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Bull. Sci. math. 130 (2006) 71–86

BULLETIN DES SCIENCES MATHÉMATIQUES

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Bifurcations of limit cycles from quintic Hamiltonian systems with a double figure eight loop ☆,☆☆

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Received 5 July 2005; accepted 31 July 2005

Available online 2 November 2005

Abstract

This paper deals with Liénard equations of the form $\dot{x} = y$, $\dot{y} = P(x) + yQ(x, y)$, with P and Q polynomials of degree 5 and 4 respectively. Attention goes to perturbations of the Hamiltonian vector fields with an elliptic Hamiltonian of degree six, exhibiting a double figure eight loop. The number of limit cycles and their distributions are given by using the methods of bifurcation theory and qualitative analysis.

Keywords: Limit cycle; Homoclinic bifurcation; Heteroclinic bifurcation; Double figure eight loop

1. Introduction and main results

A part of the well-know Hilbert's 16th problem is to consider the existence of maximal number of limit cycles for a general planar polynomial system. In general, this is a very difficult question and it has been studied by many mathematicians (see [1–15], for example). Recently many authors have a great interest for the systems which exhibit a eight figure loop and the Hamiltonian has the form

$$H_{n+1} = \frac{y^2}{2} + P_{n+1}(x),$$

☆ The work was supported in part by Australia Research Counsil under the Discovery Projects scheme (grant ID: DP0559111).

☆☆ Supported by Shanghai Leading Academic Discipline Project #T0401.

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0007-4497/$ – see front matter
doi:10.1016/j.bulsci.2005.07.001

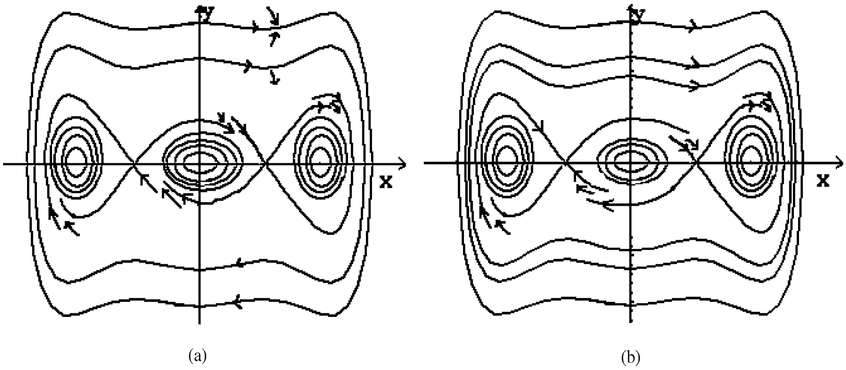

Fig. 1.

where $P_{n+1}(x)$ is a polynomial in x of degree $n+1$, see [2,3,5,6,8,12,15]. The key point used in [2–5,15] is to find simple zeros of a Melnikov function which is also called a Abelian integral.

As we know, when we study Hopf bifurcation for a planar polynomial system a typical way to find limit cycles is to change the stability of a focus. [6,10] used this idea to find limit cycles near a homoclinic loop or a heteroclinic loop. That is, a limit cycle can be bifurcated from a homoclinic loop or a heteroclinic loop when its stability changes. A final limit cycle can be obtained by making the homoclinic or heteroclinic loop broken. Then the method was developed to investigate the limit cycle bifurcation from a double homoclinic loop by Han and Chen [8], and was used to study the existence of 11 limit cycles for some cubic system by [11,13,14]. In this paper, we develop this method to study the bifurcations of limit cycles for quintic system with the double figure eight loop, consisting of two homoclinic loops and a heteroclinic loop.

Consider the following perturbed Hamiltonian system

$$\begin{aligned} \dot{x} &= y + \varepsilon\big(a_{10}x + a_{30}x^3 + a_{12}xy^2 + a_{14}xy^4 + a_{50}x^5 + a_{32}x^3y^2\big) \\ &\equiv f(x,y) + \varepsilon P(x,y), \\ \dot{y} &= -x^5 - bx^3 - x + \varepsilon\big(b_{01}y + b_{21}x^2y + b_{03}y^3 + b_{05}y^5 + b_{41}x^4y + b_{23}x^2y^3\big) \\ &\equiv g(x,y) + \varepsilon Q(x,y), \end{aligned} \tag{1.1$_\varepsilon$}$$

where b is a negative constant with $b < -\frac{4}{\sqrt{3}}$ and $\varepsilon > 0$ is small. We consider the coefficients a_{ij} and b_{ij} in $(1.1)_\varepsilon$ as parameters. Our main result can be stated as follows.

Theorem. *Let $b = -\frac{5}{2}$. Then system $(1.1)_\varepsilon$ can have* 14 *limit cycles and two different distributions are given in Fig.* 1.

2. The properties of system $(1.1)_\varepsilon$ and some preliminaries

In what follows, we are going to obtain a complete analysis of system $(1.1)_\varepsilon$. From $(1.1)_\varepsilon$, we know that system $(1.1)_0$ has the first integral of the form

$$H(x,y) = \frac{y^2}{2} + \frac{x^2}{2} + \frac{bx^4}{4} + \frac{x^6}{6}$$

and the phase portraits is shown in Fig. 2.

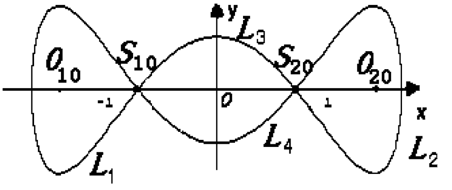

Fig. 2.

$H=0$ and $H=-\frac{b}{4}+\frac{b^3}{24}+\frac{\sqrt{b^2-4}}{6}-\frac{b^2\sqrt{b^2-4}}{24}$ correspond to the centers $O(0,0)$, $O_{10}=(-\sqrt{-b+\sqrt{b^2-4}}/\sqrt{2},0)$ and $O_{20}=(\sqrt{-b+\sqrt{b^2-4}}/\sqrt{2},0)$ respectively.

$$L_1=\left\{(x,y)\mid H(x,y)=-\frac{b}{4}+\frac{b^3}{24}-\frac{\sqrt{b^2-4}}{6}+\frac{b^2\sqrt{b^2-4}}{24},\right.$$
$$\left.-\sqrt{-\frac{b}{2}+\sqrt{b^2-4}}<x<-\frac{\sqrt{-b-\sqrt{b^2-4}}}{\sqrt{2}}\right\}$$

corresponds to the saddle point $S_{10}=(-\sqrt{-b-\sqrt{b^2-4}}/\sqrt{2},0)$ and homoclinic loop,

$$L_2=\left\{(x,y)\mid H(x,y)=-\frac{b}{4}+\frac{b^3}{24}-\frac{\sqrt{b^2-4}}{6}+\frac{b^2\sqrt{b^2-4}}{24},\right.$$
$$\left.\frac{\sqrt{-b-\sqrt{b^2-4}}}{\sqrt{2}}<x<\sqrt{-\frac{b}{2}+\sqrt{b^2-4}}\right\}$$

corresponds to the saddle point $S_{20}=(\sqrt{-b-\sqrt{b^2-4}}/\sqrt{2},0)$ and homoclinic loop, $L=L_3\cup S_{20}\cup L_4\cup S_{10}$ is a 2-polycycle, where

