator, Existence of moments.

1. INTRODUCTION

A number of recent studies have used the Hill estimator (see [1]) to measure the tail thickness of financial data and inferred from its estimates the maximum moments of the data (see, for example, [2–5]). Assuming the right tail of a distribution is asymptotically Pareto, i.e., for large x, $1 - F(x) \approx cx^{-\alpha_P}$ ($\alpha_P > 0$, $c > 0$), the Hill estimator attempts to measure tail thickness α_P. Given a sample of n observations, $X_1, X_2, \dots, X_n$, the Hill estimator is given by

$$\hat{\alpha}_H = \frac{1}{(1/k)\sum_{j=1}^{k} \ln X_{n+1-j:n} - \ln X_{n-k:n}}, \tag{1}$$

where $X_{m:n}$ denotes the m^{th} order statistic of sample $X_1, \dots, X_n$ (see [6,7]). The appropriate choice of order k is a nontrivial problem. Index k has to be small enough so that $X_{n-k:n}$ still belongs to the tail of the distribution; but if it is too small, the estimator will lack precision.

If a sample comes from a distribution in the domain of attraction of a Pareto distribution, all moments of order less than α_P exist. Hence, if the Hill estimator produces an estimate in excess of 2, one would rule out that the data come from a distribution with infinite variance. In particular, the stable Paretian or α-stable distribution, which is frequently used in financial modeling (see [8] for a survey), would be among those distributions to be excluded. The standard symmetric α-stable distribution with *stable index* $\alpha \in (0, 2]$ has characteristic function

$$\phi_\alpha(\theta) = e^{-|\theta|^\alpha}, \qquad \theta \in \mathbf{R}. \tag{2}$$

If $\alpha = 2$, the α-stable distribution corresponds to the normal distribution and, thus, has moments of infinite order. For $0 < \alpha < 2$, it is more fat-tailed than the normal distribution and only moments below order α exist.

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In empirical studies, such as those cited above, the Hill estimator typically produced estimates ranging from about 2.5 to 4, causing the authors to reject the α-stable hypothesis. In fact, both the infinite variance case $0 < \alpha < 2$ and, because $\hat{\alpha}_H < \infty$, the normal distribution (i.e., $\alpha = 2$) or, for that matter, any other distribution for which all moments exist would have to be ruled out as possible candidates. As was argued in [9], the problem with using Hill's tail-slope estimates for drawing inference about the existence of moments is that the estimator assumes that the underlying distribution has Pareto-like tails. If this assumption does not hold, such inference cannot be made.

