SOME MIXING PROPERTIES OF TIME SERIES MODELS

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Sufficient conditions are given for linear processes and ARMA processes to have the Gaswirth and Rubin mixing condition. The mixing rates are also determined.

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1. Introduction

Let $X(t), t = \ldots, -1, 0, 1, \ldots$ be a $p$-variate random process, $M_0^t$ and $M_n$ denote respectively the $\sigma$-fields generated by $X(t), t < 0$ and by $X(t), t \geq n$. Let $X = (\ldots, X(-1), X(0)), Y = (X(n), X(n+1), \ldots)$ and $P_{XY}, P_X, P_Y$ be respectively the joint distribution of $X, Y$ and the marginal distributions of $X, Y$.

Define a function $\Delta_n(x)$ by the condition that for any measurable subset $A$ of $R^p \times R^p \times \cdots$

$$
\int \Delta_n(x) P_X(dx) = \sup_{|h| \leq 1} \left\{ \int \int_A h(x, y)[P_X(dx)P_Y(dy) - P_{X,Y}(dx, dy)] \right\}. \quad (1.1)
$$

Note that $\Delta_n$ always exists since the right hand side of (1.1) defines a measure absolutely continuous with respect to $P_X(dx)$. If the conditional distribution of $P_Y(dy|x)$ of $Y$ given $X$ exists, then $\Delta_n(x)$ is just the total variation of $P_Y(dy|x) - P_Y(dy)$.

Let $\|\Delta_n\|_s$ be the $L^s$ norm of $\Delta_n$. Then $X(t)$ is said to satisfy the Gastwirth and Rubin mixing condition if $\|\Delta_n\|_s \to 0$ as $n \to \infty$ for some $0 < s < \infty$. See Gastwirth and Rubin [5]. The purpose of this paper is to study the convergence of $\|\Delta_n\|_s$ to zero for linear processes and ARMA processes.

When $\|\Delta_n\|_1 \to 0$ as $n \to \infty$, the process $X(t)$ is often referred to as absolutely regular, weak Bernoulli or completely regular. There is vast literature on processes

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of this type. See for example, Volkonskii and Rozanov [9], Yoshihara [12] and Berbee [2].

Let \( \alpha(n) = \sup_{A \in M_n^\infty, B \in M_m^\infty} |P(A \cap B) - P(A)P(B)| \). If \( \alpha(n) \to 0 \) as \( n \to \infty \), then \( X(t) \) is said to satisfy the strong mixing condition. Chanda [4], Gorodetskii [6] and Withers [11] have obtained various conditions for linear processes to be strong mixing.

It can be shown that \( \alpha_n \leq 2 \| \Delta_n \|_1 \) and hence the condition that \( \| \Delta_n \|_1 \to 0 \) is stronger than strong mixing. Some results for absolutely regular processes do not hold just under strong mixing. See, for example, Berbee [2, p. 104] or Bradley [3]. Volkonskii and Rozanov [9, p. 187] have pointed out that the condition of absolute regularity is also more suitable for research. It is thus of interest to determine whether a process is absolutely regular.

The processes considered here are multivariate while those in Chanda, Gorodetskii and Withers are univariate.

The results are presented in Sections 2 and 3. Section 2 studies the convergence of \( \| \Delta_n \|_1 \) to zero for linear processes and Section 3 studies the convergence of \( \| \Delta_n \|_s \) for ARMA processes. Section 3 is a revised version of an earlier seminar note by Pham and Tran [8].

Throughout the paper, \( K \) will denote a constant whose values are unimportant and may vary from line to line.

2. Mixing conditions for linear processes

Assume that there exists a sequence of independent random vectors \( e(t) \), and matrices \( A(j) \) such that

\[
X(t) = \sum_{j=0}^{\infty} A(j)e(t-j), \quad A(0) = I,
\]

where \( I \) is the identity matrix. We further assume that the \( e(t) \) admit a density, say \( g_\tau \). The following lemmas will be useful later.