$$L_3=\left\{(x,y)\mid H(x,y)=-\frac{b}{4}+\frac{b^3}{24}-\frac{\sqrt{b^2-4}}{6}+\frac{b^2\sqrt{b^2-4}}{24},\right.$$
$$\left.-\frac{\sqrt{-b-\sqrt{b^2-4}}}{\sqrt{2}}<x<\frac{\sqrt{-b-\sqrt{b^2-4}}}{\sqrt{2}},\ y\geqslant 0\right\}$$

and

$$L_4=\left\{(x,y)\mid H(x,y)=-\frac{b}{4}+\frac{b^3}{24}-\frac{\sqrt{b^2-4}}{6}+\frac{b^2\sqrt{b^2-4}}{24},\right.$$
$$\left.-\frac{\sqrt{-b-\sqrt{b^2-4}}}{\sqrt{2}}<x<\frac{\sqrt{-b-\sqrt{b^2-4}}}{\sqrt{2}},\ y\leqslant 0\right\}$$

are heteroclinic orbits connecting hyperbolic saddle points S_{10} and S_{20} and L is clockwise oriented. Let $\Gamma=L_1\cup L_2\cup L_3\cup L_4$ is the double figure eight loop, see Fig. 2. Obviously, system $(1.1)_\varepsilon$ has still five critical points $O(0,0)$, $O_{1\varepsilon}$, $O_{2\varepsilon}$, $S_{1\varepsilon}$, $S_{2\varepsilon}$. For ε small, system $(1.1)_\varepsilon$ has separatrices L_i^u and L_i^s, $i=1,2,3,4$, near the saddle points $S_{i\varepsilon}$, $i=1,2$, and $L_1^u\cup L_3^u$ and $L_2^u\cup L_4^u$ are the unstable manifolds of $S_{1\varepsilon}$ and $S_{2\varepsilon}$ respectively, and $L_1^s\cup L_3^s$ and $L_2^s\cup L_4^s$ are the stable manifolds of $S_{1\varepsilon}$ and $S_{2\varepsilon}$ respectively.

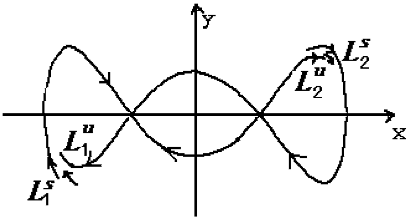

Fig. 3.

Let $\delta = (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) = (a_{10} + b_{01}, 3a_{30} + b_{21}, 5a_{50} + b_{41}, a_{12} + 3b_{03}, a_{14} + 5b_{05})$, $\delta_6 = 3(a_{32} + b_{23}) > 0$. We take δ as a vector parameter. Recall that the directed distance from L_i^u to L_i^s is measured by

$$d_i(\varepsilon, \delta) = \varepsilon N_i M_i(\delta) + \mathrm{O}(\varepsilon^2), \tag{2.1}$$

where $N_i > 0$, $i = 1, 2, 3, 4$, is a constant, and

$$M_i(\delta) = \int\limits_{L_i} Q\,\mathrm{d}x - P\,\mathrm{d}y. \tag{2.2}$$

For example, if $d_1(\varepsilon, \delta) < 0, d_2(\varepsilon, \delta) < 0$, the relative position of L_i^u and L_i^s is shown in Fig. 3, $i = 1, 2$.

For the expression of $M_i(\delta)$, $i = 1, 2, 3, 4$, by (2.1) and (2.2), we have

Lemma 2.1. *For* $(1.1)_\varepsilon$, *we have*

$$M_i(\delta) = \delta_1 B_{01} + \delta_2 B_{21} + \delta_3 B_{41} + \frac{\delta_4}{3} B_{03} + \frac{\delta_5}{5} B_{05} + \frac{\delta_6}{3} B_{23}, \quad i = 1, 2,$$

$$M_i(\delta) = \delta_1 A_{01} + \delta_2 A_{21} + \delta_3 A_{41} + \frac{\delta_4}{3} A_{03} + \frac{\delta_5}{5} A_{05} + \frac{\delta_6}{3} A_{23}, \quad i = 3, 4,$$

where

$$A_{01} = \int\limits_{L_3} y\,\mathrm{d}x = \int\limits_{L_4} y\,\mathrm{d}x, \qquad A_{21} = \int\limits_{L_3} x^2 y\,\mathrm{d}x = \int\limits_{L_4} x^2 y\,\mathrm{d}x,$$

$$A_{03} = \int\limits_{L_3} y^3\,\mathrm{d}x = \int\limits_{L_4} y^3\,\mathrm{d}x, \qquad A_{05} = \int\limits_{L_3} y^5\,\mathrm{d}x = \int\limits_{L_4} y^5\,\mathrm{d}x,$$

$$A_{23} = \int\limits_{L_3} x^2 y^3\,\mathrm{d}x = \int\limits_{L_4} x^2 y^3\,\mathrm{d}x, \qquad A_{41} = \oint\limits_{L_1} x^4 y\,\mathrm{d}x = \oint\limits_{L_2} x^4 y\,\mathrm{d}x,$$

$$B_{01} = \oint\limits_{L_1} y\,\mathrm{d}x = \oint\limits_{L_2} y\,\mathrm{d}x, \qquad B_{21} = \oint\limits_{L_1} x^2 y\,\mathrm{d}x = \oint\limits_{L_2} x^2 y\,\mathrm{d}x,$$

$$B_{03} = \oint\limits_{L_1} y^3\,\mathrm{d}x = \oint\limits_{L_2} y^3\,\mathrm{d}x, \qquad B_{05} = \oint\limits_{L_1} y^5\,\mathrm{d}x = \oint\limits_{L_2} y^5\,\mathrm{d}x,$$

$$B_{23} = \oint_{L_1} x^2 y^3 \, dx = \oint_{L_2} x^2 y^3 \, dx, \qquad B_{41} = \oint_{L_1} x^4 y \, dx = \oint_{L_2} x^4 y \, dx.$$

Proof. We only proof the expression of $M_i(\delta)$, $i = 1, 2$. Using the same arguments, we can obtain the expression of $M_3(\delta)$ and $M_4(\delta)$. Notice that

$$-\oint_{L_1} x \, dy = \oint_{L_1} y \, dx \equiv B_{01}, \qquad -\frac{1}{3}\oint_{L_1} x^3 \, dy = \oint_{L_1} x^2 y \, dx \equiv B_{21},$$
$$-3\oint_{L_1} xy^2 \, dy = \oint_{L_1} y^3 \, dx \equiv B_{03}, \qquad -5\oint_{L_1} xy^4 \, dy = \oint_{L_1} y^5 \, dx \equiv B_{05},$$
$$-\frac{1}{5}\oint_{L_1} x^5 \, dy = \oint_{L_1} x^4 y \, dx \equiv B_{41}, \qquad -\oint_{L_1} x^3 y^2 \, dy = \oint_{L_1} x^2 y^3 \, dx \equiv B_{23}.$$

Eq. (2.2) and the straightforward computation gives

$$\begin{aligned} M_1(\delta) = M_2(\delta) &= (a_{10} + b_{01}) \oint_{L_1} y \, dx + (b_{21} + 3a_{30}) \oint_{L_1} x^2 y \, dx \\ &\quad + \left(b_{03} + \frac{1}{3}a_{12}\right) \oint_{L_1} y^3 \, dx + \left(b_{05} + \frac{1}{5}a_{14}\right) \oint_{L_1} y^5 \, dx \\ &\quad + (b_{41} + 5a_{50}) \oint_{L_1} x^4 y \, dx + (a_{32} + b_{23}) \oint_{L_1} x^2 y^3 \, dx \\ &= \delta_1 B_{01} + \delta_2 B_{21} + \delta_3 B_{41} + \frac{\delta_4}{3} B_{03} + \frac{\delta_5}{5} B_{05} + \frac{\delta_6}{3} B_{23}. \end{aligned}$$

