Bifurcations of limit cycles from quintic Hamiltonian systems with a double figure eight loop

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Abstract

This paper deals with Liénard equations of the form $\dot{x} = y$, $\dot{y} = P(x) + yQ(x, y)$, with $P$ and $Q$ polynomials of degree 5 and 4 respectively. Attention goes to perturbations of the Hamiltonian vector fields with an elliptic Hamiltonian of degree six, exhibiting a double figure eight loop. The number of limit cycles and their distributions are given by using the methods of bifurcation theory and qualitative analysis.

Keywords: Limit cycle; Homoclinic bifurcation; Heteroclinic bifurcation; Double figure eight loop

1. Introduction and main results

A part of the well-know Hilbert’s 16th problem is to consider the existence of maximal number of limit cycles for a general planar polynomial system. In general, this is a very difficult question and it has been studied by many mathematicians (see [1–15], for example). Recently many authors have a great interest for the systems which exhibit a eight figure loop and the Hamiltonian has the form

$$H_{n+1} = \frac{y^2}{2} + P_{n+1}(x),$$

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where $P_{n+1}(x)$ is a polynomial in $x$ of degree $n + 1$, see [2,3,5,6,8,12,15]. The key point used in [2–5,15] is to find simple zeros of a Melnikov function which is also called a Abelian integral.

As we know, when we study Hopf bifurcation for a planar polynomial system a typical way to find limit cycles is to change the stability of a focus. [6,10] used this idea to find limit cycles near a homoclinic loop or a heteroclinic loop. That is, a limit cycle can be bifurcated from a homoclinic loop or a heteroclinic loop when its stability changes. A final limit cycle can be obtained by making the homoclinic or heteroclinic loop broken. Then the method was developed to investigate the limit cycle bifurcation from a double homoclinic loop by Han and Chen [8], and was used to study the existence of 11 limit cycles for some cubic system by [11,13,14]. In this paper, we develop this method to study the bifurcations of limit cycles for quintic system with the double figure eight loop, consisting of two homoclinic loops and a heteroclinic loop.

Consider the following perturbed Hamiltonian system

\[
\begin{align*}
\dot{x} &= y + \varepsilon(a_{10}x + a_{30}x^3 + a_{12}xy^2 + a_{14}xy^4 + a_{50}x^5 + a_{32}x^3y^2) \\
&= f(x, y) + \varepsilon P(x, y), \\
\dot{y} &= -x^5 - bx^3 - x + \varepsilon(b_{01}y + b_{21}x^2y + b_{03}y^3 + b_{05}y^5 + b_{41}x^4y + b_{23}x^2y^3) \\
&= g(x, y) + \varepsilon Q(x, y),
\end{align*}
\]

(1.1)\_e

where $b$ is a negative constant with $b < -\frac{4}{\sqrt{3}}$ and $\varepsilon > 0$ is small. We consider the coefficients $a_{ij}$ and $b_{ij}$ in (1.1)\_e as parameters. Our main result can be stated as follows.

**Theorem.** Let $b = -\frac{5}{2}$. Then system (1.1)\_e can have 14 limit cycles and two different distributions are given in Fig. 1.

2. **The properties of system (1.1)\_e and some preliminaries**

In what follows, we are going to obtain a complete analysis of system (1.1)\_e. From (1.1)\_e, we know that system (1.1)\_0 has the first integral of the form

\[
H(x, y) = \frac{y^2}{2} + \frac{x^2}{2} + \frac{b}{4}x^4 + \frac{x^6}{6}
\]

and the phase portraits is shown in Fig. 2.
Fig. 2.

\[ H = 0 \text{ and } H = -\frac{b}{4} + \frac{b^3}{24} + \frac{\sqrt{b^2 - 4}}{6} - \frac{b^2 \sqrt{b^2 - 4}}{24} \] correspond to the centers \( O(0, 0) \), \( O_{10} = (-\sqrt{-b + \sqrt{b^2 - 4}}/\sqrt{2}, 0) \) and \( O_{20} = (\sqrt{-b + \sqrt{b^2 - 4}}/\sqrt{2}, 0) \) respectively.