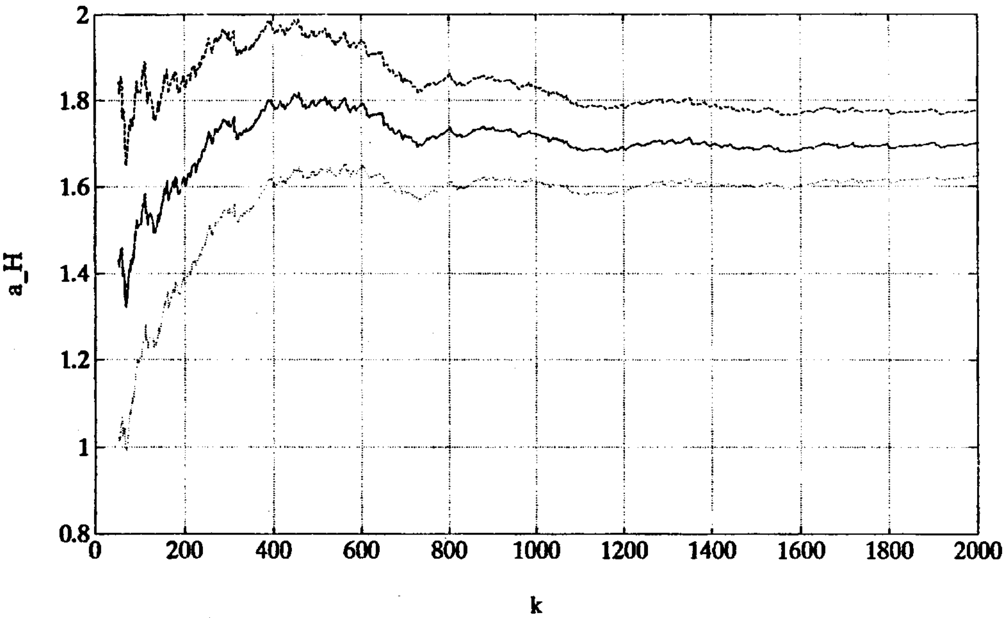

Figure 1. Hill estimates for Pareto sample.

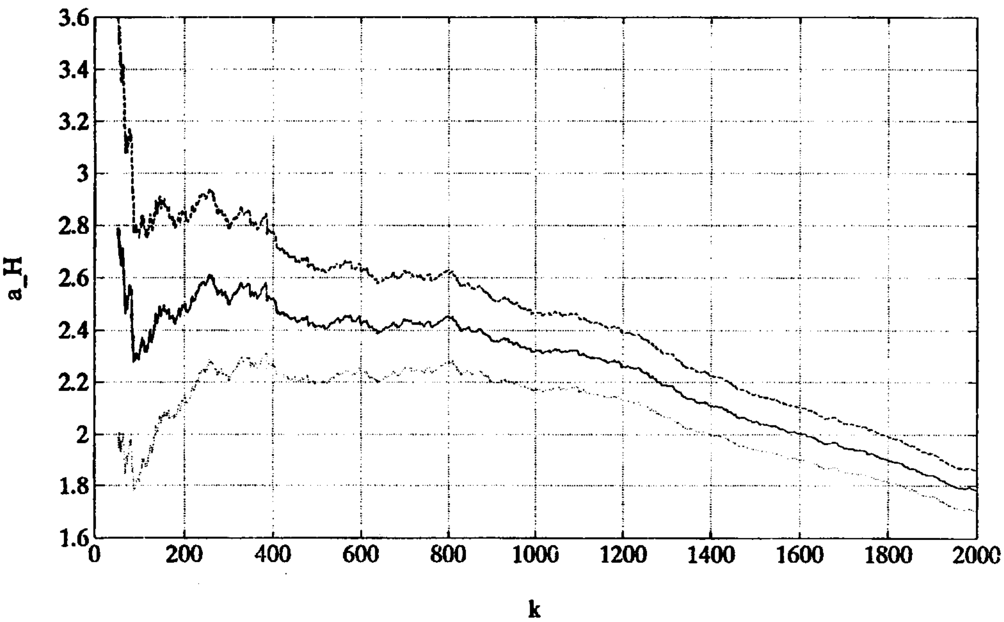

Figure 2. Hill estimates for alpha-stable sample.

To illustrate this, we generated two pseudo-random samples of size 10,000 each. The first sample was drawn from a Pareto distribution with $\alpha_P = 1.7$ and the second from a symmetric α-stable distribution with $\alpha = 1.7$. (In each case data were generated only for the positive halfline.) Thus, for both distributions all moments of order less than 1.7 exist. Figures 1 and 2 show the Hill estimates and approximate 95% confidence bounds for the Pareto and α-stable data, respectively, for $k = 50, \dots, 2000$. Clearly, the Hill estimator works well for the Pareto sample with $\hat{\alpha}_H$ stabilizing around 1.7 when $k > 500$. For the α-stable sample the estimator performs rather poorly. It does not stabilize as k grows, but falls almost linearly with k. The underlying true $\alpha = 1.7$ cannot be inferred from the estimates. In fact, most estimates exceed two, suggesting that the data do not come from an infinite-variance distribution.

In this paper, we propose a refined tail-estimation procedure designed to measure stability index α of α-stable distributions.