The proof is completed. □

Using Mathematics 4.0, for $b = -\frac{5}{2}$, we have

$$A_{01} = 2\int_0^{\frac{\sqrt{2}}{2}} \sqrt{\frac{11}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \, dx \doteq 0.44296,$$

$$A_{21} = 2\int_0^{\frac{\sqrt{2}}{2}} x^2 \sqrt{\frac{11}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \, dx \doteq 0.0433278,$$

$$A_{03} = 2\int_0^{\frac{\sqrt{2}}{2}} \left(\sqrt{\frac{11}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2}\right)^3 dx \doteq 0.0688018,$$

$$A_{05} = 2\int_0^{\frac{\sqrt{2}}{2}} \left(\sqrt{\frac{11}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2}\right)^5 dx \doteq 0.01268728,$$

$$A_{23}=2\int_0^{\frac{\sqrt{2}}{2}} x^2\left(\sqrt{\frac{11}{48}-\frac{x^6}{3}+\frac{5x^4}{4}-x^2}\right)^3 \mathrm{d}x \doteq 0.00365236,$$

$$A_{41}=2\int_0^{\frac{\sqrt{2}}{2}} x^4\sqrt{\frac{11}{48}-\frac{x^6}{3}+\frac{5x^4}{4}-x^2}\,\mathrm{d}x \doteq 0.00916792,$$

$$B_{01}=2\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{11}}{2}} \sqrt{\frac{11}{48}-\frac{x^6}{3}+\frac{5x^4}{4}-x^2}\,\mathrm{d}x \doteq 0.91058,$$

$$B_{21}=2\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{11}}{2}} x^2\sqrt{\frac{11}{48}-\frac{x^6}{3}+\frac{5x^4}{4}-x^2}\,\mathrm{d}x \doteq 1.54361,$$

$$B_{03}=2\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{11}}{2}} \left(\sqrt{\frac{11}{48}-\frac{x^6}{3}+\frac{5x^4}{4}-x^2}\right)^3 \mathrm{d}x \doteq 0.34257,$$

$$B_{05}=2\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{11}}{2}} \left(\sqrt{\frac{11}{48}-\frac{x^6}{3}+\frac{5x^4}{4}-x^2}\right)^5 \mathrm{d}x \doteq 0.1543964,$$

$$B_{23}=2\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{11}}{2}} x^2\left(\sqrt{\frac{11}{48}-\frac{x^6}{3}+\frac{5x^4}{4}-x^2}\right)^3 \mathrm{d}x \doteq 0.62931,$$

$$B_{41}=2\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{11}}{2}} x^4\sqrt{\frac{11}{48}-\frac{x^6}{3}+\frac{5x^4}{4}-x^2}\,\mathrm{d}x \doteq 2.88248.$$

Notice the symmetry of system $(1.1)_\varepsilon$, we have $d_1(\varepsilon,\delta)=d_2(\varepsilon,\delta)$ and $d_3(\varepsilon,\delta)=d_4(\varepsilon,\delta)$. Consider the equations $d_i(\varepsilon,\delta)=0$, $i=1,2,3,4$. The implicit function theorem implies that two functions

$$\phi_1(\varepsilon,\delta_2,\delta_3,\delta_4,\delta_5,\delta_6)=-\left(\frac{B_{21}}{B_{01}}\delta_2+\frac{B_{41}}{B_{01}}\delta_3+\frac{B_{03}}{3B_{01}}\delta_4+\frac{B_{05}}{5B_{01}}\delta_5+\frac{B_{23}}{3B_{01}}\delta_6\right)+\mathrm{O}(\varepsilon)$$

and

$$\begin{aligned}\phi_2(\varepsilon,\delta_3,\delta_4,\delta_5,\delta_6)=\frac{1}{A_{21}B_{01}-A_{01}B_{21}}\bigg[&(A_{01}B_{41}-A_{41}B_{01})\delta_3+\frac{A_{01}B_{03}-A_{03}B_{01}}{3}\delta_4\\&+\frac{A_{01}B_{05}-A_{05}B_{01}}{5}\delta_5+\frac{A_{01}B_{23}-A_{23}B_{01}}{3}\delta_6\bigg]+\mathrm{O}(\varepsilon)\end{aligned}$$

exist such that for $\varepsilon > 0$ small

$$d_1(\varepsilon,\delta) = d_2(\varepsilon,\delta) \geqslant 0 \text{ (resp., } < 0) \quad \text{if and only if} \quad \delta_1 \geqslant \phi_1 \text{ (resp., } < \phi_1),$$

and

$$d_3(\varepsilon,\delta) = d_4(\varepsilon,\delta) \geqslant 0 \text{ (resp., } < 0) \quad \text{if and only if} \quad \delta_2 \leqslant \phi_2 \text{ (resp., } > \phi_2).$$

Thus, two homoclinic loops $L_1^*(\varepsilon,\delta_2,\delta_3,\delta_4,\delta_5,\delta_6)$ and $L_2^*(\varepsilon,\delta_2,\delta_3,\delta_4,\delta_5,\delta_6)$ exist near L_1 and L_2 respectively as $\delta_1 = \phi_1$, and a heteroclinic loop $L^*(\delta_3,\delta_4,\delta_5,\delta_6) = L_3^* \cup L_4^*$ exists near $L = L_3 \cup L_4$ as $\delta_1 = \phi_1$ and $\delta_2 = \phi_2$. In other words, a double figure eight loop $\Gamma^*(\varepsilon,\delta_3,\delta_4,\delta_5,\delta_6) = L_1^* \cup L_2^* \cup L^*$ exists near $\Gamma = L_1 \cup L_2 \cup L$ as $\delta_1 = \phi_1$ and $\delta_2 = \phi_2$. Further we consider the stability of the homoclinic loop, heteroclinic loop and the double figure eight loop. Denote by $r_i(\varepsilon,\delta) = -\frac{\lambda_{i2}(\varepsilon,\delta)}{\lambda_{i1}(\varepsilon,\delta)}$ the hyperbolic ratio of $S_{i\varepsilon}$, where $\lambda_{i2} < 0 < \lambda_{i1}$ are the eigenvalues of $S_{i\varepsilon}$ $(i = 1,2)$, and we have $r_i(\varepsilon,\delta) = r_{i0} + \varepsilon r_i^*(\varepsilon,\delta)$. By computing, we know $r_{i0} = r_i(0,0) = 1$. From $(1.1)_\varepsilon$, the linear part of system $(1.1)_\varepsilon$ at $S_{i\varepsilon}$ is given by the matrix

$$\begin{pmatrix} \lambda - f_x(S_{i\varepsilon}) & -f_y(S_{i\varepsilon}) \\ -g_x(S_{i\varepsilon}) & \lambda - g_y(S_{i\varepsilon}) \end{pmatrix},$$

where $S_{1\varepsilon} = (x_1,y_1) = (-\frac{\sqrt{2}}{2} + \mathrm{O}(\varepsilon^2), \frac{\sqrt{2}}{2}(a_{10} + \frac{1}{2}a_{30} + \frac{1}{4}a_{50})\varepsilon + \mathrm{O}(\varepsilon^2))$, $S_{2\varepsilon} = (-x_1,-y_1)$, and

