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Phirotopes, super p-branes and qubit theory

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Abstract

The phirotope is a complex generalization of the concept of chirotope in oriented matroid theory. Our main goal in this work is to establish a link between phirotopes, super p-branes and qubit theory. For this purpose we first discuss maximally supersymmetric solutions of 11-dimensional supergravity from the point of view of the oriented matroid theory. We also clarify a possible connection between oriented matroid theory and supersymmetry via the Grassmann–Plücker relations. These links are in turn useful for explaining how our approach can be connected with qubit theory.

1. Introduction

Oriented matroid theory [1] is a combinatorial structure that has been proposed as the underlying mathematical framework for M-theory [2]. There are a number of evidences that suggest that this may be the case, including the following connections with oriented matroid theory: p-branes, qubit theory, Chern–Simons theory, supergravity and string theory, among others (see Refs. [3–10] and references therein). The key concept to realize these developments is the so-called chirotope notion which provides one of the possible axiomatizations for oriented matroid theory (see Ref. [1] and references therein). Since supersymmetry is part of M-theory one may extend such analysis to include complex structure. It turns out that when the chirotopes are combined with a complex structure one is led to the phirotope concept [11–13]. Thus, one should

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expect that when the complex structure is considered, the link between chirotopes and p-branes may be generalized to a connection between phirotopes and super p-branes.

In order to achieve our goal we first explain how phirotopes can be linked to supersymmetry (see Ref. [9]). In this case, we explain how maximally supersymmetric solutions of 11-dimensional supergravities [14,15] may be the key route to construct such a link. This is because the 4-form $F = dA$ or $F^{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}}$, with $\hat{\mu}, \hat{\nu} = 0, \ldots, 10$, of 11-dimensional supergravity satisfies the Grassmann–Plücker relations (see Ref. [9] and references therein) which in turn are used to define both chirotopes and phirotopes. In order to clarify this constructions we briefly review maximally supersymmetric solution. In particular, we focus on the algebraic identities of Englert solution [16] of 11-dimensional supergravity. We mention that not only in the case of the Freund–Rubin solution [17] of 11-dimensional supergravity the 4-form field $F^{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}}$ admits an interpretation of a chirotope, but also the Englert solution [16]. In fact, if one assumes that the only non-vanishing components of $F^{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}}$ are proportional to the completely antisymmetric symbol $\varepsilon^{\mu\nu\alpha\beta}$, with $\mu, \nu, \alpha, \beta = 0, \ldots, 3$, then the Freund–Rubin solution arises from the bosonic sector of 11-dimensional supergravity field equations. While, if in addition, one assumes non-vanishing values for F^{ijkl}, with $ijkl = 4, \ldots, 10$, one obtains the Englert solution. From this perspective it becomes evident that it is important to study, deeply, the algebraic properties of $F^{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}}$. One observes, for instance, that since in the case of maximally supersymmetric solutions $F^{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}}$ is totally decomposable, it must be possible to relate $F^{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}}$ to the chirotope concept via the Grassmann–Plücker relations (see Ref. [18] for details).

It turns out that a natural generalization of the concept of chirotope is the so-called phirotopes (see Refs. [11–13]). The main difference is that while the chirotope take values in the set $\{-1, 0, 1\}$ the phirotopes take values in the set $\{e^{i\theta} \mid 0 < \theta < 2\pi\}$. This means that the phirotopes describe a complex structure. Thus, in principle one can use phirotopes to introduce Grassmann variables and in this way to define the concept of superphirotope which in turn can be used to establish a link with super p-branes.

The above scenario can be linked with class of N-qubits (see Refs. [4,5] and references therein), with the Hilbert space in the form $C^{2^N} = C^L \otimes C^l$, with $L = 2^{N-n}$ and $l = 2^n$. In fact, such a partition allows a geometric interpretation in terms of the complex Grassmannian variety $Gr(L, l)$ of l-planes in C^L via the Plücker embedding [19]. In the case of N-rebits one can set an $L \times l$ matrix variable b^μ_a, $\mu = 1, 2, \ldots, L$, $a = 1, 2, \ldots, l$, of $2^N = L \times l$ associated with the variable $b_{a_1 a_2 \ldots a_N}$, with $a_1, a_2, \ldots$ etc. taking values in the set $\{1, 2\}$. Moreover, one can consider that the first $N - n$ terms in $b_{a_1 a_2 \ldots a_N}$ are represented by the index μ in b^μ_a, while the remaining n terms are label by the index a in b^μ_a. One of the advantage of this construction is that the Plücker coordinates associated with the real Grassmannians b^μ_a are natural invariants of the theory. Since oriented matroid theory leads to the chirotope concept which is also defined in terms Plücker coordinates these developments establishes a possible link between chirotopes and p-branes with qubit theory.

This article is organized as follows. In Section 2, we present a proof that a p-form is totally decomposable if and only if satisfies the Grassmann–Plücker relations. In Section 3, Figueroa-O'Farrill–Papadopoulos formalism of 11-supergravity is revisited. In Section 4, Englert solution of 11-dimensional supergravity is reviewed. In Section 5, the chirotope concept of oriented matroid theory is related to supergravity. In Section 6, the generalization of chirotopes to phirotopes it is discussed. In Section 7, we comment about the relation between maximally supersymmetric solutions of 11-dimensional supergravity and the chirotope concept. In Section 8, we develop the idea of superphirotopes. In Section 9, we focus on the possible relation between qubit theory and oriented matroid theory. Finally in Section 10, we make some final remarks.

2. Grassmann–Plücker relations and decomposable p-forms

It is known that the Grassmann–Plücker relation [20] is one of the key concepts in oriented matroid theory [1]. In order to better understand this notion it is first convenient to recall the mathematical definition of a Grassmannian $Gr(p,n)$ (Grassmann variety) over the real R (or any other field K). Let V a vector space of dimensions n. The space $Gr(p,n)$ over R is defined as the set of all p-dimensional subspaces of V.

Here, we are interested in considering the Plücker embedding of $Gr(p,n)$ into the projective space $P(\Lambda^p V)$. Given a subspace $W \in Gr(p,n)$ with basis $\{F^1, F^2, \dots, F^3\}$ let a map f be given by

$$f : W \longrightarrow F^1 \wedge F^2 \wedge \cdots \wedge F^p, \tag{1}$$

where the symbol $\wedge$ denotes wedge product. It not difficult to show that up to scalar multiplication, this map (called Plücker map) is injective and unique.

It is worth mentioning that, when one is classifying oriented bundles, the Grassmannian $Gr(p,n)$ can also be denoted by the coset space [20]

$$Gr(p,n) = \frac{SO(n)}{SO(n-p)SO(p)}. \tag{2}$$

It is interesting to compare (2) with the definition of the $(n-1)$-sphere S^{n-1} in terms of the orthogonal group $SO(n)$, namely

$$S^{n-1} = \frac{SO(n)}{SO(n-1)}. \tag{3}$$

Comparing (2) and (3) one sees that $Gr(p,n)$ is a generalization of S^{n-1}. Moreover, one may compute the dimension of $Gr(p,n)$ by simply recalling how is computed the dimension of any coset space $\frac{G}{H}$, with H a subgroup of G. One has $\dim \frac{G}{H} = \dim G - \dim H$. Since $\dim SO(n) = \frac{n(n-1)}{2}$ one finds the result $\dim Gr(p,n) = p(n-p)$.

A p-form $F_{\mu_1 \dots \mu_p} \in \Lambda^p V$ is totally decomposable if there exit a basis $F^1, \dots, F^p$ such that

$$F_{\mu_1 \dots \mu_p} \longrightarrow F^1 \wedge \cdots \wedge F^p. \tag{4}$$

In order to connect this definition with the Grassmannian $Gr(p,n)$ we first write

$$F = \frac{1}{p!} F_{\mu_1 \mu_2 \dots \mu_p} e^{\mu_1} \wedge e^{\mu_2} \wedge \cdots \wedge e^{\mu_p}. \tag{5}$$

The expression $e^{\mu_1} \wedge e^{\mu_2} \wedge \cdots \wedge e^{\mu_p}$ denotes a basis of $\Lambda^p V$. Similarly, one has

$$F^1 \wedge \cdots \wedge F^p = \frac{1}{p!} \varepsilon_{a_1 a_2 \dots a_p} F^{a_1}_{\mu_1} F^{a_2}_{\mu_2} \dots F^{a_p}_{\mu_p} e^{\mu_1} \wedge e^{\mu_2} \wedge \cdots \wedge e^{\mu_p}. \tag{6}$$

The ε-symbol $\varepsilon_{a_1 a_2 \dots a_p}$ in (6) is a completely antisymmetric tensor associated with the p-subspace. So, in this context (4) means that

$$F_{\mu_1 \dots \mu_p} = \varepsilon_{a_1 a_2 \dots a_p} F^{a_1}_{\mu_1} F^{a_2}_{\mu_2} \dots F^{a_p}_{\mu_p}. \tag{7}$$

This result implies that the Plücker map can also be understood by the transition

$$F^a_\mu \longrightarrow \varepsilon_{a_1 a_2 \ldots a_p} F^{a_1}_{\mu_1} F^{a_2}_{\mu_2} \ldots F^{a_p}_{\mu_p}, \tag{8}$$

where $F^a_\mu \in Gr(p,n)$.