Lemma 2.1. Let \( r(n) = \sum_{j=n}^{\infty} A(j)e(n-j) \) and \( \xi(n) = X(n) - r(n) \). Then \( (\xi(n), \ldots, \xi(n+m))' \) admits a density, say, \( f_{n,m} \) and

\[
\Delta_n(X) \leq \sup_{m > 0} \{E[\delta_{n,m}(R_{n,m})|X]| + E[\delta_{n,m}(R_{n,m})] \} \quad \text{a.s.}
\]

where \( R_{n,m} = (r(n), \ldots, r(n+m)) \), \( X = (\ldots, X(-1), X(0)) \) and

\[
\delta_{n,m}(u) = \int |f_{n,m}(z-u) - f_{n,m}(z)| \, dz.
\]

Proof. Since \( \xi(t) = \sum_{j=0}^{t-1} A(j)e(t-j) \) and \( A(0) = I \), the random vector \( (\xi(1), \ldots, \xi(n+m)) \) admits a density, and therefore \( (\xi(n+1), \ldots, \xi(n+m)) \) also
admits a density. Let \( Y_m = (X(n), \ldots, X(n + m))' \). Then \( Y_m = S_{n,m} + R_{n,m} \) where \( S_{n,m} \) is independent of \( X \). Hence the conditional distribution of \( Y_m \) given \( X \) and the marginal distribution of \( Y_m \) admit respectively the densities \( E[f_{n,m}(y - R_{n,m})|X] \) and \( E[f_{n,m}(y - R_{n,m})] \). Thus

\[
\Delta_n(x) \leq \sup_{m \geq 0} \int |E[f_{n,m}(y - R_{n,m}) - f_{n,m}(y)|X]\,dy
+ \sup_{m \geq 0} \int E[f_{n,m}(y - R_{n,m}) - f_{n,m}(y)]\,dy. \tag{2.1}
\]

The first term in the right hand side of (2.1) is bounded by

\[
\sup_{m \geq 0} \int E[|f_{n,m}(y - R_{n,m}) - f_{n,m}(y)||X]\,dy = \sup_{m \geq 0} E[\delta_{n,m}(R_{n,m})|X];
\]

and the last term is bounded by

\[
\sup_{m \geq 0} E[\delta_{n,m}(R_{n,m})].
\]

Lemma 2.2. Suppose that

(i) \( \int |g_r(v - u) - g_r(v)|\,dv < K\|u\| \) for all \( t \);

(ii) \( \sum_{j=0}^{\infty} \|A(j)\| < \infty \) and \( \sum_{j=0}^{\infty} A(j)z^j \neq 0 \) for all \( z \) with \( |z| \leq 1 \). Then

\[
\sup_{m \geq 0} \delta_{n,m}(R_{n,m}) \leq K \sum_{j=0}^{\infty} \alpha(j + n)\|e(-j)\| \text{ where } \alpha(j) = \sum_{k=j}^{\infty} \|A_k\|.
\]

Proof. Let \( B(j) \) be the Taylor coefficients of \([\sum A(j)z^j]^{-1}\). Then \( e(t) = \sum_{j=0}^{t-1} B(j)\xi(t) \) where the \( \xi(t) \) are as defined in Lemma 2.1. Therefore

\[
f_{n,m}(y_n, \ldots, y_{n+m}) = \int \cdots \int \prod_{i=1}^{n+m} g_i \left[ \sum_{j=1}^{t} B(t-j)y_j \right] dy_1 \cdots dy_{n-1}.
\]

Setting \( u_i = 0 \) for \( t < n \). Then

\[
\delta_{n,m}(u_n, \ldots, u_{n+m}) = \int \cdots \int \prod_{i=1}^{n+m} g_i \left[ \sum_{j=1}^{t} B(t-j)(y_j - u_i) \right]
- \prod_{i=1}^{n+m} g_i \left[ \sum_{j=1}^{t} B(t-j)y_j \right] dy_1 \cdots dy_{n+m}.
\]

Using the fact that

\[
\prod_{i} (a_i + \alpha_i) - \prod_{i} a_i = \sum_{i<j} \prod_{i} a_i \alpha_i \prod_{j>i} (a_j + \alpha_j),
\]

and that \( g_i \) is a density function, we get

\[
\delta_{n,m}(u_n, \ldots, u_{n+m}) \leq \sum_{i=n}^{n+m} K \left\| \sum_{j=n}^{t} B(t-j)u_i \right\|.
\]
Thus

\[ \delta_{n,m}(R_{n,m}) \leq K \sum_{i=0}^{\infty} \|B(i)\| \cdot \sum_{j=0}^{\infty} \|r(j)\| \]

\[ \leq K \left[ \sum_{i=0}^{\infty} \|B(i)\| \right] \left[ \sum_{j=0}^{\infty} \alpha(j+n)\|e(-j)\| \right]. \]

Note that \( \sum_{i=0}^{\infty} \|B(i)\| < \infty \) by Wiener's theorem. See Wiener [10, p. 91].