2. TAIL ESTIMATION FOR α-STABLE LAWS

From [10] (see also [11]) characteristic function (2) gives rise to the following asymptotic expansion of the standard symmetric α-stable distribution function, denoted by $S_\alpha(x)$,

$$S_\alpha(x) = 1 + \frac{1}{\pi} \sum_{m=1}^{\infty} (-1)^m \frac{\Gamma(\alpha m)}{m!} x^{-\alpha m} \sin \frac{\alpha m \pi}{2}, \qquad \text{as } x \to \infty. \tag{3}$$

Given sample $X_1, \dots, X_n$ this limiting expansion enables us to derive a tail estimator for stable index α, denoted by $\hat{\alpha}_S$, by solving equation

$$\frac{j-1}{n} = \frac{1}{\pi} \sum_{m=1}^{\infty} (-1)^m \frac{\Gamma(\hat{\alpha}_S m)}{m!} X_{n-j+1:n}^{-\hat{\alpha}_S m} \sin \frac{\hat{\alpha}_S m \pi}{2} \tag{4}$$

for $\hat{\alpha}_S$. To apply (4), the infinite sum has to be truncated. Truncation at $m = 1$ leads to Pickands' tail estimator (see [12])

$$\hat{\alpha}_{S,1} = \frac{\ln 2}{\ln X_{n-k+1:n} - \ln X_{n-2k+1:n}}, \tag{5}$$

which is discussed, for example, in [9,13,14]. In fact, for $j = k$ and $j = 2k$ (4) leads to

$$\begin{aligned} \tfrac{k-1}{n} &\approx c_\alpha X_{n-k+1:n}^{-\hat{\alpha}_{S,1}}, \\ \tfrac{2k-1}{n} &\approx c_\alpha X_{n-2k+1:n}^{-\hat{\alpha}_{S,1}}, \end{aligned} \qquad k \to \infty, \quad \frac{k}{n} \to \infty,$$

which clearly implies (5).

Truncating (4) at $m = 2$ leads to the following estimation problem. Find $\hat{\alpha}_{S,2}$, $\hat{\beta}$, $\hat{c}_1$, $\hat{c}_2$ as the solution to equation system

$$\frac{j-1}{n} = \hat{c}_1 X_{n-j+1:n}^{-\hat{\alpha}_{S,2}} + \hat{c}_2 X_{n-j+1:n}^{-\hat{\beta}}, \qquad j = k, 2k, 3k, 5k. \tag{6}$$

Equation system (6) consists of four equations with four unknowns. Because c_1 and c_2 enter linearly, the system can be reduced to a system of two equations with only $\hat{\alpha}_{S,2}$ and $\hat{\beta}$ being unknown. Defining

$$\mathbf{X}_{\alpha,\beta}^{(1)} = \begin{bmatrix} X_{n-k+1:n}^{-\hat{\alpha}_{S,2}} & X_{n-k+1:n}^{-\hat{\beta}} \\ X_{n-2k+1:n}^{-\hat{\alpha}_{S,2}} & X_{n-2k+1:n}^{-\hat{\beta}} \end{bmatrix}, \qquad \mathbf{X}_{\alpha,\beta}^{(2)} = \begin{bmatrix} X_{n-3k+1:n}^{-\hat{\alpha}_{S,2}} & X_{n-3k+1:n}^{-\hat{\beta}} \\ X_{n-5k+1:n}^{-\hat{\alpha}_{S,2}} & X_{n-5k+1:n}^{-\hat{\beta}} \end{bmatrix}$$

and

$$\kappa_1 = \begin{bmatrix} k-1 \\ 2k-1 \end{bmatrix}, \qquad \kappa_2 = \begin{bmatrix} 3k-1 \\ 5k-1 \end{bmatrix},$$

to solve the first two equations for c_1 and c_2, and substituting into the last two equations of (6) estimates of $\hat{\alpha}_{S,2}$ and $\hat{\beta}$ are obtained by solving

$$\kappa_2 = \mathbf{X}_{\alpha,\beta}^{(2)} \left(\mathbf{X}_{\alpha,\beta}^{(1)} \right)^{-1} \kappa_1, \tag{7}$$

for $\hat{\alpha}_{S,2}$ and $\hat{\beta}$.

The properties of the proposed estimator are currently under investigation.

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