Now, one may ask: when a p-form $F_{\mu_1 \ldots \mu_p}$ is totally decomposable? There are several ways to approach this question. For instance, one may prove that $F_{\mu_1 \ldots \mu_p}$ is totally decomposable if and only if the dimension of all the $v \in V$ dividing $F \in \Lambda^p(V)$ is p [21]. Here, however, we shall be interested to consider the Grassmann–Plücker relation

$$F_{\mu_1 \ldots [\mu_p} F_{\nu_1 \ldots \nu_p]} = F_{\mu_1 \ldots \mu_{p-1} \alpha_{p+1}} F_{\alpha_1 \ldots \alpha_p} \delta^{\alpha_1 \ldots \alpha_p \alpha_{p+1}}_{\nu_1 \ldots \nu_p \mu_p} = 0. \tag{9}$$

Here, the symbol $\delta^{\alpha_1 \ldots \alpha_p \alpha_{p+1}}_{\nu_1 \ldots \nu_p \mu_p}$ denotes a generalized delta. The idea is now to prove that a p-form $F_{\mu_1 \ldots \mu_p}$ is totally decomposable if and only if the Grassmann–Plücker relation (9) holds.

If $F_{\mu_1 \ldots \mu_p}$ is totally decomposable then one sees that using (7) the combination

$$F_{\mu_1 \ldots [\mu_p} F_{\nu_1 \ldots \nu_p]} \tag{10}$$

leads to

$$\varepsilon_{a_1 a_2 \ldots [a_p} \varepsilon_{b_1 b_2 \ldots b_p]} F^{a_1}_{\mu_1} F^{a_2}_{\mu_2} \ldots F^{a_p}_{\mu_p} F^{b_1}_{\nu_1} F^{b_2}_{\nu_2} \ldots F^{b_p}_{\nu_p}. \tag{11}$$

But one has

$$\varepsilon_{a_1 a_2 \ldots [a_p} \varepsilon_{b_1 b_2 \ldots b_p]} \equiv 0. \tag{12}$$

So, if the Grassmann–Plücker relation (9) holds then $F_{\mu_1 \ldots \mu_p}$ is totally decomposable. Perhaps, it is more difficult to prove that (9) implies (7). This can be shown using an induction method (see [21] and references therein), but here we present an alternative prove that we are not aware of its existence in the literature.

Let F^A_μ be an extended basis of V. We can define

$$F^{A_1 \ldots A_p} \equiv F^{\mu_1 \ldots \mu_p} F^{A_1}_{\mu_1} \ldots F^{A_p}_{\mu_p}. \tag{13}$$

Considering the inverse F^μ_A of F^A_μ this expression leads to

$$F_{\mu_1 \ldots \mu_p} = F_{A_1 \ldots A_p} F^{A_1}_{\mu_1} \ldots F^{A_p}_{\mu_p}. \tag{14}$$

Using (13), it is not difficult to see that (9) implies

$$F_{A_1 \ldots [A_p} F_{B_1 \ldots B_p]} = F_{A_1 \ldots A_{p-1} C_{p+1}} F_{C_1 \ldots C_p} \delta^{C_1 \ldots C_p C_{p+1}}_{B_1 \ldots B_p A_p} = 0. \tag{15}$$

Assume that (15) holds. Let us apply (15) to the particular case

$$F_{a_1 \ldots a_{p-1} C_{p+1}} F_{C_1 \ldots C_p} \delta^{C_1 \ldots C_p C_{p+1}}_{b_1 \ldots b_p A_p} = 0, \tag{16}$$

with $C_p \neq a$ and a and b running in the dimension of the p-subspace. One can show that (16) leads to

$$F_{a_1 \ldots a_{p-1} A_p} F_{b_1 \ldots b_p} = 0. \tag{17}$$

Since in general

$$F_{b_1 \ldots b_p} = \Lambda \varepsilon_{b_1 b_2 \ldots b_p} \neq 0, \tag{18}$$

with Λ an arbitrary constant, one finds that

$$F_{a_1\ldots a_{p-1}A_p}=0. \tag{19}$$

Now, considering the next particular case

$$F_{a_1\ldots a_{p-2}C_{p+2}C_{p+1}}F_{C_1\ldots C_p}\delta^{C_1\ldots C_pC_{p+1}C_{p+2}}_{b_1\ldots b_pA_{p-1}A_p}=0 \tag{20}$$

and using (19) one obtains

$$F_{a_1\ldots a_{p-2}A_{p-1}A_p}=0. \tag{21}$$

Following similar procedure one ends up with that the result that the only non-vanishing components of $F_{A_1\ldots A_p}$ are given by

$$F_{a_1a_2\ldots a_p}\neq 0. \tag{22}$$

But, one knows that $F_{a_1a_2\ldots a_p}=\Lambda\varepsilon_{a_1a_2\ldots a_p}$. Therefore, using (14) we obtain

$$F_{\mu_1\mu_2\ldots\mu_p}=\Lambda\varepsilon_{a_1a_2\ldots a_p}F^{a_1}_{\mu_1}F^{a_2}_{\mu_2}\ldots F^{a_p}_{\mu_p}. \tag{23}$$

Up to constant, this expression corresponds to (7) meaning that $F_{\mu_1\mu_2\ldots\mu_p}$ is decomposable. The expression (23) will be very useful in the next sections.

3. Figueroa-O'Farrill–Papadopoulos formalism revisited

Consider the 4-form $F_{\mu_1\mu_2\mu_3\mu_4}$. We shall assume that this form satisfies the Grassmann–Plücker relation

$$F_{\mu_1\mu_2\mu_3[\mu_4}F_{\nu_1\nu_2\nu_3\nu_4]}=0. \tag{24}$$

It turns out that (24) holds if any only if the following two the relations are satisfied

$$F_{[\mu_1\mu_2\mu_3\mu_4}F_{\nu_1\nu_2\nu_3\nu_4]}=0, \tag{25}$$

and

$$F_{\mu_1[\mu_2\mu_3\mu_4}F_{\nu_1\nu_2\nu_3]\nu_4}=0. \tag{26}$$

It is worth mentioning that (25) and (26) play a crucial role in maximally supersymmetric 11-dimensional supergravity [14,15]. Let us prove that in fact this result holds. First, one observes that in general the bracket [,] in (24)–(26) can be written as

$$G_{[\mu_1\ldots\mu_{d+1}]}\equiv G_{\alpha_1\ldots\alpha_{d+1}}\delta^{\alpha_1\ldots\alpha_{d+1}}_{\mu_1\ldots\mu_{d+1}}. \tag{27}$$

The quantity $G_{\alpha_1\ldots\alpha_{d+1}}$ is any $d+1$-rank tensor. Considering the fact that

$$\delta^{\alpha_1\ldots\alpha_{d+1}}_{\mu_1\ldots\mu_{d+1}}=\delta^{\alpha_1}_{\mu_1}\delta^{\alpha_2\ldots\alpha_{d+1}}_{\mu_2\ldots\mu_{d+1}}+\sum_{k=2}^{d+1}(-1)^k\delta^{\alpha_1}_{\mu_k}\delta^{\alpha_2\ldots\alpha_{d+1}}_{\mu_2\ldots\hat{\mu}_k\ldots\mu_{d+1}}, \tag{28}$$

where $\hat{\mu}_k$ means omitting this index, one finds that (25) follows if and only if one has

$$F_{\mu_1\alpha_2\alpha_3\alpha_4}F_{\beta_1\beta_2\beta_3\beta_4}\delta^{\alpha_2\alpha_3\alpha_4\beta_1\beta_2\beta_3\beta_4}_{\mu_2\mu_3\mu_4\nu_1\nu_2\nu_3\nu_4}=0, \tag{29}$$

which means

$$F_{\mu_1[\mu_2\mu_3\mu_4}F_{\nu_1\nu_2\nu_3\nu_4]} = 0. \tag{30}$$

Properly applying again (28) one gets

$$F_{\mu_1[\mu_2\mu_3\mu_4}F_{\nu_1\nu_2\nu_3\nu_4]} = 3F_{\mu_1\mu_2[\mu_3\mu_4}F_{\nu_1\nu_2\nu_3\nu_4]} + 4F_{\mu_1[\mu_3\mu_4\nu_1}F_{\nu_2\nu_3\nu_4]\mu_2}. \tag{31}$$

Thus, considering the fact that (26) holds the first term in (31) vanishes, that is

$$F_{\mu_1\mu_2[\mu_3\mu_4}F_{\nu_1\nu_2\nu_3\nu_4]} = 0. \tag{32}$$

Similar technique it leads us to the identity

$$F_{\mu_1\mu_2[\mu_3\mu_4}F_{\nu_1\nu_2\nu_3\nu_4]} = 2F_{\mu_1\mu_2\mu_3[\mu_4}F_{\nu_1\nu_2\nu_3\nu_4]} + 4F_{\mu_1\mu_2[\mu_4\nu_1}F_{\nu_2\nu_3\nu_4]\mu_3}, \tag{33}$$

which in turn gives,

$$F_{\mu_1\mu_2\mu_3[\mu_4}F_{\nu_1\nu_2\nu_3\nu_4]} = -2F_{\mu_1\mu_2[\mu_4\nu_1}F_{\nu_2\nu_3\nu_4]\mu_3}. \tag{34}$$

This expression implies that the right hand side of (34) is antisymmetric in the indices μ_1 and μ_3.