$$\begin{aligned}
f_x(S_{1\varepsilon}) &= f_x(S_{2\varepsilon}) = \left(a_{10} + \frac{3}{2}a_{30} + \frac{5}{4}a_{50}\right)\varepsilon + \mathrm{O}(\varepsilon^2),\\
f_y(S_{1\varepsilon}) &= f_y(S_{2\varepsilon}) = 1 + \mathrm{O}(\varepsilon^2),\\
g_x(S_{1\varepsilon}) &= g_x(S_{2\varepsilon}) = \frac{3}{2} + \mathrm{O}(\varepsilon^2),\\
g_y(S_{1\varepsilon}) &= g_y(S_{2\varepsilon}) = \left(b_{01} + \frac{1}{2}b_{21} + \frac{1}{4}b_{41}\right)\varepsilon + \mathrm{O}(\varepsilon^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lambda_{11} = \lambda_{21} &= \frac{f_x(S_{1\varepsilon}) + g_y(S_{1\varepsilon}) + \sqrt{(f_x(S_{1\varepsilon}) - g_y(S_{1\varepsilon}))^2 + 4f_y(S_{1\varepsilon})g_x(S_{1\varepsilon})}}{2},\\
\lambda_{12} = \lambda_{22} &= \frac{f_x(S_{1\varepsilon}) + g_y(S_{1\varepsilon}) - \sqrt{(f_x(S_{1\varepsilon}) - g_y(S_{1\varepsilon}))^2 + 4f_y(S_{1\varepsilon})g_x(S_{1\varepsilon})}}{2}.
\end{aligned}$$

And hence,

$$r_1(\varepsilon,\delta)r_2(\varepsilon,\delta) = \big(r_1(\varepsilon,\delta)\big)^2 = 1 - 2\,\mathrm{div}(S_{1\varepsilon})\Delta = 1 - \left(\frac{2\sqrt{6}}{3} + \mathrm{O}(\varepsilon)\right)\mathrm{div}(S_{1\varepsilon}),$$

where

$$\begin{aligned}
\Delta &= \frac{\Delta_1}{(f_x(S_{1\varepsilon}))^2 + (g_y(S_{1\varepsilon}))^2 + 2f_y(S_{1\varepsilon})g_x(S_{1\varepsilon}) + \mathrm{div}(S_{1\varepsilon})\Delta_1},\\
\Delta_1 &= \sqrt{(f_x(S_{1\varepsilon}) - g_y(S_{1\varepsilon}))^2 + 4f_y(S_{1\varepsilon})g_x(S_{1\varepsilon})}.
\end{aligned}$$

By above analysis, we have

$$r_1(\varepsilon,\delta)r_2(\varepsilon,\delta) \geqslant 1 \ (< 1) \quad \text{if and only if} \quad \mathrm{div}(S_{1\varepsilon}) = \mathrm{div}(S_{2\varepsilon}) \leqslant 0 \ (> 0). \tag{2.3}$$

Under $\delta_1=\phi_1, \delta_2=\phi_2$, we can obtain

$$\begin{aligned}
\operatorname{div}|_{S_1} &= \operatorname{div}|_{S_2} = \varepsilon(P_x+Q_y)(S_1) = \varepsilon\left(\delta_1+\frac{\delta_2}{2}+\frac{\delta_3}{4}\right)+\mathrm{O}(\varepsilon^2)\\
&= \frac{1}{A_{01}B_{21}-A_{21}B_{01}}\Bigg[\left(\frac{A_{41}B_{01}-A_{01}B_{41}}{2}+A_{21}B_{41}-A_{41}B_{21}\right.\\
&\quad \left.+\frac{A_{01}B_{21}-A_{21}B_{01}}{4}\right)\delta_3+\frac{1}{3}\left(\frac{A_{03}B_{01}-A_{01}B_{03}}{2}+A_{21}B_{03}-A_{03}B_{21}\right)\delta_4\\
&\quad +\frac{1}{5}\left(\frac{A_{05}B_{01}-A_{01}B_{05}}{2}+A_{21}B_{05}-A_{05}B_{21}\right)\delta_5\\
&\quad +\frac{1}{3}\left(\frac{A_{23}B_{01}-A_{01}B_{23}}{2}+A_{21}B_{23}-A_{23}B_{21}\right)\delta_6\Bigg]\varepsilon+\mathrm{O}(\varepsilon^2)\\
&= \frac{1}{A_{01}B_{21}-A_{21}B_{01}}(X_0\delta_3+X_1\delta_4+X_2\delta_5+X_3\delta_6)\\
&\equiv \varepsilon\sigma_0(\varepsilon,\delta_3,\delta_4,\delta_5,\delta_6),
\end{aligned}$$

where

$$\begin{aligned}
X_0 &= \frac{A_{41}B_{01}-A_{01}B_{41}}{2}+A_{21}B_{41}-A_{41}B_{21}+\frac{A_{01}B_{21}-A_{21}B_{01}}{4},\\
X_1 &= \frac{1}{3}\left(\frac{A_{03}B_{01}-A_{01}B_{03}}{2}+A_{21}B_{03}-A_{03}B_{21}\right),\\
X_2 &= \frac{1}{5}\left(\frac{A_{05}B_{01}-A_{01}B_{05}}{2}+A_{21}B_{05}-A_{05}B_{21}\right),\\
X_3 &= \frac{1}{3}\left(\frac{A_{23}B_{01}-A_{01}B_{23}}{2}+A_{21}B_{23}-A_{23}B_{21}\right).
\end{aligned}$$

By computing, we can obtain $X_0 \doteq -0.36242176 < 0$. The implicit function theorem implies that a unique function

$$\phi_3(\varepsilon,\delta_4,\delta_5,\delta_6) = -\frac{1}{X_0}(X_1\delta_4+X_2\delta_5+X_3\delta_6)+\mathrm{O}(\varepsilon)$$

exists such that for $\varepsilon>0$ small

$$\sigma_0(\varepsilon,\delta)\geqslant 0\ (<0) \quad \text{if and only if} \quad \delta_3\leqslant\phi_3\ (>\phi_3).$$

From (2.3),

$$r_1(\varepsilon,\delta)r_2(\varepsilon,\delta)\geqslant 1\ (<1) \quad \text{if and only if} \quad \delta_3\geqslant\phi_3\ (<\phi_3).$$

By [7], we know that if $\delta_3=\phi_3$, then the integrals $\oint_{L_i^*}(P_x+Q_y)\,\mathrm{d}t\equiv\sigma_{1i}(\varepsilon)$, $i=1,2$, and $\int_{L_i^*}(P_x+Q_y)\,\mathrm{d}t\equiv\sigma_{1i}(\varepsilon)$, $i=3,4$, converge finitely, and they hold that

$$\sigma_{1i}(\varepsilon) = \oint\limits_{L_i^*}(P_x+Q_y)\,\mathrm{d}t = \oint\limits_{L_i}(P_x+Q_y)\,\mathrm{d}t+\mathrm{O}(\varepsilon), \quad i=1,2,$$

$$\sigma_1(\varepsilon) = \int\limits_{L^*}(P_x+Q_y)\,\mathrm{d}t = \int\limits_{L}(P_x+Q_y)\,\mathrm{d}t+\mathrm{O}(\varepsilon) = \sum_{i=3}^{4}\int\limits_{L_i}(P_x+Q_y)\,\mathrm{d}t+\mathrm{O}(\varepsilon).$$

Lemma 2.2. (1) *Assume* $\delta_i = \phi_i$, $i = 1, 2, 3$. *Then*

$$\sigma_{11} = \sigma_{12} = Y_1\delta_4 + Y_2\delta_5 + Y_3\delta_6 + \mathrm{O}(\varepsilon);$$