\[ L_1 = \left\{ (x, y) \mid H(x, y) = -\frac{b}{4} + \frac{b^3}{24} - \frac{\sqrt{b^2 - 4}}{6} + \frac{b^2 \sqrt{b^2 - 4}}{24}, \right. \]
\[ \left. -\sqrt{-\frac{b}{2} + \sqrt{b^2 - 4}} < x < -\frac{\sqrt{-b - \sqrt{b^2 - 4}}}{\sqrt{2}} \right\} \]

corresponds to the saddle point \( S_{10} = (-\sqrt{-b - \sqrt{b^2 - 4}}/\sqrt{2}, 0) \) and homoclinic loop,

\[ L_2 = \left\{ (x, y) \mid H(x, y) = -\frac{b}{4} + \frac{b^3}{24} - \frac{\sqrt{b^2 - 4}}{6} + \frac{b^2 \sqrt{b^2 - 4}}{24}, \right. \]
\[ \left. \frac{\sqrt{-b - \sqrt{b^2 - 4}}}{\sqrt{2}} < x < \sqrt{-\frac{b}{2} + \sqrt{b^2 - 4}} \right\} \]

corresponds to the saddle point \( S_{20} = (\sqrt{-b - \sqrt{b^2 - 4}}/\sqrt{2}, 0) \) and homoclinic loop, \( L = L_3 \cup S_{20} \cup L_4 \cup S_{10} \) is a 2-polycycle, where

\[ L_3 = \left\{ (x, y) \mid H(x, y) = -\frac{b}{4} + \frac{b^3}{24} - \frac{\sqrt{b^2 - 4}}{6} + \frac{b^2 \sqrt{b^2 - 4}}{24}, \right. \]
\[ \left. -\frac{\sqrt{-b - \sqrt{b^2 - 4}}}{\sqrt{2}} < x < \frac{\sqrt{-b - \sqrt{b^2 - 4}}}{\sqrt{2}}, \quad y \geq 0 \right\} \]

and

\[ L_4 = \left\{ (x, y) \mid H(x, y) = -\frac{b}{4} + \frac{b^3}{24} - \frac{\sqrt{b^2 - 4}}{6} + \frac{b^2 \sqrt{b^2 - 4}}{24}, \right. \]
\[ \left. -\frac{\sqrt{-b - \sqrt{b^2 - 4}}}{\sqrt{2}} < x < \frac{\sqrt{-b - \sqrt{b^2 - 4}}}{\sqrt{2}}, \quad y \leq 0 \right\} \]

are heteroclinic orbits connecting hyperbolic saddle points \( S_{10} \) and \( S_{20} \) and \( L \) is clockwise oriented. Let \( \Gamma = L_1 \cup L_2 \cup L_3 \cup L_4 \) is the double figure eight loop, see Fig. 2. Obviously, system \((1.1)_\varepsilon\) has still five critical points \( O(0, 0), O_{1\varepsilon}, O_{2\varepsilon}, S_{1\varepsilon}, S_{2\varepsilon}. \) For \( \varepsilon \) small, system \((1.1)_\varepsilon\) has separatrices \( L_i^u \) and \( L_i^s, i = 1, 2, 3, 4, \) near the saddle points \( S_{1\varepsilon}, i = 1, 2, \) and \( L_1^u \cup L_3^u \) and \( L_2^s \cup L_4^s \) are the unstable manifolds of \( S_{1\varepsilon} \) and \( S_{2\varepsilon} \) respectively, and \( L_1^s \cup L_3^s \) and \( L_2^s \cup L_4^s \) are the stable manifolds of \( S_{1\varepsilon} \) and \( S_{2\varepsilon} \) respectively.
Let $\delta = (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) = (a_{10} + b_{01}, 3a_{30} + b_{21}, 5a_{50} + b_{41}, a_{12} + 3b_{03}, a_{14} + 5b_{05})$, $\delta_6 = 3(a_{32} + b_{23}) > 0$. We take $\delta$ as a vector parameter. Recall that the directed distance from $L_{u_i}$ to $L_{s_i}$ is measured by

$$d_i(\varepsilon, \delta) = \varepsilon N_i M_i(\delta) + O(\varepsilon^2),$$  

(2.1)

where $N_i > 0, i = 1, 2, 3, 4$, is a constant, and

$$M_i(\delta) = \int_{L_i} Q \, dx - P \, dy.$$  

(2.2)