On the other hand one obtains

$$\begin{aligned} F_{\mu_1[\mu_2\mu_4\nu_1}F_{\nu_2\nu_3\nu_4]\mu_3} &= 3F_{\mu_1\mu_2[\mu_4\nu_1}F_{\nu_2\nu_3\nu_4]\mu_3} - 3F_{\mu_1[\mu_4\nu_1\nu_2}F_{\nu_3\nu_4]\mu_2\mu_3} \\ &= 3F_{\mu_1\mu_2[\mu_4\nu_1}F_{\nu_2\nu_3\nu_4]\mu_3} - 3F_{\mu_3\mu_2[\mu_4\nu_1}F_{\nu_2\nu_3\nu_4]\mu_1}. \end{aligned} \tag{35}$$

From (26) one sees that the left hand side of (35) vanishes and therefore we obtain

$$F_{\mu_1\mu_2[\mu_4\nu_1}F_{\nu_2\nu_3\nu_4]\mu_3} = F_{\mu_3\mu_2[\mu_4\nu_1}F_{\nu_2\nu_3\nu_4]\mu_1}. \tag{36}$$

This means that $F_{\mu_1\mu_2[\mu_4\nu_1}F_{\nu_2\nu_3\nu_4]\mu_3}$ is symmetric in the indices μ_1 and μ_3 which contradicts the conclusion below (34). Thus, we have found that the only consistent possibility is to set

$$F_{\mu_1\mu_2[\mu_4\nu_1}F_{\nu_2\nu_3\nu_4]\mu_3} = 0, \tag{37}$$

which implies (24) via (34). Summarizing, we have shown that (25) and (26) imply (24) which is the Grassmann–Plücker relation. Conversely, using once again the properties of the generalized delta $\delta^{\alpha_1\ldots\alpha_{d+1}}_{\mu_1\ldots\mu_{d+1}}$ one can show that both $F_{\mu_1[\mu_2\mu_3\mu_4}F_{\nu_1\nu_2\nu_3\nu_4]}$ and $F_{\mu_1[\mu_3\mu_4\nu_1}F_{\nu_2\nu_3\nu_4]\mu_2}$ can be written in terms of $F_{\mu_1\mu_2\mu_3[\mu_4}F_{\nu_1\nu_2\nu_3\nu_4]}$ and therefore (24) implies (25) and (26). This means that the expression (24) is equivalent to the two formulae (25) and (26). Thus, we have complete an alternative proof of such an equivalence.

The formula (24) implies that $F_{\mu_1\mu_2\mu_3\mu_4}$ is totally decomposable. This means that there exist $(4 \times d + 1)$-matrices F^{μ}_{a} in $Gr(p, n)$ such that $F_{\mu_1\mu_2\mu_3\mu_4}$ can be written in the form

$$F^{\mu_1\mu_2\mu_3\mu_4} = \varepsilon^{a_1a_2a_3a_4}F^{\mu_1}_{a_1}F^{\mu_2}_{a_2}F^{\mu_3}_{a_3}F^{\mu_4}_{a_4}. \tag{38}$$

Thus, one may conclude that maximally supersymmetric solutions of 11-dimensional supergravity implies that $F^{\mu_1\mu_2\mu_3\mu_4}$ can be written as (38).

It turns out convenient to briefly mention how the above result is linked to maximally supersymmetric solution of 11-dimensional supergravity. In fact, Figueroa-O'Farrill and Papadopoulos proved that such a solution must be isometric to either $AdS_4 \times S^7$ or $AdS_7 \times S^4$. Their starting point in this result is the vanishing of the curvature $\mathcal{R}$ of the supercovariant connection $\mathcal{D}$ living in (M^{11}, g, F). In fact, demanding the vanishing of the curvature $\mathcal{R}$ they found that (M^{11}, g, F) is maximally supersymmetric solution if and only if (M^{11}, g) is locally symmetric space and F is parallel and decomposable.

Let us clarify further this theorem. In the non-degenerate case, spontaneous compactification allows to assume that the only non-vanishing components of F_a^μ are $F_a^\mu \sim \delta_a^\mu$, with $\mu = 0, 1, 2, 3$ or $F_a^{\hat\mu} \sim F_a^{\hat\mu}$, with $\hat\mu = 8, 9, 10, 11$ leading to the two possible solutions $AdS_4 \times S^7$ or $AdS_7 \times S^4$, respectively. In fact, in the first case one gets that the only non-vanishing components of $F^{\mu_1\mu_2\mu_3\mu_4}$ are $F^{\mu\nu\alpha\beta} \sim \varepsilon^{\mu\nu\alpha\beta}$. Thus, as seen from the 11-dimensional field equations

$$\frac{1}{3!}\varepsilon_{\mu_1\mu_2\mu_3\mu_4\nu_1\nu_2\nu_3\nu_4 NPQ} F^{NPQM}{}_{;M} = \frac{1}{2(4!)^2} F_{[\mu_1\mu_2\mu_3\mu_4} F_{\nu_1\nu_2\nu_3\nu_4]},$$
$$R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{6} F_{MPQR} F_N^{PQR} - \frac{1}{48} g_{MN} F_{SPQR} F^{SPQR}, \tag{39}$$

one obtains the Freund–Rubin solution $AdS_4 \times S^7$. While in the second case one assumes the solution $F^{\hat\mu\hat\nu\hat\alpha\hat\beta} \sim \varepsilon^{\hat\mu\hat\nu\hat\alpha\hat\beta}$ and the field equations (39) lead to the solution $AdS_7 \times S^4$.

Perhaps, it is also convenient to write the three equations (24)–(26) in abstract notation. From the formula $\mathcal{R} = 0$ one can essentially derive two algebraic formulae for F (25) and (26) which in abstract notation become

$$F \wedge F = 0 \tag{40}$$

and

$$\iota_X F \wedge \iota_Y F = 0, \tag{41}$$

respectively. Here ι_X and ι_Y denote an inner product for the two arbitrary vectors X and Y, respectively. From (25) and (26) we proved that F satisfies (24) which in abstract notation is written as

$$\iota_Z \iota_Y \iota_X F \wedge F = 0. \tag{42}$$

It is interesting to mention the way that Figueroa-O'Farrill and Papadopoulos prove that (25) and (26) imply (24). They first observe that contracting (25) with respect to the three vectors X, Y and Z one obtains

$$\iota_Z \iota_Y \iota_X F \wedge F = -\iota_Y \iota_X F \wedge \iota_Z F. \tag{43}$$

While, contracting equation (26) with a third vector field one gets

$$\iota_Y \iota_X F \wedge \iota_Z F = \iota_Y \iota_Z F \wedge \iota_X F. \tag{44}$$

Thus, comparing (43) and (44) one sees that whereas (42) implies that the expression $\iota_Y \iota_X F \wedge \iota_Z F$ is symmetric in X and Z, (44) means that it is skew-symmetric. This means that the term $\iota_Y \iota_X F \wedge \iota_Z F$ must vanish and therefore (42) follows (see Refs. [14] and [15] for details).

4. Englert solution revisited

Consider the octonionic identity [22],

$$f^{ijkl} f_{mnrl} = \delta_m^{[i} \delta_n^j \delta_r^{k]} + \frac{1}{4} f_{[mn}^{[ij} \delta_{r]}^{k]}, \tag{45}$$

with the indices $i, j, \ldots$ etc. running from 4 to 11. Here, f_{ijkl} is a self dual object. Furthermore, f_{ijkl} is defined in terms of the octonionic structure constants ψ_{ijk} and its dual φ_{ijkl} through the relations

$$f_{ijk11} = \psi_{ijk} \tag{46}$$

and

$$f_{ijkl} = \varphi_{ijkl}. \tag{47}$$

From (45) it is not difficult to see that

$$f^r_{[ijk} f_{lmn]r} = 0. \tag{48}$$

This expression can be understood as a solution for

$$f_{s[ijk} f_{lmn]r} = 0, \tag{49}$$

which remains us the formula (26) reduced to seven dimensions. In fact, introducing a siebenbein h^i_k one can make this identification more transparent [22]. In fact, one has

$$F_{ijkl} = h^r_i h^s_j h^t_k h^m_l f_{rstm} \tag{50}$$

and therefore (49) leads to

$$F_{s[ijk} F_{lmn]r} = 0. \tag{51}$$

Starting from (45) and following similar arguments we may establish that

$$F_{s[ijk} F_{lmnr]} = 0 \tag{52}$$

and

$$F_{[sijk} F_{lmnr]} = 0. \tag{53}$$

Thus, according to the discussion of previous sections (52) and (53) imply that F_{ijkl} satisfies the relation

$$F_{sij[k} F_{lmnr]} = 0, \tag{54}$$

which means that F_{ijkl} is decomposable.

