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Determinants and current flows in electric networks*

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Abstract

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We consider electrical networks in which current enters at a single node and leaves at another. It has long been known that the currents and potential differences in such networks can be expressed in terms of determinants, or alternatively as counts of trees. Here we give alternative determinantal expressions.

1. Introduction

An electrical network consists of a finite set of 'wires' E_r forming the edges of a connected graph, possibly with multiple edges and loops. (No current flows along a loop, since there is no potential difference along it. Thus a loop may be removed without affecting other features of the network.)

* This paper is intended to celebrate not only the 100th anniversary of Petersen's paper, but also (approximately) the 50th anniversary of the original paper by Brooks, Smith, Stone and Tutte.

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The laws connecting currents c_r in wires E_r and the potentials p_h at nodes N_h in such a network are as follows:

(a) each wire (edge) E_r has associated with it a nonzero number Q_r, its 'conductance'. (More often the resistance $R_r = 1/Q_r$, is given, but the conductance is more convenient here.)

(b) Each node N_h of the network has an associated potential p_h (a number).

(c) Let the end nodes of a wire E_r be N_h and N_k.
The current c_r along E_r (in the direction from N_h to N_k) is the product of the potential difference and conductance:

$$c_r = (p_h - p_k)Q_r. \tag{1}$$

(d) The total current flowing into a node must equal that leaving it, or, with appropriate signs, the total current flowing into a node must be zero. (Law of conservation of electricity, or Kirchhoff's first law.) Current can enter the network at certain nodes, or 'sources', and leave at others, 'sinks'. We consider here only networks with just one source, N_1, and just one sink, N_n, where $n \geqslant 2$ is the number of nodes. From the conservation law it follows that the current C entering at the source must be equal to that leaving at the sink. We call C the 'total current flow' through the network. The potential difference (PD) between the source N_1 and sink N_n will be called the 'total potential drop' P in the network:

$$P = p_1 - p_n. \tag{2}$$

Kirchhoff's second law, stating that the total potential drop round a circuit is zero, follows immediately.

What is in our terminology a 'wire' or edge E_r may be very complicated in engineering terms. But here we concern ourselves only with its conductance Q_r, current, c_r, and the potential drop along it. It is a standard result (Jeans [2]) that if all conductances are positive, all currents and potential differences in the network are uniquely determined if either the total current C or total potential drop P are given. With negative conductances allowed, 'singular' cases can in principle occur, in which that is not so. We suppose here that the networks considered are nonsingular and connected. Alternating current networks can be covered by using complex 'conductances', as explained in textbooks.

2. Networks with unit conductances

We begin with the special case in which all wires E_r have unit conductance, so that the current c_r in the wire is equal to the potential drop along it.

We illustrate the argument with the network G shown in Fig. 1, with $n = 5$ nodes, N_1 to N_5, and $e = 6$ wires (edges), E_1 to E_6. In any case, we suppose the network under consideration drawn in the plane, with the source N_1 placed

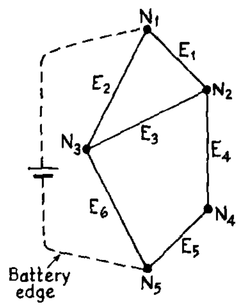


Fig. 1.

highest, the sink N_n lowest, and no two nodes at the same height. (If the network should be nonplanar, this would result in apparent crossings of wire other than at nodes. These crossings are, of course, to be ignored.) Each wire is assigned a downward direction, so that if, for example, in Fig. 1 the current in wire E_3 turns out to flow from N_2 to N_3 it will be counted as positive, if in the opposite direction, as negative.

For certain purposes it is convenient to add an extra 'battery' wire E_0 (shown interrupted in Fig. 1) from the sink N_n to the source N_1, containing a battery to provide the electromotive force voltage to drive the current through the network.