For example, if $d_1(\varepsilon, \delta) < 0, d_2(\varepsilon, \delta) < 0$, the relative position of $L_{u_i}$ and $L_{s_i}$ is shown in Fig. 3, $i = 1, 2$.

For the expression of $M_i(\delta), i = 1, 2, 3, 4$, by (2.1) and (2.2), we have

**Lemma 2.1.** For (1.1)$_\varepsilon$, we have

$$M_i(\delta) = \delta_1 B_{01} + \delta_2 B_{21} + \delta_3 B_{41} + \frac{\delta_4}{3} B_{03} + \frac{\delta_5}{5} B_{05} + \frac{\delta_6}{3} B_{23}, \quad i = 1, 2,$$

$$M_i(\delta) = \delta_1 A_{01} + \delta_2 A_{21} + \delta_3 A_{41} + \frac{\delta_4}{3} A_{03} + \frac{\delta_5}{5} A_{05} + \frac{\delta_6}{3} A_{23}, \quad i = 3, 4,$$

where

$$A_{01} = \int_{L_3} y \, dx = \int_{L_4} y \, dx, \quad A_{21} = \int_{L_3} x^2 y \, dx = \int_{L_4} x^2 y \, dx,$$

$$A_{03} = \int_{L_3} y^3 \, dx = \int_{L_4} y^3 \, dx, \quad A_{05} = \int_{L_3} y^5 \, dx = \int_{L_4} y^5 \, dx,$$

$$A_{23} = \int_{L_3} x^2 y^3 \, dx = \int_{L_4} x^2 y^3 \, dx, \quad A_{41} = \oint_{L_1} x^4 y \, dx = \oint_{L_2} x^4 y \, dx,$$

$$B_{01} = \oint_{L_1} y \, dx = \oint_{L_2} y \, dx, \quad B_{21} = \oint_{L_1} x^2 y \, dx = \oint_{L_2} x^2 y \, dx,$$

$$B_{03} = \oint_{L_1} y^3 \, dx = \oint_{L_2} y^3 \, dx, \quad B_{05} = \oint_{L_1} y^5 \, dx = \oint_{L_2} y^5 \, dx,$$

$$B_{23} = \oint_{L_1} x^2 y^3 \, dx = \oint_{L_2} x^2 y^3 \, dx.$$
\[ B_{23} = \oint_{L_1} x^2 y^3 \, dx = \oint_{L_2} x^2 y^3 \, dx, \quad B_{41} = \oint_{L_1} x^4 y \, dx = \oint_{L_2} x^4 y \, dx. \]

**Proof.** We only prove the expression of \( M_i(\delta) \), \( i = 1, 2 \). Using the same arguments, we can obtain the expression of \( M_3(\delta) \) and \( M_4(\delta) \). Notice that

\[ -\oint_{L_1} x \, dy = \oint_{L_1} y \, dx \equiv B_{01}, \quad -\frac{1}{3} \oint_{L_1} x^3 \, dy = \oint_{L_1} x^2 y \, dx \equiv B_{21}, \]

\[ -3 \oint_{L_1} xy^2 \, dy = \oint_{L_1} y^3 \, dx \equiv B_{03}, \quad -5 \oint_{L_1} xy^4 \, dy = \oint_{L_1} y^5 \, dx \equiv B_{05}, \]

\[ -\frac{1}{5} \oint_{L_1} x^5 \, dy = \oint_{L_1} x^4 y \, dx \equiv B_{41}, \quad -\oint_{L_1} x^3 y^2 \, dy = \oint_{L_1} x^2 y^3 \, dx \equiv B_{23}. \]