On the other hand, in four dimensions as we already mentioned, we can take

$$F^{\mu\nu\alpha\beta} = \Lambda \varepsilon^{\mu\nu\alpha\beta}, \tag{55}$$

where Λ is an arbitrary function. Since $\varepsilon^{\mu\nu\alpha\beta}$ is a maximally completely antisymmetric object in four dimensions we get the formula

$$F_{\mu\nu\alpha[\beta} F_{\sigma\rho\tau\gamma]} = 0, \tag{56}$$

which implies

$$F_{[\mu\nu\alpha\beta} F_{\sigma\rho\tau\gamma]} = 0. \tag{57}$$

Thus, $F^{\mu\nu\alpha\beta}$ is also decomposable.

Our main observation is that despite both F_{ijkl} and $F_{\mu\nu\alpha\beta}$ are both decomposable, the 11-dimensional components $F_{A\nu\alpha D}$ are not. The reason comes from the fact that in spite that F_{ijkl} and $F_{\mu\nu\alpha\beta}$ are decomposable the components of $F_{A\nu\alpha D}$ not necessarily satisfies the relation $F_{A_1A_2A_3[A_4} F_{\nu_1\nu_2\nu_3\nu_4]} = 0$. The result follows from the expression

$$F_{\mu\nu\alpha[\beta} F_{ijkm]} \neq 0, \tag{58}$$

or

$$F_{[\mu\nu\alpha\beta}F_{ijkm]} \neq 0. \tag{59}$$

So, it turns out that full F_{ABCD} is not decomposable. In fact, since $\varepsilon^{\mu\nu\alpha\beta}$ and f^{ijkm} take values in the set $\{-1, 0, 1\}$ in general we have that

$$\varepsilon_{\mu\nu\alpha[\beta}f_{ijkm]} \neq 0, \tag{60}$$

or

$$\varepsilon_{[\mu\nu\alpha\beta}f_{ijkm]} \neq 0. \tag{61}$$

In turn this means that $F_{[A_1A_2A_3A_4}F_{\nu_1\nu_2\nu_3\nu_4]} \neq 0$ or $F \wedge F \neq 0$. Consequently we no longer have maximally supersymmetric solution. Nevertheless, as Englert showed, although the right hand side of the first field equation in (39) is not vanishing the field equations still admit the solution $AdS_4 \times S^7$. This means that maximally supersymmetric solutions can be considered as a broken symmetry (see Ref. [18] and references therein).

5. Connection with chirotopes

The aim of this section is to discuss part of the formalism described in Sections 2, 3 and 4 from the point of view of the oriented matroid theory. Indeed, our discussion will focus on the chirotope concept which provides one possible definition of an oriented matroid [1]. In fact, chirotopes has been a major subject of investigation in mathematics during the last 25 years [1]. Roughly speaking a chirotope is a combinatorial abstraction of subdeterminants of a given matrix. More formally, a realizable p-rank chirotope is an alternating function $\chi : \{1, \dots, n\}^p \to \{-1, 0, 1\}$ satisfying the Grassmann–Plücker relation

$$\chi_{\hat{A}_1\dots\hat{A}_{n-1}[\hat{A}_p}\chi_{\hat{B}_1\dots\hat{B}_p]} = 0, \tag{62}$$

while nonrealizable p-rank chirotope corresponds to the case

$$\chi_{\hat{A}_1\dots\hat{A}_{n-1}[\hat{A}_p}\chi_{\hat{B}_1\dots\hat{B}_p]} \neq 0. \tag{63}$$

It is worth mentioning that there is a close connection between chirotopes and Grassmann variety. In fact, the Grassmann–Plücker relations describe a projective embedding of the Grassmannian of planes via decomposable p-forms (see Ref. [1] for details).

Thanks to our revisited review of Freund–Rubin and Englert solutions given in the previous sections we find that the link between these solutions and the chirotope is straightforward. In fact, our first observation is that any ε-symbol is in fact a realizable chirotope (see Ref. [23]), since it is always true that

$$\varepsilon_{\hat{A}_1\dots\hat{A}_{p-1}[\hat{A}_p}\varepsilon_{\hat{B}_1\dots\hat{B}_p]} = 0. \tag{64}$$

From this perspective we recognize that the formula (24) indicates that in the case of maximally supersymmetric solutions, in 11-dimensional supergravity, the 4-form F_{ABCD} is a realizable 4-rank chirotope. While in the case of Freund–Rubin–Englert solution, from (39) and (42) one discovers that according to our discussion of section 4 one may identify F_{ABCD} with a nonrealizable 4-rank chirotope. From this connections one may expect that there may be many possible 4-rank chirotopes in 11-dimensions and therefore there must be many new and unexpected solutions for 11-dimensional supergravity.

One of our key tools in our formalism is the octonionic structure. This division algebra was already related to the Fano matroid and therefore, a possible connection with supergravity was established (see Ref. [2] and references therein). Here, we have been more specific and through the chirotope concept we established the relation between the Freund–Rubin–Englert solution and oriented matroid theory. However, it may be interesting to understand the possible role of the Fano matroid in this scenario.

Moreover, here we focused on 11-dimensional supergravity but, in principle, one may expect to apply similar procedure in the case of 10-dimensional supergravity and other higher dimensional supergravities such as Type I supergravity and massive IIA supergravity.

An important property in the oriented matroid theory is that one can associate any chirotopes with its dual. Thus, working on the framework of oriented matroids we can assure that any possible solution for 11-dimensional supergravity in terms of chirotopes will have a dual solution. This means that this kind of solution contains automatically a dual symmetry.

It is worth mentioning that using the idea of matroid bundle [24–28], Guha [29] has observed that chirotopes can be related to Nambu–Poisson structure. It may be interesting to see whether this Nambu–Poisson structure is related to 11-dimensional supergravity.

6. Chirotope and phirotope concepts

Let us start considering again the completely antisymmetric symbol

$$\varepsilon^{a_1\ldots a_d} \in \{-1, 0, 1\}. \tag{65}$$

In this section, the indices $a_1, \ldots, a_d$ run from 1 to d. This is a d-rank tensor which values are $+1$ or -1 depending on even or odd permutations of $\varepsilon^{12\ldots d}$, respectively. Moreover, $\varepsilon^{a_1\ldots d_d}$ takes the value 0 unless $a_1 \ldots a_d$ are all different. Let v_a^i be any $d \times n$ matrix over some field F, where the index i takes values in the set $E = \{1, \ldots, n\}$. Consider the object

$$\Sigma^{i_1\ldots i_d} = \varepsilon^{a_1\ldots a_d} v_{a_1}^{i_1} \ldots v_{a_d}^{i_d}, \tag{66}$$

which can also be written as

$$\Sigma^{i_1\ldots i_d} = \det(\mathbf{v}^{i_1}, \ldots, \mathbf{v}^{i_d}). \tag{67}$$

Using the ε-symbol property

$$\varepsilon^{a_1\ldots[a_d} \varepsilon^{b_1\ldots b_d]} = 0. \tag{68}$$

It is not difficult to prove that $\Sigma^{i_1\ldots i_d}$ satisfies the Grassmann–Plücker relations, namely

$$\Sigma^{i_1\ldots[i_d} \Sigma^{j_1\ldots j_d]} = 0. \tag{69}$$

We recall that the brackets in the indices of (68) and (69) mean completely antisymmetrized.

A realizable chirotope χ is defined as

$$\chi^{i_1\ldots i_d} = sign \Sigma^{i_1\ldots i_d}. \tag{70}$$

From the point of view of exterior algebra one finds that there is a close connection between Grassmann algebra and a chirotope. Let us denote by $\wedge_d R^n$ the $\binom{n}{d}$-dimensional real vector space of alternating d-forms on R^n. We recall that an element $\boldsymbol{\Sigma}$ in $\wedge_d R^n$ is said to be decomposable if

$$\boldsymbol{\Sigma} = \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_d, \tag{71}$$

for some $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d \in R^n$. It is not difficult to see that (71) can also be written as

$$\boldsymbol{\Sigma} = \frac{1}{r!} \Sigma^{i_1 \dots i_d} e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_d}, \tag{72}$$

where $e_{i_1}, e_{i_2}, \dots, e_{i_d}$ are 1-form bases in R^n and $\Sigma^{i_1 \dots i_d}$ is given in (66). This shows that $\Sigma^{i_1 \dots i_d}$ can be identified with an alternating decomposable d-form.

In order to define non-realizable chirotopes it is convenient to write the expression (69) in the alternative form

$$\sum_{k=1}^{d+1} s_k = 0, \tag{73}$$

where

$$s_k = (-1)^k \Sigma^{i_1 \dots i_{d-1} j_k} \Sigma^{j_1 \dots \hat{j}_k \dots j_{d+1}}. \tag{74}$$

Here, $j_{d+1} = i_d$ and $\hat{j}_k$ establish the notation for omitting this index. Thus, for a general definition one defines a d-rank chirotope $\chi : E^d \to \{-1, 0, 1\}$ if there exist $r_1, \dots, r_{d+1} \in R^+$ such that

$$\sum_{k=1}^{d+1} r_k s_k = 0, \tag{75}$$

with

$$s_k = (-1)^k \chi^{i_1 \dots i_{d-1} j_k} \chi^{j_1 \dots \hat{j}_k \dots j_{d+1}}, \tag{76}$$

and $k = 1, \dots, d+1$. It is evident that (73) is a particular case of (76). Therefore, there are chirotopes that may be non-realizable. Moreover, this definition of a chirotope is equivalent to various others (see Refs. [11–13] for details), but it seems that the present one is more convenient for a generalization to the complex structure setting.