We introduce an incidence matrix $\boldsymbol{i}$ by the definition:

$$i_{hr} = \begin{cases} 1 & \text{when } N_h \text{ is the upper end node of } E_r, \\ -1 & \text{when } N_h \text{ is the lower end node of } E_r, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Thus, for the network of Fig. 1,

$$\boldsymbol{i} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

Each column of $\boldsymbol{i}$ contains just one '1' and one '-1', all other entries being 0. We calculate

$$\boldsymbol{j} = \boldsymbol{i}\boldsymbol{i}^{\mathrm{T}}. \tag{4}$$

Thus, for the network of Fig. 1,

$$\boldsymbol{j} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

The interpretation of $\boldsymbol{j}$ is:

• a diagonal element j_{hh} is the number of wires having N_h as an end node, excluding loops,

• a nondiagonal element j_{hk} is minus the number of wires connecting N_h to N_k.

The column sums of $\boldsymbol{i}$ are all zero. If $\mathbf{1}$ is a column vector with all elements 1, then $\mathbf{1}^{\mathrm{T}}\boldsymbol{i} = \mathbf{0}$. Hence $\mathbf{1}^{\mathrm{T}}\boldsymbol{j} = \mathbf{0}$, so the column totals of $\boldsymbol{j}$ are all zero. Since $\boldsymbol{j}$ is symmetric, all row sums are also zero. It readily follows that all first cofactors in $\boldsymbol{j}$ are equal, having the value f, say.

Let N_h be any node, other than the source or sink. The currents flowing out from N_h sum to zero:

$$\sum_k (p_h - p_k) = 0. \tag{5}$$

summed over all N_k adjacent to N_h. That is,

$$\sum_{k=1}^{n} j_{hk} p_k = 0 \qquad (h \neq 1, n) \tag{6}$$

so that the vector $\boldsymbol{p} = [p_1, p_2, \ldots, p_n]$ is orthogonal to all rows of $\boldsymbol{j}$ except the first and last.

Equation (5) shows that these relations involve only potential differences, so that the potentials p_h themselves are indeterminate by an additive constant. We can fix p_n, the potential of the sink, to be 0. Equations (6) then give $n-2$ linear constraints on $n-1$ parameters p_h. It is shown later (in Section 4) that the determinant $f > 0$ and hence the constraints are linearly independent so that the p_h are then fixed to within a multiplicative constant.

Let J_{gh} denote the cofactor in $\boldsymbol{j}$ obtained by deleting rows 1 and n and columns g and h, and multiplying by -1 when appropriate. Then

$$\sum_{k=1}^{n-1} j_{hk} J_{nk} = \delta_{hn} f, \tag{7}$$

where δ_{hk} is the Kronecker delta, using the standard expression for a determinant (here, first cofactor) in terms of its cofactors. A comparison with (6) shows that $p_k = J_{kn}$ (for $k \neq n$), $p_n = 0$ is one solution of (6). In the nonsingular case, any solution will be

$$p_k = \lambda J_{kn}. \tag{8}$$

The particular solution with $\lambda = 1$ is called the 'full potential' (and from it we get the 'full currents', etc.) The full currents in the network G of Fig. 1 are shown in Fig. 2. (If the network has only 2 nodes, this definition is inapplicable, and we then define the full PD between the 2 nodes to be 1.)

If we put $h = n$ in (7), we find that the total full current C flowing out at the sink is equal to the common first cofactor f. It is also readily shown that P, the potential drop between source and sink, is equal to the minor (=cofactor)

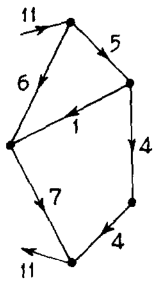


Fig. 2.

obtained by deleting the first and last rows and columns of j and taking the determinant.

3. Representation as a squared rectangle

Some 50 years ago we showed (Brooks et al. [1]) that when such an electrical network is planar, it can alternatively be represented as a rectangle divided up into squares, as shown in Fig. 3. Each wire E_r is replaced by a square, whose side is equal to the current (=PD) in that wire. The horizontal side of the rectangle is equal to the total current C in the network, and the vertical side is equal to the total potential drop P.