Eq. (2.2) and the straightforward computation gives

\[ M_1(\delta) = M_2(\delta) = (a_{10} + b_{01}) \oint_{L_1} y \, dx + (b_{21} + 3a_{30}) \oint_{L_1} x^2 y \, dx \]

\[ + \left( b_{03} + \frac{1}{3}a_{12} \right) \oint_{L_1} y^3 \, dx + \left( b_{05} + \frac{1}{5}a_{14} \right) \oint_{L_1} y^5 \, dx \]

\[ + (b_{41} + 5a_{50}) \oint_{L_1} x^4 y \, dx + (a_{32} + b_{23}) \oint_{L_1} x^2 y^3 \, dx \]

\[ = \delta_1 B_{01} + \delta_2 B_{21} + \delta_3 B_{41} + \frac{\delta_4}{3} B_{03} + \frac{\delta_5}{5} B_{05} + \frac{\delta_6}{3} B_{23}. \]

The proof is completed. \( \square \)

Using Mathematics 4.0, for \( b = -\frac{5}{2} \), we have

\[ A_{01} = 2 \int_{0}^{\frac{\sqrt{2}}{2}} \sqrt{\frac{11}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \, dx \doteq 0.44296, \]

\[ A_{21} = 2 \int_{0}^{\frac{\sqrt{2}}{2}} x^2 \sqrt{\frac{11}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \, dx \doteq 0.0433278, \]

\[ A_{03} = 2 \int_{0}^{\frac{\sqrt{2}}{2}} \left( \sqrt{\frac{11}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \right)^3 \, dx \doteq 0.0688018, \]

\[ A_{05} = 2 \int_{0}^{\frac{\sqrt{2}}{2}} \left( \sqrt{\frac{11}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \right)^5 \, dx \doteq 0.01268728, \]
\[
\begin{align*}
A_{23} &= 2 \int_0^{\sqrt{\frac{\pi}{2}}} x^2 \left( \sqrt{\frac{11}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \right)^3 \, dx \doteq 0.00365236, \\
A_{41} &= 2 \int_0^{\sqrt{\frac{\pi}{2}}} x^4 \sqrt{\frac{11}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \, dx \doteq 0.00916792, \\
B_{01} &= 2 \int_0^{\sqrt{\frac{\pi}{2}}} x^2 \sqrt{\frac{11}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \, dx \doteq 0.91058, \\
B_{21} &= 2 \int_0^{\sqrt{\frac{\pi}{2}}} x^4 \sqrt{\frac{11}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \, dx \doteq 1.54361, \\
B_{03} &= 2 \int_0^{\sqrt{\frac{\pi}{2}}} \left( \sqrt{\frac{11}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \right)^3 \, dx \doteq 0.34257, \\
B_{05} &= 2 \int_0^{\sqrt{\frac{\pi}{2}}} \left( \sqrt{\frac{11}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \right)^5 \, dx \doteq 0.1543964, \\
B_{23} &= 2 \int_0^{\sqrt{\frac{\pi}{2}}} x^2 \left( \sqrt{\frac{11}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \right)^3 \, dx \doteq 0.62931, \\
B_{41} &= 2 \int_0^{\sqrt{\frac{\pi}{2}}} x^4 \sqrt{\frac{11}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \, dx \doteq 2.88248.
\end{align*}
\]

Notice the symmetry of system (1.1), we have \(d_1(\epsilon, \delta) = d_2(\epsilon, \delta)\) and \(d_3(\epsilon, \delta) = d_4(\epsilon, \delta)\). Consider the equations \(d_i(\epsilon, \delta) = 0, \ i = 1, 2, 3, 4\). The implicit function theorem implies that two functions

\[
\phi_1(\epsilon, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6) = -\left( \frac{B_{21}}{B_{01}} \delta_2 + \frac{B_{41}}{B_{01}} \delta_3 + \frac{B_{03}}{3B_{01}} \delta_4 + \frac{B_{05}}{5B_{01}} \delta_5 + \frac{B_{23}}{3B_{01}} \delta_6 \right) + O(\epsilon)
\]