The generalization of a chirotope to a phirotope is straightforward. A function $\varphi : E^d \to S^1 \cup \{0\}$ on all d-tuples of $E = \{1, \dots, n\}$ is called a d-rank phirotope if (a) φ is alternating and (b) for

$$\omega_k = (-1)^k \varphi^{i_1 \dots i_{d-1} j_k} \varphi^{j_1 \dots \hat{j}_k \dots j_{d+1}} = 0, \tag{77}$$

for $k = 1, \dots, d+1$ there exist $r_1, \dots, r_{d+1} \in R^+$ such that

$$\sum_{k=1}^{d+1} r_k \omega_k = 0. \tag{78}$$

In the case of a realizable phirotope we have

$$\Omega^{i_1 \dots i_d} = \omega\big(\det\big(\mathbf{u}^{i_1}, \dots, \mathbf{u}^{i_d}\big)\big), \tag{79}$$

where $\omega(z) \in S^1 \cup \{0\}$ and $(\mathbf{u}^{i_1} \dots \mathbf{u}^{i_d})$ are a set of complex vectors in C^d. We observe that one of the main differences between a chirotope and a phirotope is that the image of a phirotope is no longer a discrete set (see Refs. [11–13] for details).

7. Supergravity and phirotopes

As we mentioned in Section 3, maximally supersymmetric solution of 11-dimensional supergravity leads to the two conditions

$$F_{M[L_1L_2L_3}F_{L_4L_5L_6L_7]} = 0 \tag{80}$$

and

$$F_{M[P_1P_2P_3}F_{Q_1Q_2Q_3]N} = 0, \tag{81}$$

for the 4-form field strength $F = dA$ which are equivalent to the Grassmann–Plücker relations

$$F_{MP_1P_2[P_3}F_{Q_1Q_2Q_3Q_4]} = 0, \tag{82}$$

meaning that F is decomposable. Thus, according to the discussion of the previous sections one discovers that (82) establishes that F is a realizable 4-rank chirotope with a ground set $E = \{1,\ldots,11\}$. This in turn means that maximal supersymmetry in 11-dimensional supergravity is related to oriented matroid theory. Similar conclusion can be obtained for the case of 10-dimensional supergravity. Hence, one may understand the chirotope concept as the bridge between supersymmetry and the oriented matroid theory. Thus, one should expect a generalization of oriented matroid theory which would include supersymmetry. But in order to develop this idea it turns out more convenient to consider a complex structure, and this means that we need to focus on the superphirotope notion rather than on the superchirotope concept which must arise as a particular case of the former.

8. Superphirotope

The main goal of this section is to outline a possible supersymmetrization of a phirotope. By convenience we shall call superphirotope such a supersymmetric phirotope. Inspired in super p-brane theory one finds that one way to define a superphirotope, which assures supersymmetry, is as follows. First, we need to locally consider the expressions (77)–(79) in the sense that $\varphi^{i_1,\ldots,j_d}(\xi)$ is a local phirotope if

$$\omega_k = (-1)^k\varphi^{i_1\ldots i_{d-1}j_k}(\xi)\varphi^{j_1,\ldots\hat{j}_k\ldots j_{d+1}}(\xi), \tag{83}$$

for $k = 1,\ldots,d+1$ there exist $r_1,\ldots,r_{d+1} \in R^+$ such that

$$\sum_{k=1}^{d+1} r_k\omega_k(\xi) = 0. \tag{84}$$

In the case of a realizable local phirotope we have

$$\Omega^{i_1\ldots i_d}(\xi) = \omega\big(\det\big(\mathbf{u}^{i_1}(\xi),\ldots,\mathbf{u}^{i_d}(\xi)\big)\big), \tag{85}$$

where $\xi = (\xi^1,\ldots,\xi^d)$ are local coordinates of some d-dimensional manifold B. The vectors $\mathbf{v}^{i_1}(\xi),\ldots,\mathbf{v}^{i_d}(\xi)$ can be thought as vectors in the tangent space $T_\xi(B)$ at ξ. One can assume that the possibility of considering the expressions (77)–(79) in a local context may be justified in principle by the so-called matroid bundle notion (see Refs. [24–28]). Let us recall that the projective variety of decomposable forms is isomorphic to the Grassmann variety of d-dimensional linear subspaces in R^n. In turn, the Grassmann variety is the classifying space for vector bundle structures. Taking these ideas as a motivation, MacPherson developed the combinatorial differential manifold concept. The matroid bundle notion arises as a generalization of the MacPherson proposal. Roughly speaking, a matroid bundle is a structure in which at each point of the differentiable manifold an oriented matroid is attached as a fiber (see [24–28] for details).

Now, let us consider a supermanifold $\mathcal{B}$ parametrized by the local coordinates (ξ, θ) where θ are elements of the odd Grassmann algebra (anticommuting variables). We shall now consider the supersymmetric prescription

$$\mathbf{v}^i \to \pi^i = \mathbf{v}^{i_1} - i\bar{\theta}\gamma^i \partial\theta. \tag{86}$$

Here, γ^i are elements of a Clifford algebra. Using (86) one can generalize (85) in the form

$$\Psi^{i_1 \ldots i_d}(\xi, \theta) = \omega\big(\det\big(\pi^{i_1}(\xi, \theta), \ldots, \pi^{i_d}(\xi, \theta)\big)\big). \tag{87}$$

The symbol det means the superdeterminant. One should expect that (87) satisfies a kind of supersymmetric Grassmann–Plücker relations. It is not difficult to see that up to total derivative (87) is invariant under the global supersymmetric transformations

$$\delta\theta = \epsilon \tag{88}$$

and

$$\delta\mathbf{v}^{i_1} = i\bar{\epsilon}\gamma^i \partial\theta, \tag{89}$$

where ϵ is a constant complex spinor parameter.

Similarly, one can generalize the superphirotope to the non-representable case by assuming that if

$$\omega_k = (-1)^k \varphi^{i_1 \ldots i_{d-1} j_k}(\xi, \theta)\varphi^{j_1 \ldots \hat{j}_k \ldots . j_{d+1}}(\xi, \theta), \tag{90}$$

for $k = 1, \ldots, d+1$ there exist $r_1, \ldots, r_{d+1} \in R^+$ such that

$$\sum_{k=1}^{d+1} r_k \omega_k(\xi, \theta) = 0. \tag{91}$$

Of course, in the case that the complex structure is projected to the real structure one should expect that the superphirotope is reduced to the superchirotope.

With the superphirotope $\Psi^{i_1 \ldots i_d}(x, \theta)$ at hand one may consider a possible partition function

$$Z = \int D\Psi \exp(iS), \tag{92}$$

where

$$S = \frac{1}{2}\int d^d\xi\, d\theta\big(\lambda^{-1}\Psi^{i_1 \ldots i_d}(\xi, \theta)\Psi_{i_1 \ldots i_d}(\xi, \theta) - \lambda T_d^2\big) \tag{93}$$

is a Schild type action for a superphirotope. Here, λ is a Lagrange multiplier and T_d is the $(d-1)$-phirotope tension. Moreover, in a more general context the action may have the form

$$S = \frac{1}{2}\int d^d\xi\, d\theta\big(\lambda^{-1}\varphi^{i_1 \ldots i_d}(\xi, \theta)\varphi_{i_1 \ldots i_d}(\xi, \theta) - \lambda T_d^2\big). \tag{94}$$

The advantage of the actions (93) and (94) is that duality is automatically assured. In fact, in the oriented matroid theory duality is a main subject in the sense that any chirotope has an associated dual chirotope. This means that a theory described in the context of an oriented matroid automatically contains a duality symmetry. Therefore, with our prescription one is assuring not only the supersymmetry for the action (93) or (94) but also the duality symmetry.

The action (93) can be related to an ordinary super p-brane by assuming that $\Psi^{i_1\cdots i_d}(\xi,\theta)$ is a closed d-form because in that case we can write

$$\pi_a^i = \partial_a x^i - i\bar{\theta}\gamma^i\partial_a\theta. \tag{95}$$

The coordinates x^i are the p-brane bosonic coordinates. It is worth mentioning that the bosonic sector of $\Psi^{i_1\cdots i_d}(\xi,\theta)$ is a constraint of the Nambu–Poisson geometry which has been related to oriented matroid theory (see Ref. [29] for details).

It may be interesting for further research to consider the action (93) from the point of view of a superfield formalism instead of using the prescription (95). In this case one may consider a supersymmetrization in the form $\pi_a^i(\xi,\theta) = \partial_a X^i$, with X^i as a scalar superfield admitting a finite expansion in terms of θ. For instance, in four dimensions one may have

$$X^i(\xi,\theta) = x^i(\xi) + i\theta\psi^i(\xi) + \frac{i}{2}\bar{\theta}\theta B^i(\xi). \tag{96}$$

The state ψ^i denotes a Majorana spinor field, while B^i refers to an auxiliary field. By substituting (96) into (93) one should expect a splitting of (93) in several terms containing the variables $x^i(\xi)$, $\psi^i(\xi)$ and $B^i(\xi)$. The important thing is that using the prescription (96) supersymmetry becomes evident in the sense that the algebra of supersymmetry transformations is closed off the mass-shell.