(2) *Assume* $\delta_i = \phi_i$, $i = 1, 2, 3$, *and* $\sigma_{11} = \sigma_{12} = 0$, *and then*

$$\sigma_1 = 2\sigma_{13} = 2\sigma_{14} = Z_1\delta_5 + Z_2\delta_6 + \mathrm{O}(\varepsilon),$$

where

$$Y_1 = \frac{b_1(A_{41}B_{03} - A_{03}B_{41} + \frac{A_{03}B_{01} - A_{01}B_{03}}{4})}{3X_0} - \frac{X_1}{X_0}\left(\frac{b_1}{2} + b_2\right) + B_{01},$$

$$Y_2 = \frac{b_1(A_{41}B_{05} - A_{05}B_{41} + \frac{A_{05}B_{01} - A_{01}B_{05}}{4})}{5X_0} - \frac{X_2}{X_0}\left(\frac{b_1}{2} + b_2\right) + B_{03},$$

$$Y_3 = \frac{b_1(A_{41}B_{23} - A_{23}B_{41} + \frac{A_{23}B_{01} - A_{01}B_{23}}{4})}{3X_0} - \frac{X_3}{X_0}\left(\frac{b_1}{2} + b_2\right) + B_{21},$$

$$Z_1 = \frac{a_1(A_{05}B_{41} - A_{41}B_{05} + \frac{A_{01}B_{05} - A_{05}B_{01}}{4})}{5X_0} + \frac{X_2}{X_0}\left(\frac{a_1}{2} + a_2\right) + A_{03}$$
$$- \frac{Y_2}{Y_1}\left[\frac{a_1(A_{03}B_{41} - A_{41}B_{03} + \frac{A_{01}B_{03} - A_{03}B_{01}}{4})}{3X_0} + \frac{X_1}{X_0}\left(\frac{a_1}{2} + a_2\right) + A_{01}\right],$$

$$Z_2 = \frac{a_1(A_{23}B_{41} - A_{41}B_{23} + \frac{A_{01}B_{23} - A_{23}B_{01}}{4})}{3X_0} + \frac{X_3}{X_0}\left(\frac{a_1}{2} + a_2\right) + A_{21}$$
$$- \frac{Y_3}{Y_1}\left[\frac{a_1(A_{03}B_{41} - A_{41}B_{03} + \frac{A_{01}B_{03} - A_{03}B_{01}}{4})}{3X_0} + \frac{X_1}{X_0}\left(\frac{a_1}{2} + a_2\right) + A_{01}\right],$$

$$a_1 = 2\int_0^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{\frac{11}{12} - \frac{x^2}{3}}}\,\mathrm{d}x \doteq 1.525974, \qquad a_2 = 2\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{\sqrt{\frac{11}{12} - \frac{x^2}{3}}}\,\mathrm{d}x \doteq 0.261096,$$

$$b_1 = 2\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{11}}{2}} \frac{1}{\sqrt{\frac{11}{12} - \frac{x^2}{3}}}\,\mathrm{d}x \doteq 3.91542, \qquad b_2 = 2\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{11}}{2}} \frac{x^2}{\sqrt{\frac{11}{12} - \frac{x^2}{3}}}\,\mathrm{d}x \doteq 7.22082.$$

Proof. We need only to prove $\sigma_{1i}(0) = Y_1\delta_4 + Y_2\delta_5 + Y_3\delta_6$ if $\delta_i = \phi_i|_{\varepsilon=0}$. In fact, the equations $\delta_i = \phi_i|_{\varepsilon=0}$, for $i = 1, 2$ imply that

$$\delta_1 = -\left(\frac{B_{02}}{B_{01}}\delta_2 + \frac{B_{41}}{B_{01}}\delta_3 + \frac{B_{03}}{3B_{01}}\delta_4 + \frac{B_{05}}{5B_{01}}\delta_5 + \frac{B_{23}}{3B_{01}}\delta_6\right),$$

$$\delta_2 = \frac{1}{A_{21}B_{01} - A_{01}B_{21}}\left[(A_{01}B_{41} - A_{41}B_{01})\delta_3 + \frac{A_{01}B_{03} - A_{03}B_{01}}{3}\delta_4\right.$$
$$\left. + \frac{A_{01}B_{05} - A_{05}B_{01}}{5}\delta_5 + \frac{A_{01}B_{23} - A_{23}B_{01}}{3}\delta_6\right],$$

$$\delta_3 = -\frac{1}{X_0}(X_1\delta_4 + X_2\delta_5 + X_3\delta_6).$$

Hence, by symmetry, we have

$$\begin{aligned}
\sigma_{11}(0) = \sigma_{12}(0) &= \oint_{L_1} (P_x + Q_y)\,\mathrm{d}t \\
&= \delta_1 \oint_{L_1} \frac{1}{y}\,\mathrm{d}x + \delta_2 \oint_{L_1} \frac{x^2}{y}\,\mathrm{d}x + \delta_3 \oint_{L_1} \frac{x^4}{y}\,\mathrm{d}x \\
&\quad + \delta_4 \oint_{L_1} y\,\mathrm{d}x + \delta_5 \oint_{L_1} y^3\,\mathrm{d}x + \delta_6 \oint_{L_1} x^2 y\,\mathrm{d}x \\
&= \left(\delta_1 + \frac{\delta_2}{2} + \frac{\delta_3}{4}\right) \oint_{L_1} \frac{1}{y}\,\mathrm{d}x + \left(\delta_2 + \frac{\delta_3}{2}\right) b_1 + \delta_3 b_2 + B_{01}\delta_4 + B_{03}\delta_5 + B_{21}\delta_6 \\
&= \left(\delta_2 + \frac{\delta_3}{2}\right) b_1 + \delta_3 b_2 + B_{01}\delta_4 + B_{03}\delta_5 + B_{21}\delta_6.
\end{aligned}$$

Substituting $\delta_1, \delta_2, \delta_3$ into the above equality, we obtain

$$\sigma_{11}(0) = \sigma_{12}(0) = Y_1\delta_4 + Y_2\delta_5 + Y_3\delta_6.$$

By computing, we know $Y_1 \doteq 0.546349 > 0$. Therefore the implicit function theorem implies that a unique function $\phi_4(\varepsilon, \delta_5, \delta_6) = -\frac{Y_2}{Y_1}\delta_5 - \frac{Y_3}{Y_1}\delta_6 + \mathrm{O}(\varepsilon)$ exists such that for $\varepsilon > 0$ small

$$\sigma_{11} \geqslant 0\ (< 0) \quad \text{if and only if} \quad \delta_4 \geqslant \phi_4\ (< \phi_4).$$

In the same way, under $\delta_i = \phi_i$, $i = 1, 2, 3, 4$, we can obtain $\sigma_1 = Z_1\delta_5 + Z_2\delta_6 + \mathrm{O}(\varepsilon)$. This completes the proof. □

From Lemma 2.2, we know $Z_1 \doteq -0.066157 < 0$. By the implicit function theorem again there exists a unique function

$$\phi_5(\varepsilon, \delta_6) = -\frac{Z_2}{Z_1}\delta_6 + \mathrm{O}(\varepsilon) \doteq -5.34766561\delta_6 + \mathrm{O}(\varepsilon)$$

such that

$$\sigma_1 \geqslant 0\ (< 0) \quad \text{if and only if} \quad \delta_5 \leqslant \phi_5\ (> \phi_5).$$

If denotes by R_{1i} the first order saddle value at the saddle points $S_{i\varepsilon}$ of the system $(1.1)_\varepsilon$, $i = 1, 2$, then by [9], we have