and

\[
\phi_2(\epsilon, \delta_3, \delta_4, \delta_5, \delta_6) = \frac{1}{A_{21}B_{01} - A_{01}B_{21}} \left[ (A_{01}B_{41} - A_{41}B_{01})\delta_3 + \frac{A_{01}B_{03} - A_{03}B_{01}}{3} \delta_4 + \frac{A_{01}B_{05} - A_{05}B_{01}}{5} \delta_5 + \frac{A_{01}B_{23} - A_{23}B_{01}}{3} \delta_6 \right] + O(\epsilon)
\]
exist such that for $\varepsilon > 0$ small
\[ d_1(\varepsilon, \delta) = d_2(\varepsilon, \delta) \geq 0 \text{ (resp., } < 0) \text{ if and only if } \delta_1 \geq \phi_1 \text{ (resp., } < \phi_1) , \]
and
\[ d_3(\varepsilon, \delta) = d_4(\varepsilon, \delta) \geq 0 \text{ (resp., } < 0) \text{ if and only if } \delta_2 \leq \phi_2 \text{ (resp., } > \phi_2). \]
Thus, two homoclinic loops $L^*_1(\varepsilon, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6)$ and $L^*_2(\varepsilon, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6)$ exist near $L_1$ and $L_2$ respectively as $\delta_1 = \phi_1$, and a heteroclinic loop $L^*(\delta_3, \delta_4, \delta_5, \delta_6) = L^*_3 \cup L^*_4$ exists near $L = L_3 \cup L_4$ as $\delta_1 = \phi_1$ and $\delta_2 = \phi_2$. In other words, a double figure eight loop $\Gamma^*(\varepsilon, \delta_3, \delta_4, \delta_5, \delta_6) = L^*_1 \cup L^*_2 \cup L^*$ exists near $\Gamma = L_1 \cup L_2 \cup L$ as $\delta_1 = \phi_1$ and $\delta_2 = \phi_2$. Further we consider the stability of the homoclinic loop, heteroclinic loop and the double figure eight loop. Denote by $\lambda_{i1}(\varepsilon, \delta) = -\frac{\lambda_{i2}(\varepsilon, \delta)}{\lambda_{i1}(\varepsilon, \delta)}$ the hyperbolic ratio of $S_{i\varepsilon}$, where $\lambda_{i2} < 0 < \lambda_{i1}$ are the eigenvalues of $S_{i\varepsilon}$ ($i = 1, 2$), and we have $r_i(\varepsilon, \delta) = r_{i0} + \varepsilon r_i^*(\varepsilon, \delta)$. By computing, we know $r_{i0} = r_1(0, 0) = 1$. From (1.1)$_\varepsilon$, the linear part of system (1.1)$_\varepsilon$ at $S_{i\varepsilon}$ is given by the matrix
\[ \begin{pmatrix}
\lambda - f_x(S_{i\varepsilon}) & -f_y(S_{i\varepsilon}) \\
-g_x(S_{i\varepsilon}) & \lambda - g_y(S_{i\varepsilon})
\end{pmatrix}, \]
where $S_{1\varepsilon} = (x_1, y_1) = (-\frac{\sqrt{2}}{2} + O(\varepsilon^2), \frac{\sqrt{2}}{2} (a_{10} + \frac{1}{2}a_{30} + \frac{1}{4}a_{50})\varepsilon + O(\varepsilon^2)), S_{2\varepsilon} = (-x_1, -y_1)$, and
\[ f_x(S_{1\varepsilon}) = f_x(S_{2\varepsilon}) = \left( a_{10} + \frac{3}{2}a_{30} + \frac{5}{4}a_{50} \right) \varepsilon + O(\varepsilon^2), \]
\[ f_y(S_{1\varepsilon}) = f_y(S_{2\varepsilon}) = 1 + O(\varepsilon^2), \]
\[ g_x(S_{1\varepsilon}) = g_x(S_{2\varepsilon}) = \frac{3}{2} + O(\varepsilon^2), \]
\[ g_y(S_{1\varepsilon}) = g_y(S_{2\varepsilon}) = \left( b_{01} + \frac{1}{2}b_{21} + \frac{1}{4}b_{41} \right) \varepsilon + O(\varepsilon^2). \]
Therefore,
\[ \lambda_{11} = \lambda_{21} = \frac{f_x(S_{1\varepsilon}) + g_y(S_{1\varepsilon}) + \sqrt{(f_x(S_{1\varepsilon}) - g_y(S_{1\varepsilon}))^2 + 4f_y(S_{1\varepsilon})g_x(S_{1\varepsilon})}}{2}, \]
\[ \lambda_{12} = \lambda_{22} = \frac{f_x(S_{1\varepsilon}) + g_y(S_{1\varepsilon}) - \sqrt{(f_x(S_{1\varepsilon}) - g_y(S_{1\varepsilon}))^2 + 4f_y(S_{1\varepsilon})g_x(S_{1\varepsilon})}}{2}. \]
And hence,
\[ r_1(\varepsilon, \delta)r_2(\varepsilon, \delta) = (r_1(\varepsilon, \delta))^2 = 1 - 2\text{div}(S_{1\varepsilon})\Delta = 1 - \left( \frac{2\sqrt{6}}{3} + O(\varepsilon) \right) \text{div}(S_{1\varepsilon}), \]
where
\[ \Delta = \frac{\Delta_1}{(f_x(S_{1\varepsilon}))^2 + (g_y(S_{1\varepsilon}))^2 + 2f_y(S_{1\varepsilon})g_x(S_{1\varepsilon}) + \text{div}(S_{1\varepsilon})\Delta_1}, \]
\[ \Delta_1 = \sqrt{(f_x(S_{1\varepsilon}) - g_y(S_{1\varepsilon}))^2 + 4f_y(S_{1\varepsilon})g_x(S_{1\varepsilon})}. \]
By above analysis, we have
\[ r_1(\varepsilon, \delta)r_2(\varepsilon, \delta) \geq 1 \text{ (} < 1 \text{) if and only if } \text{div}(S_{1\varepsilon}) = \text{div}(S_{2\varepsilon}) \leq 0 \text{ (} > 0 \text{)}, \]
\[ (2.3) \]
Under $\delta_1 = \phi_1$, $\delta_2 = \phi_2$, we can obtain