Although in Section 3 we focused on 11-dimensional supergravity similar arguments can be applied to the case of 10-dimensional supergravity. Specifically, as we already mentioning by studying maximal supersymmetry in IIB supergravity Figueroa-O'Farrill and Papadopoulos [14, 15] used the vanishing of the curvature of the supercovariant derivative to derive the analogue Grassmann–Plücker formula

$$F_{LP_1P_2P_3[P_4}F^L_{Q_1Q_2Q_3Q_4]} = 0, \tag{97}$$

for the five-form $F_{LP_1P_2P_3P_4}$. Moreover, in Refs. [14,15] is proved that (97) implies that

$$F = G + {}^*G, \tag{98}$$

where G is a decomposable 5-form and ${}^*G$ denotes the 10-dimensional dual of G. This means that G and ${}^*G$ satisfy the Grassmann–Plücker relations and therefore can be identified with a 5-rank chirotope.

9. Connection with qubit theory

A connection between 4-rebits (real qubits) and the Nambu–Goto action with target 'space–time' of four time and four space dimensions ($(4+4)$-dimensions) was proposed in Ref. [5]. The motivation for this proposal came three observations. The first one is that a 4-rebit contains exactly the same number of degree of freedom as a complex 3-qubit and therefore 4-rebits are special in the sense of division algebras. Secondly, the $(4+4)$-dimensions can be splitted as $(4+4) = (3+1) + (1+3)$ and therefore they are connected with an ordinary $(1+3)$-space–time and with changed signature associated with $(3+1)$-space–time [30]. Moreover it was shown how geometric aspects of 4-rebits can be related to the chirotope concept of oriented matroid theory (see Ref. [4]).

It is worth mentioning that the discovery of new hidden discrete symmetries of the Nambu–Goto action (through the identification of the coordinates x^μ of a bosonic string, in target space of $(2+2)$-signature, with a 2×2 matrix x^{ab}) [31] leads to increase the interest in qubit theory. It turns out that the key mathematical tool in this development is the Cayley hyperdeterminant $Det(b)$ [32] of the hypermatrix $b_a{}^{bc} = \partial_a x^{bc}$. A striking result is that $Det(b)$ can also be associated with the four electric charges and four magnetic charges of an STU black hole in four dimensional string theory [33] (see also Ref. [34]). Even more surprising is the fact that $Det(b)$ makes also its appearance in quantum information theory by identifying $b_a{}^{bc}$ with a complex 3-qubit system $a_a{}^{bc}$ [35]. These coincidences, among others, have increased the interest on the qubit/black hole correspondence [36,37].

Additional motivation concerning a connection between the $(4+4)$-signature and qubit theory may arise from the following observation that $(4+4)$-dimensions can also be understood as $(4+4) = ((2+2)+(2+2))$. The importance of the signature $(2+2)$ appears in different physical scenarios, including $N=2$ strings (see Ref. [38] and references therein).

It turns out that in information theory 4-qubit is just subclass of N-qubit entanglement. In fact, the Hilbert space can be broken into the form $C^{2^N} = C^L \otimes C^l$, with $L = 2^{N-1}$ and $l=2$. Such a partition it allows a geometric interpretation in terms of the complex Grassmannian variety $Gr(L,l)$ of 2-planes in C^L via the Plücker embedding. In this case, the Plücker coordinates of Grassmannians $Gr(L,l)$ are natural invariants of the theory (see Refs. [19] and [39] for details).

However, in this context, it has been mentioned in Ref. [40], and proved in Refs. [41] and [42], that for normalized qubits the complex 1-qubit, 2-qubit and 3-qubit are deeply related to division algebras via the Hopf maps, $S^3 \xrightarrow{S^1} S^2$, $S^7 \xrightarrow{S^3} S^4$ and $S^{15} \xrightarrow{S^7} S^8$, respectively.

Consider the general complex state $|\psi\rangle \in C^{2^N}$,

$$|\psi\rangle = \sum_{a_1,a_2,\ldots,a_N=0}^{1} a_{a_1a_2\ldots a_N}|a_1a_2\ldots a_N\rangle, \tag{99}$$

where the states $|a_1a_2\ldots a_N\rangle = |a_1 > \otimes|a_2\rangle\cdots\otimes|a_N\rangle$ correspond to a standard basis of the N-qubit. For a 3-qubit (99) becomes

$$|\psi\rangle = \sum_{a_1,a_2,a_3=0}^{1} a_{a_1a_2a_3}|a_1a_2a_3\rangle, \tag{100}$$

while for 4-qubit one has

$$|\psi\rangle = \sum_{a_1,a_2,a_3,a_4=0}^{1} a_{a_1a_2a_3a_4}|a_1a_2a_3a_4\rangle. \tag{101}$$

It is interesting to make the following observations. First, one notes that $a_{a_1a_2a_3}$ has 8 complex degrees of freedom, that is 16 real degrees of freedom, while $a_{a_1a_2a_3a_4}$ contains 16 complex degrees of freedom, that is 32 real degrees of freedom. Let us denote N-rebit system (real N-qubit) by $b_{a_1a_2\ldots a_N}$. So we shall denote the corresponding 3-rebit, 4-rebit by $b_{a_1a_2a_3}$ and $b_{a_1a_2a_3a_4}$, respectively. One observes that $b_{a_1a_2a_3}$ has 8 real degrees of freedom, while $b_{a_1a_2a_3a_4}$ has 16 real degrees of freedom. Thus, by this simple (degree of freedom) counting one note that it seems more natural to associate the 4-rebit $b_{a_1a_2a_3a_4}$ with the complex 3-qubit, $a_{a_1a_2a_3}$, than with the complex 4-qubit, $a_{a_1a_2a_3a_4}$. Of course, by imposing some constraints one can always reduce the 32 real degrees of freedom of $a_{a_1a_2a_3a_4}$ to 16, and this is the kind of embedding discussed in Ref. [19].

The main idea in Ref. [5] was to make sense out of a 4-rebit in the Nambu–Goto context without loosing the important connection with a division algebra via the Hopf map $S^{15} \xrightarrow{S^7} S^8$.

Let us first show the formalism concerning the Nambu–Goto action/qubits correspondence in a space–time of $(2+2)$-signature. In the $(2+2)$-dimensions one may introduce the matrix

$$x^{ab} = \begin{pmatrix} x^1 + x^3 & x^2 + x^4 \\ x^2 - x^4 & -x^1 + x^3 \end{pmatrix}. \tag{102}$$

Using (102) the line element

$$ds^2 = dx^\mu \, dx^\nu \, \eta_{\mu\nu}, \tag{103}$$

can also be written as

$$ds^2 = \frac{1}{2} dx^{ab} \, dx^{cd} \, \varepsilon_{ac}\varepsilon_{bd}, \tag{104}$$

where

$$\eta_{\mu\nu} = diag(-1, -1, 1, 1), \tag{105}$$

is a flat metric corresponding to $(2+2)$-signature and ε_{ab} is the completely antisymmetric symbol (ε-symbol) with $\varepsilon_{12} = 1$.

On the other hand in a target space of $(4+4)$-signature one may introduce the matrices

$$x^{ab1} = \begin{pmatrix} x^1 + x^5 & x^2 + x^6 \\ x^2 - x^6 & -x^1 + x^5 \end{pmatrix}, \tag{106}$$

and

$$x^{ab2} = \begin{pmatrix} x^3 + x^7 & x^4 + x^8 \\ x^4 - x^8 & -x^3 + x^7 \end{pmatrix}. \tag{107}$$

At first sight one may consider the line element

$$ds^2 = \frac{1}{2} dx^{abc} \, dx^{def} \, \varepsilon_{ad}\varepsilon_{be}\varepsilon_{cf} \tag{108}$$

as the analogue of (104). But this vanishes identically because

$$s^{cf} \equiv dx^{abc} \, dx^{def} \, \varepsilon_{ad}\varepsilon_{be} \tag{109}$$

is a symmetric quantity, while ε_{cf} is antisymmetric. In fact, the correct line element in $(4+4)$-dimensions turns out to be

$$ds^2 = \frac{1}{2} dx^{abc} \, dx^{def} \, \varepsilon_{ad}\varepsilon_{be}\eta_{cf}. \tag{110}$$

Notice that we have changed the last ε-symbol in (110) for the η-symbol. Here, $\eta_{cf} = diag(-1, 1)$. Moreover, one can prove that (103), with

$$\eta_{\mu\nu} = (-1, -1, -1, -1, +1, +1, +1, +1), \tag{111}$$

follows from (110).