4. Trees and circuits

Consider a spanning tree T in an electrical network, and add to it the battery edge E_0. The resulting configuration will contain a unique circuit γ. Run a unit current round this circuit, from sink to source in the battery edge. Repeat for all spanning trees, and sum the resulting currents. Since the current in any circuit obeys Kirchhoff's first law, so does the total. It can also be shown to obey Kirchhoff's second law, so this constitutes a correct current in the network. This

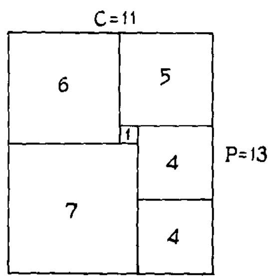


Fig. 3.

can also be shown to be the same as the full current as defined above. The simplest proof is inductive. It is easy to verify that the two definitions agree for networks of not more than 3 nodes. Also, if G is an electrical network, E_r an edge in it, and G' denotes the network with E_r removed and G'' the network with E_r contracted, then according to either definition the currents in G are the sums of the currents in G' and those in G''. This provides the inductive step.

Since each spanning tree gives rise to a unit current entering the network at the source, the total current through the network is equal to the number f of spanning trees, i.e., the complexity of the network. Hence $f > 0$.

5. The new matrix and determinant

So far we have described what is already known, though possibly in a slightly unusual way. See Kirchhoff [3] and Brooks et al. [1].

Now take a network G drawn out as in Fig. 1, and draw horizontal lines L_1 to L_{n-1} between the nodes, as in Fig. 4. The line L_u lies below node N_u and above N_{u+1}. Construct a matrix $\boldsymbol{H}$ as follows. The element H_{uv} is the number of wires which intersect both horizontal lines L_u and L_v. (Thus H_{uu} is the number of wires intersected by L_u.) For the network of Fig. 1, the matrix $\boldsymbol{H}$ is shown to the right of the network in Fig. 4.

Find the adjoint matrix adj $\boldsymbol{H}$ (the matrix of cofactors of $\boldsymbol{H}$), and

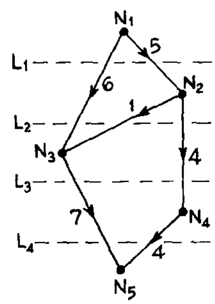
$$\boldsymbol{g} = \mathbf{1}^{\mathrm{T}} \operatorname{adj} \boldsymbol{H}. \tag{9}$$

For the network shown, $\boldsymbol{g} = [5\ \ 1\ \ 3\ \ 4]$. Then (with, if as usual, the nodes are numbered downwards from N_1 to N_n) we assert that

$$g_h = \text{the P.D. between } N_h \text{ and } N_{h+1}, \tag{10}$$

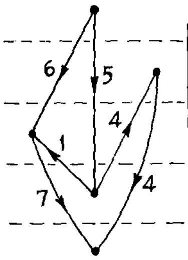
$$\det \boldsymbol{H} = \text{the complexity } f \text{ of the network.} \tag{11}$$

We prove (11) as follows. Take the matrix $\boldsymbol{j}$ and delete the last row and column to get a matrix $\boldsymbol{j}'$ whose determinant is f. Add the first row of $\boldsymbol{j}'$ to the second



$$\begin{array}{c} (L_1\quad L_2\quad L_3\quad L_4) \\ \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \end{array}$$

Fig. 4.



$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 2 & 4 & 3 & 1 \\ 1 & 3 & 5 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

Fig. 5.

row, the new second row to the third, and so on, and then do similarly with the columns. It will be found that the resulting matrix is $\boldsymbol{H}$. But adding columns to columns and rows to rows in this way does not alter the value of the determinant. A further detailed consideration of this process proves (10).