\[
\text{div } |S_1| = \text{div } |S_2| = \varepsilon (P_x + Q_y) (S_1) = \varepsilon \left( \frac{\delta_1}{2} + \frac{\delta_3}{4} \right) + O(\varepsilon^2)
\]

\[
= \frac{1}{A_{01} B_{21} - A_{21} B_{01}} \left( \frac{A_{41} B_{01} - A_{01} B_{41}}{2} + A_{21} B_{41} - A_{41} B_{21} \right)
+ \frac{A_{01} B_{21} - A_{21} B_{01}}{4} \left( \frac{A_{03} B_{01} - A_{01} B_{03}}{2} + A_{21} B_{03} - A_{03} B_{21} \right) \delta_3
+ \frac{1}{5} \left( \frac{A_{05} B_{01} - A_{01} B_{05}}{2} + A_{21} B_{05} - A_{05} B_{21} \right) \delta_5
+ \frac{1}{3} \left( \frac{A_{23} B_{01} - A_{01} B_{23}}{2} + A_{21} B_{23} - A_{23} B_{21} \right) \delta_6 \varepsilon + O(\varepsilon^2)
\]

\[
= \frac{1}{A_{01} B_{21} - A_{21} B_{01}} (X_0 \delta_3 + X_1 \delta_4 + X_2 \delta_5 + X_3 \delta_6)
\equiv \varepsilon \sigma_0(\varepsilon, \delta_3, \delta_4, \delta_5, \delta_6),
\]