Similarly, the world sheet metric in $(2+2)$-dimensions

$$\gamma_{ab} = \partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu} = \gamma_{ba}, \tag{112}$$

can be written as

$$\gamma_{ab} = \frac{1}{2}\partial_a x^{cd}\partial_b x^{ef}\varepsilon_{ce}\varepsilon_{df}. \tag{113}$$

While in $(4+4)$-dimensions, one has

$$\gamma_{ab} = \frac{1}{2}b_a{}^{cdg}b_b{}^{fhl}\varepsilon_{cf}\varepsilon_{dh}\eta_{gl}, \tag{114}$$

with

$$b_a{}^{cdg} \equiv \partial_a x^{cdg}. \tag{115}$$

In $(2+2)$-dimensions one can write the determinant of γ_{ab},

$$\det\gamma = \frac{1}{2}\varepsilon^{ab}\varepsilon^{cd}\gamma_{ac}\gamma_{bd}, \tag{116}$$

in the hyperdeterminant form

$$\det\gamma = \frac{1}{2}\varepsilon^{ab}\varepsilon^{cd}\varepsilon_{eg}\varepsilon_{fh}\varepsilon_{ru}\varepsilon_{sv}b_a{}^{ef}b_c{}^{gh}b_b{}^{rs}b_d{}^{uv} = Det(b), \tag{117}$$

with

$$b_a{}^{cd} \equiv \partial_a x^{cd}. \tag{118}$$

Thus, this proves that the Nambu–Goto action

$$S = \frac{1}{2}\int d^2\xi\sqrt{\det\gamma}, \tag{119}$$

for a flat target "spacetime" with $(2+2)$-signature can also be written as [31]

$$S = \frac{1}{2}\int d^2\xi\sqrt{Det(b)}. \tag{120}$$

While in $(4+4)$-dimensions the determinant

$$\det\gamma = \frac{1}{2}\varepsilon^{ab}\varepsilon^{cd}\gamma_{ac}\gamma_{bd}, \tag{121}$$

can be written as

$$\det\gamma = \frac{1}{2}c^{efrs}c^{ghuv}\varepsilon_{eg}\varepsilon_{fh}\varepsilon_{ru}\varepsilon_{sv}, \tag{122}$$

where

$$c^{efrs} \equiv \left(-\varepsilon^{ab}b_a{}^{ef1}b_b{}^{rs1} + \varepsilon^{ab}b_a{}^{ef2}b_b{}^{rs2}\right), \tag{123}$$

One recognizes in (121) the hyperdeterminant of the hypermatrix c^{efrs}. So, we can write

$$\det\gamma = Det(c). \tag{124}$$

This proves that the Nambu–Goto action in $(4+4)$-dimensions

$$S = \frac{1}{2}\int d^2\xi\sqrt{\det\gamma}, \tag{125}$$

can also be written as [5]

$$S = \frac{1}{2}\int d^2\xi \sqrt{Det(c)}. \tag{126}$$

Moreover, one may connect qubits with the chirotope concept in oriented matroid theory. In space of $(2+2)$-signature one writes

$$\det\gamma = \frac{1}{2}\sigma^{\mu\nu}\sigma^{\alpha\beta}\eta_{\mu\alpha}\eta_{\nu\beta}, \tag{127}$$

where

$$\sigma^{\mu\nu} = \varepsilon^{ab} b_a^\mu b_b^\nu. \tag{128}$$

Here, we have used the definition

$$b_a^\mu \equiv \partial_a x^\mu. \tag{129}$$

Since $\sigma^{\mu\nu}$ satisfies the identity $\sigma^{\mu[\nu}\sigma^{\alpha\beta]} \equiv 0$, one can verify that $\chi^{\mu\nu} = sign\,\sigma^{\mu\nu}$ satisfies the Grassmann–Plücker relation

$$\chi^{\mu[\nu}\chi^{\alpha\beta]} = 0, \tag{130}$$

and therefore $\chi^{\mu\nu}$ is a realizable chirotope (see Refs. [4,5] and references therein).

The Grassmann–Plücker relation (130) implies that the ground set is

$$E = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\} \tag{131}$$

and the alternating map

$$\chi^{\mu\nu} \to \{-1,0,1\}, \tag{132}$$

determine a 2-rank realizable oriented matroid $M = (E,\chi^{\mu\nu})$. The collection of bases for this oriented matroid is

$$\mathcal{B} = \big\{\{\mathbf{1},\mathbf{2}\},\{\mathbf{1},\mathbf{3}\},\{\mathbf{1},\mathbf{4}\},\{\mathbf{2},\mathbf{3}\},\{\mathbf{2},\mathbf{4}\},\{\mathbf{3},\mathbf{4}\}\big\}, \tag{133}$$

which can be obtained by just given values to the indices μ and ν in $\chi^{\mu\nu}$. Indeed, the pair $(E,\mathcal{B})$ determines a 2-rank uniform non-oriented ordinary matroid.

In the case of qubits, one may introduce the underlying ground bitset (from bit and set)

$$\mathcal{E} = \{1,2\} \tag{134}$$

and the pre-ground set

$$E_0 = \big\{(1,1),(1,2),(2,1),(2,2)\big\}. \tag{135}$$

It turns out that E_0 and E can be related by establishing the identification

$$\begin{aligned} &(1,1) \leftrightarrow \mathbf{1}, \qquad (1,2) \leftrightarrow \mathbf{2},\\ &(2,1) \leftrightarrow \mathbf{3}, \qquad (2,2) \leftrightarrow \mathbf{4}. \end{aligned} \tag{136}$$

Observe that (136) is equivalent of making the identification of indices $\{a,b\} \leftrightarrow \mu,\ldots$, etc. In fact, considering these identifications the family of bases (133) becomes

$$\begin{aligned} \mathcal{B}_0 = \big\{&\big\{(1,1),(1,2)\big\},\big\{(1,1),(2,1)\big\},\big\{(1,1),(2,2)\big\},\\ &\big\{(1,2),(2,1)\big\},\big\{(1,2),(2,2)\big\},\big\{(2,1),(2,2)\big\}\big\}. \end{aligned} \tag{137}$$

Using the definition

$$\sigma^{efrs} \equiv \varepsilon^{ab} b_a{}^{ef} b_b{}^{rs}, \tag{138}$$

one can show that

$$\det \gamma = \frac{1}{2} \sigma^{efrs} \sigma^{ghuv} \varepsilon_{eg} \varepsilon_{fh} \varepsilon_{ru} \varepsilon_{sv} = Det(b). \tag{139}$$

This establishes a link between the hyperdeterminant and "chirotope" structure.

In $(4+4)$-dimensions one may introduce the quantity

$$c^{\mu\nu} = \left(-\varepsilon^{ab} b_a^{\mu 1} b_b^{\nu 1} + \varepsilon^{ab} b_a^{\mu 2} b_b^{\nu 2}\right). \tag{140}$$

Here, one has considered the definitions

$$b_a^{\mu 1} = \partial_a x^\mu \tag{141}$$

and

$$b_a^{\mu 2} = \partial_a y^\mu. \tag{142}$$

In turn this means that we can write

$$c^{\mu\nu} = \left(-\varepsilon^{ab} \partial_a x^\mu \partial_b x^\nu + \varepsilon^{ab} \partial_a y^\mu \partial_b y^\nu\right). \tag{143}$$

One recognizes in this expression the Plücker coordinates for both cases $u_a^\mu = \partial_a x^\mu$ and $v_a^\mu = \partial_a y^\mu$.

This proves that both quantities σ^{efrs} and c^{efrs} (qubitopes), belongs to an underlying structure $Q = (\mathcal{E}, E_0, B_0)$ called qubitoid. The word "qubitoid" is a short word for qubit-matroid.

One may now be interested to see how the 4-rebit $b_{a_1 a_2 a_3 a_4}$ is connected with $a_{a_1 a_2 a_3}$. The simplest (but no the most general) possibility seems to be

$$a_{a_1 a_2 a_3} = b_{a_1 a_2 a_3 1} + i b_{a_1 a_2 a_3 2}. \tag{144}$$

In turn this implies

$$a_{a_1}{}^{a_2 a_3} = \partial_{a_1} x^{a_2 a_3 1} + i \partial_{a_1} x^{a_2 a_3 2} = \partial_{a_1}\left(x^{a_2 a_3 1} + i x^{a_2 a_3 2}\right). \tag{145}$$

Hence the 3-qubit $a_{a_1}{}^{a_2 a_3}$ is related to the two 2-rebits states $x^{a_2 a_3 1}$ and $x^{a_2 a_3 2}$. This observation may help eventually to relate 4-rebit $b_{a_1 a_2 a_3 a_4}$ with the Hopf fibration $S^{15} \xrightarrow{S^7} S^8$. In fact, a normalization of the complex states $a_{a_1 a_2 a_3}$ leads to the 15-dimensional sphere S^{15} which under the Hopf map, admit parametrization of the parallelizable sphere S^7 fibration over S^8.