**Theorem 2.1.** Assume that the conditions of Lemma 2.2 hold and \( E\|e(t)\|^{\delta} < K \) for some \( \delta > 0 \) and for all \( t \). If \( \sum_{j=1}^{\infty} \alpha(j)^{\delta/1+\delta} < \infty \) where the \( \alpha(j) \) are as in Lemma 2.2. Then \( \|\Delta_n\|_1 \leq K \sum_{j=0}^{\infty} \alpha(j)^{(1+\delta)/(1+\delta)} \) and \( X_t \) is absolutely regular.

**Proof.** Let \( \{c_j\} \) be a sequence of positive numbers. Since \( \Delta_n \equiv 2 \) a.s., we have by Lemmas 2.1 and 2.2

\[ \|\Delta_n\|_1 \leq K \sum_{j=n}^{\infty} \alpha(j)c_j + 2 \sum_{j=n}^{\infty} P\{\|e(j)\| > c_j\}. \]

By Schwartz inequality, \( P\{\|e(j)\| > c_j\} \leq K/c_j^{\delta} \). The theorem then follows by putting \( c_j = \alpha(j)^{-1/(1+\delta)} \).

**Remark 2.1.** Note that \( [\sum_i \|A(i)\|^{\delta} \leq \sum_i \|A(i)\|^{\delta} \). See Loève [7, p. 155] if \( \delta = 1 \). Then

\[ \sum_{j=r}^{\infty} \alpha(j)^{\delta/(1+\delta)} \approx \sum_{j=r}^{\infty} \left[ \sum_{i=j}^{\infty} \|A(i)\|^{\delta} \right]^{1/(1+\delta)}, \]

which equals the strong mixing rate for linear processes obtained by Gorodetskii [6].

### 3. Mixing conditions for ARMA processes

Assume now that \( X(t) \) is an autoregressive moving average (ARMA) process with values in \( \mathbb{R}^p \). Then it admits a Markovian representation

\[ X(t) = HZ(t), \quad Z(t) = FZ(t-1) + Ge(t) \quad (3.1) \]

where \( Z(t) \) are random vectors, \( H, F, G \) are appropriate matrices and \( e(t) \) are i.i.d. random vectors. (See for example Akaike [1]). Assume that the \( e(t) \) have a density \( g \). Here \( A(j) = HF^jB \) and hence \( r(n) \) and \( \xi(n) \) of Lemma 2.1 equals \( HF^nZ(0); \) and \( \sum_{j=1}^{n} F^{n-j}Be(j) = H\xi(n) \), say. Now for \( u = (u_{n+1}, \ldots, u_{n+m}) \),

\[ \delta_{n,m}(u) = \sup_{|h|=1} E[h(\xi_{n,m} - u) - h(\xi_{n,m})] \]

where \( \xi_{n,m} = (\xi(n), \ldots, \xi(n+m)) \). Take \( u_j = HF^jz \), then

\[ \xi(j) - u_j = HF^{j-n}[\xi(n) - F^nz] + \sum_{i=n}^{j} HF^{i-n}e(i) \quad \text{for } j \geq n. \]
Thus
\[ h(\xi_{n,m} - u) = \tilde{h}[(\xi(n) - F^n z, e(n+1), \ldots, e(n+m))]. \]
\[ h(\xi_{n,m}) = \tilde{h}[(\xi(n) - F^n z, e(n+1), \ldots, e(n+m))] \quad \text{for some function } \tilde{h}. \]
Hence
\[ \delta_{n,m}(u) \leq \sup_{|\tilde{h}| \leq 1} E\{ \tilde{h}[(\xi(n) - F^n z, e(n+1), \ldots, e(n+m))] \}
- \tilde{h}[(\xi(n), e(n+1), \ldots, e(n+m))]. \]
Since the \( e(t), \ t > n \) are independent of \( \xi(n) \), if \( \xi(n) \) admits a density \( \phi_n \), then
\[ \delta_{n,m}(u) \leq \int |\phi_n(z - F^n u) - \phi_n(z)| \, dz. \]
Thus Lemma 2.1 becomes

**Lemma 3.1.** Let \( X(t) \) be the ARMA process as in (3.1). Suppose that \( \xi(n) = \sum_{j=1}^{n-1} F^{n-j} B_j \) admits a density \( \phi_n \). Then
\[ \Delta_n \leq E\{ \delta_n[F^n Z(0)]X \} + E\delta_n[F^n Z(0)], \]
where \( \delta_n(z) = \int |\phi_n(v - z) - \phi_n(v)| \, dv \) and \( X \) is as in (2.1).