where

\[
X_0 = \frac{A_{41} B_{01} - A_{01} B_{41}}{2} + A_{21} B_{41} - A_{41} B_{21} + \frac{A_{01} B_{21} - A_{21} B_{01}}{4},
\]

\[
X_1 = \frac{1}{3} \left( \frac{A_{03} B_{01} - A_{01} B_{03}}{2} + A_{21} B_{03} - A_{03} B_{21} \right),
\]

\[
X_2 = \frac{1}{5} \left( \frac{A_{05} B_{01} - A_{01} B_{05}}{2} + A_{21} B_{05} - A_{05} B_{21} \right),
\]

\[
X_3 = \frac{1}{3} \left( \frac{A_{23} B_{01} - A_{01} B_{23}}{2} + A_{21} B_{23} - A_{23} B_{21} \right).
\]

By computing, we can obtain $X_0 \doteq -0.36242176 < 0$. The implicit function theorem implies that a unique function

\[
\phi_3(\varepsilon, \delta_4, \delta_5, \delta_6) = -\frac{1}{X_0} (X_1 \delta_4 + X_2 \delta_5 + X_3 \delta_6) + O(\varepsilon)
\]

exists such that for $\varepsilon > 0$ small

\[
\sigma_0(\varepsilon, \delta) \geq 0 \quad < 0 \quad \text{if and only if} \quad \delta_3 \leq \phi_3 \quad (> \phi_3).
\]

From (2.3),

\[
\sigma_1(\varepsilon, \delta) \geq 1 \quad < 1 \quad \text{if and only if} \quad \delta_3 \geq \phi_3 \quad (< \phi_3).
\]

By [7], we know that if $\delta_3 = \phi_3$, then the integrals $\int_{L_i^*} (P_x + Q_y) dt = \sigma_{1i}(\varepsilon)$, $i = 1, 2$, and $\int_{L_i^*} (P_x + Q_y) dt = \sigma_{1i}(\varepsilon)$, $i = 3, 4$, converge finitely, and they hold that

\[
\sigma_{1i}(\varepsilon) = \int_{L_i^*} (P_x + Q_y) dt = \int_{L_i} (P_x + Q_y) dt + O(\varepsilon), \quad i = 1, 2,
\]

\[
\sigma_1(\varepsilon) = \int_{L^*} (P_x + Q_y) dt = \int_{L} (P_x + Q_y) dt + O(\varepsilon) = \sum_{i=3}^{4} \int_{L_i} (P_x + Q_y) dt + O(\varepsilon).
\]
Lemma 2.2. (1) Assume $\delta_i = \phi_i$, $i = 1, 2, 3$. Then

$$\sigma_{11} = \sigma_{12} = Y_1 \delta_4 + Y_2 \delta_5 + Y_3 \delta_6 + O(\varepsilon);$$

(2) Assume $\delta_i = \phi_i$, $i = 1, 2, 3$, and $\sigma_{11} = \sigma_{12} = 0$, and then

$$\sigma_1 = 2\sigma_{13} = 2\sigma_{14} = Z_1 \delta_5 + Z_2 \delta_6 + O(\varepsilon),$$

where

$$Y_1 = \frac{b_1(A_{41}B_{03} - A_{03}B_{41} + \frac{A_{03}B_{01} - A_{01}B_{03}}{4})}{3X_0} - \frac{X_1}{X_0} \left(\frac{b_1}{2} + b_2\right) + B_{01},$$

$$Y_2 = \frac{b_1(A_{41}B_{05} - A_{05}B_{41} + \frac{A_{05}B_{01} - A_{01}B_{05}}{4})}{5X_0} - \frac{X_2}{X_0} \left(\frac{b_1}{2} + b_2\right) + B_{03},$$

$$Y_3 = \frac{b_1(A_{41}B_{23} - A_{23}B_{41} + \frac{A_{23}B_{01} - A_{01}B_{23}}{4})}{3X_0} - \frac{X_3}{X_0} \left(\frac{b_1}{2} + b_2\right) + B_{21},$$