It turns out, that just as the norm group of quaternions is $SO(4) = S^3 \times S^3$, the norm group of octonions is $SO(8) = S^7 \times S^7 \times G_2$ (see Ref. [22]). This is due to the fact that considering the 28 generators $J_{\mu\nu}$ of $SO(8)$ and the octonionic o_i structure constants ψ_{ijk} $(o_i o_j = (\psi_i)_j^k o_k)$ one can choose a basis $M_i = J_{0i}$, $K_i = \frac{1}{2}\psi_{ijk} J^{jk}$ and $\Gamma_{ij} = 2J_{ij} - \frac{1}{3!2}\varepsilon_{ijklmns}\psi^{mns} J^{kl}$ for $SO(8)$ satisfying the algebra,

$$\begin{aligned}
[M_i, M_j] &= \frac{1}{3}\left(\psi_{ijk} K^k + \Gamma_{ij}\right), \\
[K_i, K_j] &= -\psi_{ijk} K^k + \Gamma_{ij}, \\
[K_i, M_j] &= \psi_{ijk} M^k, \\
[\Gamma_{ij}, \Gamma_{ij}] &= C_{ijklms} \Gamma^{ms}.
\end{aligned} \tag{146}$$

Here,

$$C_{ijklms} = \mathcal{A}\left(\frac{3}{2}\delta_{il}\delta_{jm}\delta_{ls} - \frac{1}{8}\psi_{ijm}\psi_{kls}\right). \tag{147}$$

The $\mathcal{A}$ in (147) stands for antisymmetrization of i and j, k and l, and m and s. The relevant aspect is that from the algebra (146) one discovers that the operators $M_i + K_i$ and $M_i - K_i$ commute and therefore they corresponds to independent 7-spheres S_R^7 and S_L^7. In this way the decomposition $M_i + K_i$, $M_i - K_i$ and L_{ij} of the generators of the group $SO(8)$ leads to the decomposition $SO(8) = S_R^7 \times S_L^7 \times G_2$. Roughly speaking one can say that the coset $SO(8)/SO(7)$ is associated with M_i, the coset $SO(7)/G_2$ with K_i and C_{ijklms} determines the structure constants of G_2 (see Ref. [22] for details).

Since in the $4+4$-signature the relevant group is $SO(4,4)$. One find that 8-dimensional spinor representation associated with *spin*(8) can be written as

$$\begin{pmatrix} 0 & (\psi_i)_j^k \\ -(\psi_i)_j^k & 0 \end{pmatrix}. \tag{148}$$

This means that when $SO(8)$ decomposed under the subgroup $SO(4) \times SO(4)$ one gets irreducible representation

$$8 \longrightarrow (4,1) + (1,4). \tag{149}$$

Thus, in the case of $SO(4,4)$ one may consider decomposition under the subgroup $SO(2,2) \times SO(2,2)$ obtaining,

$$(4+4) \longrightarrow \big((2+2),1\big) + \big(1,(2+2)\big). \tag{150}$$

It turns out that these two direct summands correspond to the variables x^{ab1} and x^{ab2}. This explains why dx^{abc}, is contracted with η_{ab}, and no with ε_{ab}.

The above scenario can be generalized for class of N-qubits, with the Hilbert space in the form $C^{2^N} = C^L \otimes C^l$, with $L = 2^{N-n}$ and $l = 2^n$. Such a partition allows a geometric interpretation in terms of the complex Grassmannian variety $Gr(L,l)$ of l-planes in C^L via the Plücker embedding [19]. In the case of N-rebits one can set an $L \times l$ matrix variable b_a^μ, $\mu = 1,2,\dots,L$, $a = 1,2\dots,l$, of $2^N = L \times l$ associated with the variable $b_{a_1a_2\dots a_N}$, with $a_1, a_2, \dots$ etc. taking values in the set $\{1,2\}$. In fact, one can take the first $N-n$ terms in $b_{a_1a_2\dots a_N}$ are represented by the index μ in b_a^μ, while the remaining n terms are considered by the index a in b_a^μ. One of the advantage of this construction is that the Plücker coordinates associated with the real Grassmannians b_a^μ are natural invariants of the theory. Since oriented matroid theory leads to the chirotope concept which is also defined in terms Plücker coordinates these developments establishes a possible link between chirotopes, qubitoids and p-branes.

Moreover, since it has been shown [43] that the Duff discovery of manifest $SL(2,R) \times SL(2,R) \times SL(2,R)$ symmetry of the Nambu–Goto action can be extended to the Green–Schwarz $N=2$ string action it seems interesting to see whether the developments presented in this section may provide a useful mathematical tools in this context. The central observation in this case is that the Cayley hyperdeterminant in the supersymmetric system is also related to ordinary determinant in the form

$$\mathcal{D}et(\Pi_{i\alpha\dot\beta}) = \epsilon^{ij}\epsilon^{kl}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}\epsilon^{\dot\alpha\dot\beta}\epsilon^{\dot\gamma\dot\delta}\Pi_{i\alpha\dot\alpha}\Pi_{i\beta\dot\beta}\Pi_{i\gamma\dot\gamma}\Pi_{i\delta\dot\delta} = \det(\Sigma_{ij}), \tag{151}$$

with

$$\Sigma_{ij} = \eta_{\bar{a}\bar{b}} \Pi_i^{\bar{a}} \Pi_j^{\bar{b}}. \tag{152}$$

Here, $\eta_{\bar{a}\bar{b}} = diag(-,-,+,+)$ is the $(2+2)$-dimensional flat space–time metric and $\Pi_i^{\bar{a}} = (\partial_i Z^M) E_M^{\bar{a}}$, with the target superspace coordinates Z^M (see Ref. [43] for details). This means that it must be possible to make the expression (117), and therefore the action (119), supersymmetric. In turn this may motivate to consider supersymmetric aspects in the context of qubit theory for a space–time in $(4+4)$-dimensions.

10. Final remarks

It is evident from the discussion of the previous sections that the Grassmann–Plücker relations play a central role on a number of links between different physical and mathematical scenarios including Grassmannian varieties, 11-dimensional supergravity, qubit theory, p-branes and oriented matroid theory. If a p-form satisfies the Grassmann–Plücker relations then such a p-form is decomposable. An application of this result to maximally supersymmetric solutions of 11-dimensional supergravity opens the possibility for writing the 4-form field strength $F_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4}$ in terms of some kind of gauge field $F_{\hat{\mu}}^a$. Thus, following this kind of though one is lead to look for the analogue $F_{\hat{\mu}}^a$ for the Bianchi identities $dF = 0$ associated with $F_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4}$. In this case, it is found that $F_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4}$ can be written in terms of a gauge field $A_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3}$ in the form $F_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4} = \partial_{[\hat{\mu}_1} A_{\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4]}$. So, the question arises whether the Bianchi identities and the Grassmann–Plücker relations associated with a general p-form field strength $F_{\mu_1\ldots\mu_p}$ are connected. In other word, the challenge is to know what is the relation between $F_{\hat{\mu}_1}^{a_1}$ and $A_{\mu_1\ldots\mu_{p-1}}$.

Let us assume that a local p-form $F_{\mu_1\ldots\mu_p}$ satisfies the Grassmann–Plücker relations

$$F_{\mu_1\ldots[\mu_p} F_{\nu_1\ldots\nu_p]} = 0. \tag{153}$$

According to our previous discussion one knows that (153) implies that $F_{\mu_1\ldots\mu_p}$ is decomposable. This means that $F_{\mu_1\ldots\mu_p}$ can be written as

$$F_{\mu_1\ldots\mu_p} = \varepsilon_{a_1\ldots a_p} F_{\mu_1}^{a_1} \ldots F_{\mu_p}^{a_p}. \tag{154}$$

Now, let us assume that $dF = 0$. In tensorial notation this means

$$\partial_{\mu_{p+1}} F_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \wedge dx^{\mu_{p+1}} = 0. \tag{155}$$

But from (154) one obtains

$$\varepsilon_{a_1\ldots a_p} \partial_{\mu_{p+1}} \left(F_{\mu_1}^{a_1} \ldots F_{\mu_p}^{a_p}\right) dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \wedge dx^{\mu_{p+1}} = 0, \tag{156}$$

which implies

$$\varepsilon_{a_1\ldots a_p} \partial_{\mu_{p+1}} \left(F_{\mu_1}^{a_1}\right) F_{\mu_2}^{a_2} \ldots F_{\mu_p}^{a_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \wedge dx^{\mu_{p+1}} = 0. \tag{157}$$

So, one discovers that a solution of (157) is given by

$$F_\mu^a = \frac{\partial \lambda^a}{\partial x^\mu}. \tag{158}$$

Consequently, the expression (154) becomes

$$F_{\mu_1\ldots\mu_p} = \varepsilon_{a_1\ldots a_p} \frac{\partial \lambda^{a_1}}{\partial x^{\mu_1}} \cdots \frac{\partial \lambda^{a_p}}{\partial x^{\mu_p}}. \tag{159}$$

In turn this expression leads to

$$F^{\mu_1\ldots\mu_p} = \varepsilon^{a_1\ldots a_p}\frac{\partial x^{\mu_1}}{\partial \lambda^{a_1}}\cdots\frac{\partial x^{\mu_p}}{\partial \lambda^{a_p}}. \tag{160}$$

On the other hand (155) implies that

$$F_{\mu_1\ldots\mu_p} = \partial_{[\mu_p} A_{\mu_1\ldots\mu_{p-1}]}. \tag{161}$$

Hence, using (159) one obtains

$$A_{\mu_1\ldots\mu_{p-1}} = \varepsilon_{a_1\ldots a_p}\lambda^{a_p}\frac{\partial \lambda^{a_1}}{\partial x^{\mu_1}}\cdots\frac{\partial \lambda^{a_{p-1}}}{\partial x^{\mu_{p-1}}}. \tag{162}$$

This establishes a connection between the Bianchi identity $dF = 0$ and the Grassmann–Plücker relations. It is worth mentioning that the expression (160) is a key tool in the p-brane theory, when one writes the Nambu–Goto action for p-branes in Schild type formalism (see Ref. [3] for details).

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