To find the PDs between 2 other nodes, such as N_1 and N_3, we simply add the PDs calculated as above between intermediate nodes, e.g., here between N_1 and N_2 and between N_2 and N_3, i.e., $5+1=6$. In particular, P, the total potential drop between source and sink is

$$p = \boldsymbol{g}\mathbf{1} = \mathbf{1}^{\mathrm{T}}\mathrm{adj}\,\boldsymbol{H}\mathbf{1}. \tag{12}$$

Provided that the source is placed at the top and the sink at the bottom, the other nodes can be placed in any order. Thus Fig. 5 shows another way of drawing the network, giving

$$\boldsymbol{g} = [9 \ \ -3 \ \ -1 \ \ 8].$$

6. Weighted networks and multiple edges

We now show how to modify the method for networks in which conductances are not all equal to 1. If the conductance c_r of a wire E_r is a positive integer, it makes no essential difference whether we regard this as a single wire of conductance c_r, or as a set of c_r parallel wires, each with conductance 1. If c_r is not a positive integer, we do not, of course, have such a choice. Fig. 6 shows a modification of the network of Fig. 1 in which we replace the wire E_2 by 2 parallel wires of conductance 1, or, virtually equivalently, give E_2 conductance 2. The wire E_5 is given conductance 3.

We introduce a diagonal matrix,

$$\mathbf{c} = \mathrm{diag}[c_1\, c_2 \cdots c_n], \tag{13}$$

and calculate $\boldsymbol{j}$ as

$$\boldsymbol{j} = \boldsymbol{ici}^{\mathrm{T}}. \tag{14}$$

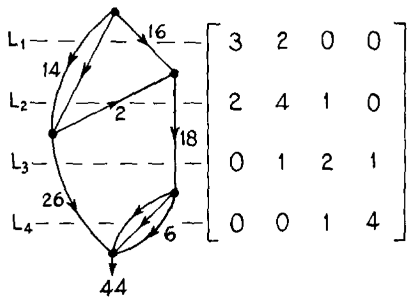


Fig. 6.

Exactly as before, the PDs between nodes are found as second cofactors in j, and the total PD, P, is the minor (cofactor) got by deleting the first and last rows and columns of j. The total current C, or weighted complexity, is equal to any first cofactor of j. We exclude from consideration the 'singular case' (=0, when there may be other possible values of the currents and PDs.

When expressing PDs and currents in terms of trees and circuits, each spanning tree must first be weighted by the product of the conductances of all wires belonging to the tree. The current in any circuit found by adding the battery edge to the tree must be given a value equal to the weight of the tree. The total current, or weighted complexity, is the sum of the weights of all spanning trees. The PDs in our example are shown in Fig. 6; the currents must be found by multiplying these by the respective conductances. It is easy to verify in this diagram that Kirchhoff's laws hold.

The network, if planar, can be represented as a rectangle divided into rectangles, as in Fig. 7. Each wire is replaced by a rectangle whose width is equal to the current in the wire, and whose height is equal to the potential drop in the wire. If the wire E_2 is considered as a single wire of conductance 2, the corresponding rectangle at the top left hand corner of Fig. 7 is a rectangle of height 14 and width $2 \times 14 = 28$. If it is considered as 2 parallel edges, we get 2 squares of side 14 lying side by side at the top left hand corner.

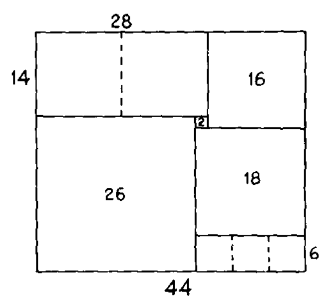


Fig. 7.

In the alternative approach the matrix $\boldsymbol{H}$ is constructed as before by drawing horizontal lines L_u between the nodes, as in Fig. 6. But H_{uv} is now defined as the total conductance of all wires which intersect L_u and L_v. As before,

$$\boldsymbol{g} = \mathbf{1}\operatorname{adj} \boldsymbol{H},$$

and $P = \boldsymbol{g}\mathbf{1}$. For the network of Fig. 6,

$$\boldsymbol{g} = [16 \ \ -2 \ \ 20 \ \ 6],$$

in accordance with the PDs shown in the figure.