Lemma 2.3. *Assume* $\delta_i = \phi_i$, $i = 1, \ldots, 5$, *then*

$$R_{11} = R_{21} = \left(-\frac{7}{9}\delta_1 - \frac{23}{90}\delta_2 - \frac{83}{180}\delta_3 + \frac{3}{10}\delta_4 + \frac{3}{20}\delta_6\right)\varepsilon + \mathrm{O}(\varepsilon^2).$$

Proof. If let T_i be an reversible matrix such that $\det T_i = 1$, $T_i D_i T_i^{-1} = \mathrm{diag}(\lambda_{i1}, \lambda_{i2})$, where $D_i = \frac{\partial(f,g)}{\partial(x,y)}(S_i)$, $i = 1, 2$, and $T_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$T_2 \begin{pmatrix} (a_{10} + \frac{3a_{30}}{2} + \frac{5a_{50}}{4})\varepsilon + \mathrm{O}(\varepsilon^2) & 1 + \mathrm{O}(\varepsilon^2) \\ \frac{3}{2} + \mathrm{O}(\varepsilon^2) & (b_{01} + \frac{b_{21}}{2} + \frac{b_{41}}{4})\varepsilon + \mathrm{O}(\varepsilon^2) \end{pmatrix} = \begin{pmatrix} \lambda_{21} & 0 \\ 0 & \lambda_{22} \end{pmatrix} T_2$$

and $ad - bc = 1$. Thus, we can obtain the following equations

$$a\lambda_{21} = \frac{3b}{2} + a\left(a_{10} + \frac{3}{2}a_{30} + \frac{5}{4}a_{50}\right)\varepsilon + \mathrm{O}(\varepsilon^2),$$
$$b\lambda_{21} = a + b\left(b_{01} + \frac{b_{21}}{2} + \frac{b_{41}}{4}\right)\varepsilon + \mathrm{O}(\varepsilon^2),$$
$$c\lambda_{22} = \frac{3d}{2} + c\left(a_{10} + \frac{3}{2}a_{30} + \frac{5}{4}a_{50}\right)\varepsilon + \mathrm{O}(\varepsilon^2),$$
$$d\lambda_{22} = c + d\left(b_{01} + \frac{b_{21}}{2} + \frac{b_{41}}{4}\right)\varepsilon + \mathrm{O}(\varepsilon^2).$$

Therefore, we have

$$T_2 = \begin{pmatrix} 1 & \frac{\sqrt{6}}{3} + \frac{b_{01}+\frac{1}{2}b_{21}+\frac{1}{4}b_{41}-a_{10}-\frac{3}{2}a_{30}-\frac{5}{4}a_{50}}{3}\varepsilon + \mathrm{O}(\varepsilon^2) \\ -\frac{\sqrt{6}}{4} + \mathrm{O}(\varepsilon^2) & \frac{1}{2} + \frac{\sqrt{6}(a_{10}+\frac{3}{2}a_{30}+\frac{5}{4}a_{50}-b_{01}-\frac{1}{2}b_{21}-\frac{1}{4}b_{41})}{12}\varepsilon + \mathrm{O}(\varepsilon^2) \end{pmatrix}.$$

Now making a linear transformation of the form

$$\begin{pmatrix} u \\ v \end{pmatrix} = T_2 \begin{pmatrix} x - x_2 \\ y - y_2 \end{pmatrix},$$

where

$$S_{2\varepsilon} = (x_2, y_2) = \left(\frac{\sqrt{2}}{2} + \mathrm{O}(\varepsilon^2), -\frac{\sqrt{2}(a_{10} + \frac{3}{2}a_{30} + \frac{5}{4}a_{50})}{2}\varepsilon + \mathrm{O}(\varepsilon^2)\right).$$

For (u, v) near the origin, we obtain from $(1.1)_\varepsilon$,

$$\begin{aligned}
\dot{u} = {} & Y + \frac{\sqrt{6}}{3}\left(-X^5 + \frac{5}{2}X^3 - X\right) + \Bigg[-\frac{\sqrt{2}}{2}A + a_{10}X + a_{30}X^3 + a_{12}XY^2 + a_{14}XY^4 \\
& + a_{50}X^5 + a_{32}X^3Y^2 + \frac{\sqrt{6}}{3}\Big(-5X^4N + \frac{15}{2}X^2N - N + b_{01}Y + b_{21}X^2Y + b_{03}Y^3 \\
& + b_{05}Y^5 + b_{41}X^4Y + b_{23}X^2Y^3\Big) + \frac{1}{3}\left(b_{01} + \frac{1}{2}b_{21} + \frac{1}{4}b_{41} - a_{10} - \frac{3}{2}a_{30} - \frac{5}{4}a_{50}\right) \\
& \times \left(-X^5 + \frac{5}{2}X^3 - X\right)\Bigg]\varepsilon + \mathrm{O}(\varepsilon^2) \\
\equiv {} & \lambda_{21}\left[u + \sum_{k=2}^{3}\sum_{j+l=k} m_{jl}u^jv^l + \mathrm{O}\big(|u, v|^4\big)\right],
\end{aligned}$$

$$\begin{aligned}
\dot{v} = {} & -\frac{\sqrt{6}}{4}Y + \frac{1}{2}\left(-X^5 + \frac{5}{2}X^3 - X\right) + \Bigg[-\frac{\sqrt{6}}{4}\Big(-\frac{\sqrt{2}}{2}A + a_{10}X + a_{30}X^3 + a_{12}XY^2 \\
& + a_{14}XY^4 + a_{50}X^5 + a_{32}X^3Y^2\Big) + \frac{1}{2}\Big(-5X^4N + \frac{15}{2}X^2N - N + b_{01}Y + b_{21}X^2Y \\
& + b_{03}Y^3 + b_{05}Y^5 + b_{41}X^4Y + b_{23}X^2Y^3\Big) + \frac{\sqrt{6}}{12}\Big(a_{10} + \frac{3}{2}a_{30} + \frac{5}{4}a_{50} - b_{01} - \frac{1}{2}b_{21} \\
& - \frac{1}{4}b_{41}\Big)\left(-X^5 + \frac{5}{2}X^3 - X\right)\Bigg]\varepsilon + \mathrm{O}(\varepsilon^2) \\
\equiv {} & -\lambda_{22}\left[-v + \sum_{k=2}^{3}\sum_{j+l=k} n_{jl}u^jv^l + \mathrm{O}\big(|u, v|^4\big)\right],
\end{aligned}$$

where

$$X=\frac{\sqrt{2}}{2}+\frac{u}{2}-\frac{\sqrt{6}}{3}v,\qquad Y=\frac{\sqrt{6}}{4}u+v,\qquad A=a_{10}+\frac{a_{30}}{2}+\frac{a_{50}}{4},$$
$$N=\left(\frac{\sqrt{6}}{12}u+\frac{1}{3}v\right)\left(a_{10}+\frac{3}{2}a_{30}+\frac{5}{4}a_{50}-b_{01}-\frac{1}{2}b_{21}-\frac{1}{4}b_{41}\right).$$