**Lemma 3.2.** Let \( g \) be an integrable and \( f \) be a bounded function on \( \mathbb{R}^d \). Assume that
\[ \int \|x\|^\gamma |g(x)| \, dx < \infty \quad \text{and} \quad f(x) = O(\|x\|^{\gamma}), \ x \to 0. \]
Then, as the \( d \times d \) matrix \( u \) tends to the zero matrix,
\[ \int f(ux)g(x) \, dx = O(\|u\|^{\gamma}) \]
where \( \|u\| = \sup_{\|x\| = 1} \|ux\| \).

The proof of Lemma 3.2 is simple and is omitted.

**Theorem 3.1.** Suppose that the eigenvalues of \( F \) are of modulus strictly less than 1, then \( \|\Delta_n\|_s \to 0 \) as \( n \to \infty \) for all \( s \). If moreover \( \int \|x\|^\delta |g(x)| \, dx < \infty \) and \( \int |g(x) - g(x - \theta)| \, dx = O(|\theta|^{\gamma}) \) for some \( \delta > 0 \) and \( \gamma > 0 \), then \( \|\Delta_n\|_s \to 0 \) at an exponential rate for all \( 0 < s < \infty \).

**Proof.** Without loss of generality, assume that the matrix \( [G, FG, \ldots, F^{n-1} G] \) is of full rank for \( n \) large enough, otherwise one can always find an equivalent (minimal) representation
\[ X(t) = \tilde{H} \tilde{Z}(t), \quad \tilde{Z}(t) = F \tilde{Z}(t-1) + \tilde{G} e(t) \]
for which the above rank condition is fulfilled. Thus we can add some rows to the matrix \( [G, FG, \ldots, F^{n-1} G] \) to get an invertible matrix, say \( U \). Hence the random
vector $\tilde{\xi}(n) = U[e(n), \ldots, e(1)]$ admits a density and so does $\xi(n)$. By Lemma 2.1 and the Lebesgue Dominated Convergence Theorem, $\|\Delta_n\|_s \to 0$.

Let $\hat{\phi}_n, \phi_n$ be the densities of $\tilde{\xi}(n)$ and $\xi(n)$. Let $h$ denote the point in $(\mathbb{R}^d)^n$ obtained from $h \in \mathbb{R}^d$ ($d = \text{dimension of } \xi(n)$) by adding zeroes. Then

$$\int |\phi_n(v - h) - \phi_n(v)| \, dv = \int |\hat{\phi}_n(u - \hat{h}) - \hat{\phi}_n(u)| \, du$$

$$= |\det U| \int \left| \prod_{j=1}^n g(x_j - \theta_j) - \prod_{j=1}^n g(x_j) \right| \, dx_1 \cdots dx_m,$$

where $\theta = U^{-1}h$. Using the assumption of the theorem and following the same line of argument as in the proof of Lemma 2.2, we get $\delta_n(h) = O(\|h\|^\gamma)$. The above relation holds for a fixed $n$. However it is easy to show that $\delta_n(h)$ decreases with $n$ and hence this relation holds uniformly in $n$.

Let $\rho$ be the maximum modulus of the eigenvalues of $F$, then $\|F^n\| < K\rho^n$. By Lemma 3.2,

$$\left\{E[\delta_n(F^nZ(0))]^s\right\}^{1/s} = O(\rho^{\eta n})$$

where $\eta = \min(\gamma, s\gamma)/s$. For $s \geq 1$, using Jensen inequality and Lemma 3.1,

$$\Delta_n^s \approx 2^{s-1} E\{\delta_n(F^nZ(0))^s \mid X\} + 2^{s-1}\{E[\delta_n(F^nZ(0))]^s\}.$$

Hence $\|\Delta_n\|_s \to 0$ at an exponential rate. Since $\|\Delta_n\|_r \leq \|\Delta_n\|_s$ for $r \leq s$, the result holds for all $s$.

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References