This method is convenient for computerization, since, for any symmetric matrix $\boldsymbol{m}$,

$$\operatorname{adj} \boldsymbol{m} = (\det \boldsymbol{m})\boldsymbol{m}^{-1},$$

involving only standard routines.

7. Dual currents

Kirchhoff's second law states that the total potential difference summed round a circuit is zero. When all wires have unit conductance, this implies that the total current flowing round the circuit is zero.

Assign a number ± 1 indicating direction to each wire in the circuit, so that '1' means a consistent orientation round the circuit, and '-1' the opposite orientation. Then Kirchhoff's second law is equivalent to the statement that currents and circuits are mutually orthogonal. (If the resistance R of a wire is not 1, we assign a number $\pm R$ to the wire.)

Kirchhoff's first law states that the total current flowing away from a node is zero. This can be generalized. A cocircuit g is a cutset of wires in the graph. Provided that the cutset does not separate the source from the sink, the total current flowing across any cutset (cocircuit) must be zero. By assigning appropriately a number ± 1 to each wire in the cocircuit, this implies that currents and cocircuits are mutually orthogonal. The source defines a cocircuit consisting of all wires incident with it, and the current flowing across it is the total current C flowing through the network; similarly for the sink.

Note that circuits and cocircuits are matroid properties, so the discussion here is close to matroid theory.

Now let e be the number of wires in the network, excluding loops, n the number of nodes, and r its rank, so that $r = n - 1$, and $r_D = e - r = e - n + 1$ is the corank. The construction for a current flow described above involves finding $n - 2 = r - 1$ independent cocircuits (1 for every node except the source and sink), and r_D independent circuits, such that the current is orthogonal to these cocircuits and circuits. (Since other circuits are linearly dependent on these, orthogonality with any circuit immediately follows.) We can define a dual current

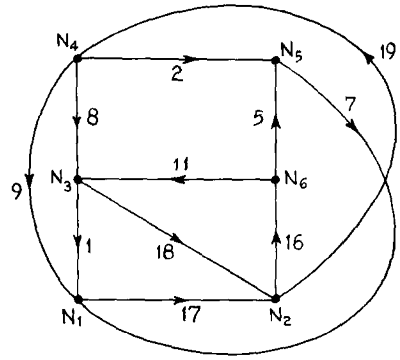


Fig. 8.

flow as one which is orthogonal to all cocircuits and to $r-2$ independent circuits. This dual flow provides a natural definition of a current flow in the dual of a graph, composed of the same wires, but with circuits and cocircuits interchanged, and with conductances becoming resistances in the dual.

For example, if we take an electrical network with one source and one sink, and add to it the battery wire, then the current flow will then obey Kirchhoff's first law at all nodes, and the second law will hold for all circuits not containing the battery wire. So we now have a dual flow.

This has a simple interpretation in terms of a rectangle divided into squares (or rectangles). In its original position the rectangle defines a complete network, i.e., one containing the battery wire. Rotate the rectangle through a right angle, interchanging horizontal and vertical, and we get correspondingly the dual network. Thus, if the network is planar, both current flows and dual flows can be readily calculated using the technique described above.

But, with a nonplanar network, it is much less easy to find how to apply the techniques described above. The network in Fig. 8 is nonplanar, for on contracting the wire N_5N_6 it becomes a K_5. It has $e=11$ wires, $n=6$ nodes, and thus rank $r=6-1=5$ and corank $r_D=11-5=6$. The currents shown obey Kirchhoff's first law at all nodes. It obeys Kirchhoff's second law in the circuits $N_1N_2N_3$, $N_1N_3N_4$, $N_1N_4N_5$, $N_3N_4N_5N_6$, $N_2N_4N_5N_6$. These circuits are independent, since each contains a wire not contained in the others. Hence Fig. 8 shows a dual current flow. Since all the relevant equations are linear, they can be solved to give the currents as determinants (equal to the currents shown in Fig. 8 multiplied by 5.) But it is not obvious how we can find a matrix like $\boldsymbol{H}$ to give a simple method of calculation.

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