According to [9] the first saddle value of $(1.1)_\varepsilon$ at $S_{i\varepsilon}$ is given by

$$R_{i1}=m_{21}+n_{12}-m_{20}m_{11}+n_{02}n_{11},\quad i=1,2. \tag{2.4}$$

The straightforward computing give

$$m_{21}=\frac{5\sqrt{6}}{12}+\left[-\frac{5(a_{10}+b_{01})}{12}+\frac{(a_{12}+3b_{03})}{4}+\frac{3(a_{32}+b_{23})}{8}\right.$$
$$\left.-\frac{3(3a_{30}+b_{21})}{8}-\frac{29(5a_{50}+b_{41})}{48}\right]\varepsilon+\mathrm{O}(\varepsilon^2),$$
$$m_{20}=\frac{5\sqrt{2}}{24}+\frac{1}{48\sqrt{3}}\left[(18a_{12}+36a_{30}+9a_{32}+60a_{50}-20b_{01}+14b_{21}+19b_{41})\varepsilon\right]$$
$$+\mathrm{O}(\varepsilon^2),$$
$$m_{11}=-\frac{5\sqrt{3}}{9}+\frac{1}{18\sqrt{2}}\left[(20a_{10}+18a_{12}-6a_{30}+9a_{32}-35a_{50})\varepsilon\right]+\mathrm{O}(\varepsilon^2),$$
$$n_{02}=\frac{5\sqrt{3}}{18}-\frac{1}{36\sqrt{2}}\left[(18a_{12}+36a_{30}+9a_{32}+60a_{50}-20b_{01}+14b_{21}+19b_{41})\varepsilon\right]$$
$$+\mathrm{O}(\varepsilon^2),$$
$$n_{11}=-\frac{5\sqrt{2}}{12}-\frac{1}{24\sqrt{3}}\left[(20a_{10}+18a_{12}-6a_{30}+9a_{32}-35a_{50})\varepsilon\right]+\mathrm{O}(\varepsilon^2),$$
$$n_{12}=-\frac{5\sqrt{6}}{12}+\left[-\frac{5(a_{10}+b_{01})}{12}+\frac{a_{12}+3b_{03}}{4}+\frac{3(a_{32}+b_{23})}{8}\right.$$
$$\left.-\frac{3(3a_{30}+b_{21})}{8}-\frac{29(5a_{50}+b_{41})}{48}\right]\varepsilon+\mathrm{O}(\varepsilon^2).$$

Substituting $m_{21}, m_{10}, m_{11}, n_{02}, n_{11}, n_{12}$ into (2.4), we have

$$R_{21}=\left(-\frac{7}{9}\delta_1-\frac{23}{90}\delta_2-\frac{83}{180}\delta_3+\frac{3}{10}\delta_4+\frac{3}{20}\delta_6\right)\varepsilon+\mathrm{O}(\varepsilon^2).$$

Using the same arguments, we obtain

$$R_{11}=R_{21}=\left(-\frac{7}{9}\delta_1-\frac{23}{90}\delta_2-\frac{83}{180}\delta_3+\frac{3}{10}\delta_4+\frac{3}{20}\delta_6\right)\varepsilon+\mathrm{O}(\varepsilon^2).$$

This is the end of Lemma 2.3. □

By [7,9,10], we know if $r_{10}=r_{20}=1, \sigma_1=0$, and then $\sigma_2=R_{11}+R_{21}=2R_{11}$, where σ_2 is defined as [9]. Hence, we have $\sigma_2=2R_{11}>0$ for $(1.1)_\varepsilon$ when $\delta_i=\phi_i$, $i=1,\dots,5$. Further we have the following rule to discriminate the stability of L_i^* $(i=1,2)$, L^* and the double figure eight loop Γ^*.

Lemma 2.4. *For $\varepsilon > 0$ small, the homoclinic loops L_1^* and L_2^* are stable inside (resp., unstable) if $\sigma_0(\varepsilon,\delta) < 0$ (resp., > 0) or $\sigma_0 = 0$ and $\sigma_{11} = \sigma_{12} < 0$ (resp., $\sigma_{11} = \sigma_{12} > 0$) or $\sigma_0 = \sigma_{11} = \sigma_{12} = 0$ and $R_{11} = R_{21} > 0$ (resp., $R_{11} = R_{21} < 0$).*

Lemma 2.5. *For $\varepsilon > 0$ small, the heteroclinic loop L^* is stable inside (resp., unstable) if $r_1(\varepsilon,\delta)r_2(\varepsilon,\delta) > 1$ (resp., $r_1(\varepsilon,\delta)r_2(\varepsilon,\delta) < 1$) or $r_1(\varepsilon,\delta)r_2(\varepsilon,\delta) = 1$ and $\sigma_1 < 0$ (resp., $\sigma_1 > 0$) or $r_1(\varepsilon,\delta)r_2(\varepsilon,\delta) = 1$ and $\sigma_1 = 0$ and $\sigma_2 > 0$ (resp., $\sigma_2 < 0$).*

Lemma 2.6. *For $\varepsilon > 0$ small, the double figure eight loop Γ^* is stable outside (resp., unstable) if $r_1(\varepsilon,\delta)r_2(\varepsilon,\delta) > 1$ (resp., $r_1(\varepsilon,\delta)r_2(\varepsilon,\delta) < 1$) or $r_1(\varepsilon,\delta)r_2(\varepsilon,\delta) = 1$ and $\sigma_1 < 0$ (resp., $\sigma_1 > 0$) or $r_1(\varepsilon,\delta)r_2(\varepsilon,\delta) = 1$ and $\sigma_1 = 0$ and $\sigma_2 < 0$ (resp., $\sigma_2 > 0$).*

3. Proof of the main results

In the following, we will find a larger limit cycles which surrounds all five singular points. By [11,14] we need to consider the relative position of separatrices near the double figure eight loop Γ^* and the behavior of orbits near a large periodic orbit L_h^*. The first order Melnikov function of L_h^* is

$$M^*(h) = \oint_{L_h^*} Q\,\mathrm{d}x - P\,\mathrm{d}y, \quad h > \frac{11}{96}.$$

Here L_h^*: $H(x,y) = h$ surrounds all singular points of system $(1.1)_\varepsilon$ for $h > \frac{11}{96}$ and $\varepsilon > 0$ small. From Lemmas 2.1 and 2.2, when $\delta_i = \phi_i$, $i = 1,\dots,5$, we can obtain

$$M^*\left(\frac{13}{96}\right) = \delta_1 \oint_{L_h^*} y\,\mathrm{d}x + \delta_2 \oint_{L_h^*} x^2 y\,\mathrm{d}x + \frac{1}{4}\delta_4 \oint_{L_h^*} y^3\,\mathrm{d}x + \frac{1}{5}\delta_5 \oint_{L_h^*} y^5\,\mathrm{d}x + \delta_3 \oint_{L_h^*} x^4 y\,\mathrm{d}x + \frac{1}{3}\delta_6 \oint_{L_h^*} x^2 y^3\,\mathrm{d}x + \mathrm{O}(\varepsilon).$$

Using Mathematica 4.0, for $h = \frac{13}{96}$, we have

$$\oint_{L_h^*} y\,\mathrm{d}x = 2\int_{x_1}^{x_2} \sqrt{\frac{13}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2}\,\mathrm{d}x \doteq 3.12974,$$

$$\oint_{L_h^*} x^2 y\,\mathrm{d}x = 2\int_{x_1}^{x_2} x^2\sqrt{\frac{13}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2}\,\mathrm{d}x \doteq 3.48546,$$

$$\oint_{L_h^*} x^4 y\,\mathrm{d}x = 2\int_{x_1}^{x_2} x^4\sqrt{\frac{13}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2}\,\mathrm{d}x \doteq 6.22568,$$

$$\oint_{L_h^*} y^3\,\mathrm{d}x = 2\int_{x_1}^{x_2} \left(\sqrt{\frac{13}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2}\right)^3\mathrm{d}x \doteq 1.006234,$$

$$\oint_{L_h^*} x^2 y^3 \, \mathrm{d}x = 2\int_{x_1}^{x_2} x^2 \left(\sqrt{\frac{13}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2}\right)^3 \mathrm{d}x \doteq 1.47457,$$

$$\oint_{L_h^*} y^5 \, \mathrm{d}x = 2\int_{x_1}^{x_2} \left(\sqrt{\frac{13}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2}\right)^5 \mathrm{d}x \doteq 0.429204,$$

where

$$x_1 = -x_2 = -\frac{1}{4}\sqrt{\frac{60 + (53568 - 3456\sqrt{58})^{1/3} + 12(31 + 2\sqrt{58})^{1/3}}{3}}.$$

Hence, $M^*(h) = -0.00226\delta_6 + \mathrm{O}(\varepsilon)$. By the Poincaré–Bendixson theorem one large cycle Γ_1 exists which surrounds five singular points since the double figure eight loop Γ^* is unstable for $\sigma_2 > 0$ (see Fig. 4).

Now we are in a position to prove our main result. From the above analysis, we know the single homoclinic loop L_i^* $(i = 1, 2)$ and the heteroclinic loop L^* are stable, and the double figure eight loop is unstable, when $R_{11} > 0$. Keep (ε, δ_6) fixed and let $\delta_5 < \phi_5$ and $0 < \phi_5 - \delta_5 \ll \varepsilon$. Thus L^* has changed their stability from stable into unstable, and hence one small stable limit cycle L_{31} has appeared inside L^*. Keep δ_5 fixed and let δ_4 satisfy $0 < \delta_4 - \phi_4 \ll \phi_5 - \delta_5 \ll \varepsilon$, thus L_1^* and L_2^* have changed their stability from stable into unstable, and hence two small stable limit cycles L_{11} and L_{21} have appeared with $L_{11} \subset L_1^*$ and $L_{21} \subset L_2^*$, see Fig. 5.

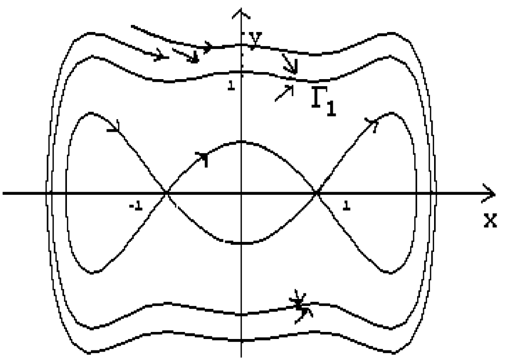

Fig. 4.

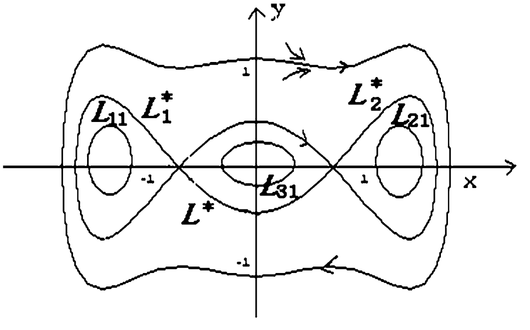

Fig. 5.

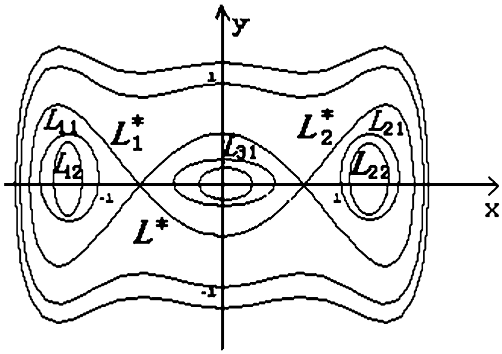

Fig. 6.

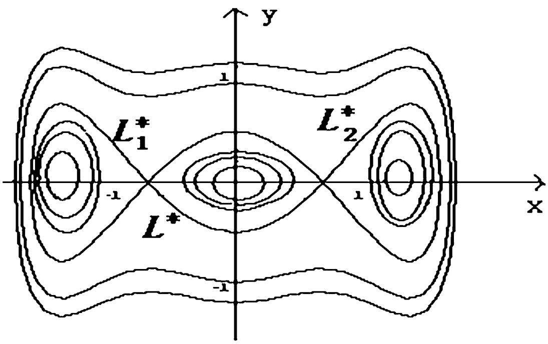

Fig. 7.

Now keep δ_4 fixed and let δ_3 satisfy $0 < \delta_3 - \phi_3 \ll \delta_4 - \phi_4 \ll \phi_5 - \delta_5 \ll \varepsilon$. Again the stability of L_1^* and L_2^* changed, and then another two small unstable limit cycles L_{12} and L_{22} are born out with $L_{11} \subset L_{12} \subset L_1^*$ and $L_{21} \subset L_{22} \subset L_2^*$. Simultaneously the stability of heteroclinic loop L^* has got changed again, and then a small unstable limit cycle L_{32} has appeared with $L_{31} \subset L_{32} \subset L^*$. Also the double eight figure loop Γ^* has changed its stability from unstable into stable, and hence a large unstable limit cycle Γ_2 has appeared outside Γ^* with $\Gamma_2 \subset \Gamma_1$, see Fig. 6.

Keep δ_3 fixed, if we change δ_2 by $0 < \delta_2 - \phi_2 \ll \delta_3 - \phi_3 \ll \delta_4 - \phi_4 \ll \phi_5 - \delta_5 \ll \varepsilon$. And then L_3^* and L_4^* have broken, and hence generated another one small stable limit cycle L_{33} with $L_{31} \subset L_{32} \subset L_{33} \subset L^*$. Finally keep δ_2 fixed and let δ_1 satisfy $0 < \phi_1 - \delta_1 \ll \delta_2 - \phi_2 \ll \delta_3 - \phi_3 \ll \delta_4 - \phi_4 \ll \phi_5 - \delta_5 \ll \varepsilon$ so that L_1^* and L_2^* have broken and two small stable limit cycles L_{13} and L_{23} have appeared with $L_{11} \subset L_{12} \subset L_{13} \subset L_1^*$ and $L_{21} \subset L_{22} \subset L_{23} \subset L_2^*$, see Fig. 7.

Under $\delta_i = \phi_i$, $i = 1, \ldots, 5$, we have

$$\operatorname{div}\big(O(0,0)\big) = \delta_1\varepsilon + \mathrm{O}(\varepsilon^2) \doteq -0.0226\delta_6\varepsilon + \mathrm{O}(\varepsilon^2) < 0,$$

$$\operatorname{div}(O_{1\varepsilon}) = \operatorname{div}(O_{2\varepsilon}) = (\delta_1 + 2\delta_2 + 4\delta_3)\varepsilon + \mathrm{O}(\varepsilon^2) \doteq -0.20055\delta_6\varepsilon + \mathrm{O}(\varepsilon^2) < 0.$$

Hence, the singular points O, $O_{1\varepsilon}$ and $O_{2\varepsilon}$ are stable. Notice that L_{11}, L_{21} and L_{31} are also stable. By the Poincaré–Bendixson theorem, we know there are three small unstable limit cycles L_{10}, L_{20} and L_{30} with $L_{10} \subset L_{11}$, $L_{20} \subset L_{21}$ and $L_{30} \subset L_{31}$ respectively. The proof of Fig. 1(a) is completed. Using the same arguments, we can obtain the second distribution.

This is the end of proof for the main result.

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