$$Z_1 = \frac{a_1(A_{05}B_{41} - A_{41}B_{05} + \frac{A_{01}B_{05} - A_{05}B_{01}}{4})}{5X_0} + \frac{X_2}{X_0} \left(\frac{a_1}{2} + a_2\right) + A_{03}$$

$$- \frac{Y_2}{Y_1} \left[a_1(A_{03}B_{41} - A_{41}B_{03} + \frac{A_{01}B_{03} - A_{03}B_{01}}{4})/3X_0 + \frac{X_1}{X_0} \left(\frac{a_1}{2} + a_2\right) + A_{01}\right],$$

$$Z_2 = \frac{a_1(A_{23}B_{41} - A_{41}B_{23} + \frac{A_{01}B_{23} - A_{23}B_{01}}{4})}{3X_0} + \frac{X_3}{X_0} \left(\frac{a_1}{2} + a_2\right) + A_{21}$$

$$- \frac{Y_3}{Y_1} \left[a_1(A_{03}B_{41} - A_{41}B_{03} + \frac{A_{01}B_{03} - A_{03}B_{01}}{4})/3X_0 + \frac{X_1}{X_0} \left(\frac{a_1}{2} + a_2\right) + A_{01}\right],$$

$$a_1 = 2 \int_0^{\sqrt{\frac{11}{12} - \frac{x^2}{3}}} \frac{1}{x^2} \, dx \approx 1.525974, \quad a_2 = 2 \int_0^{\sqrt{\frac{11}{12} - \frac{x^2}{3}}} \frac{x^2}{\sqrt{\frac{11}{12} - \frac{x^2}{3}}} \, dx \approx 0.261096,$$

$$b_1 = 2 \int_{\sqrt{\frac{x^2}{3}}}^{\sqrt{\frac{11}{12}} - \frac{x^2}{3}} \frac{1}{x^2} \, dx \approx 3.91542, \quad b_2 = 2 \int_{\sqrt{\frac{x^2}{3}}}^{\sqrt{\frac{11}{12}} - \frac{x^2}{3}} \frac{x^2}{\sqrt{\frac{11}{12} - \frac{x^2}{3}}} \, dx \approx 7.22082.$$

Proof. We need only to prove $\sigma_{11}(0) = Y_1 \delta_4 + Y_2 \delta_5 + Y_3 \delta_6$ if $\delta_i = \phi_i|_{\varepsilon = 0}$. In fact, the equations $\delta_i = \phi_i|_{\varepsilon = 0}$, for $i = 1, 2$ imply that

$$\delta_1 = -\left(\frac{B_{02}}{B_{01}} \delta_2 + \frac{B_{41}}{B_{01}} \delta_3 + \frac{B_{03}}{3B_{01}} \delta_4 + \frac{B_{05}}{5B_{01}} \delta_5 + \frac{B_{23}}{3B_{01}} \delta_6\right),$$

$$\delta_2 = \frac{1}{A_{21}B_{01} - A_{01}B_{21}} \left[(A_{01}B_{41} - A_{41}B_{01}) \delta_3 + \frac{A_{01}B_{03} - A_{03}B_{01}}{3} \delta_4ight.$$}

$$+ \frac{A_{01}B_{05} - A_{05}B_{01}}{5} \delta_5 + \frac{A_{01}B_{23} - A_{23}B_{01}}{3} \delta_6\right],$$

$$\delta_3 = -\frac{1}{X_0} (X_1 \delta_4 + X_2 \delta_5 + X_3 \delta_6).$$

Hence, by symmetry, we have
\[ \sigma_{11}(0) = \sigma_{12}(0) = \oint (P_x + Q_y) \, dt \]

\[ = \delta_1 \oint \frac{1}{y} \, dx + \delta_2 \oint \frac{x^2}{y} \, dx + \delta_3 \oint \frac{x^4}{y} \, dx \]

\[ + \delta_4 \oint y \, dx + \delta_5 \oint y^3 \, dx + \delta_6 \oint x^2 y \, dx \]

\[ = \left( \delta_1 + \frac{\delta_2}{2} + \frac{\delta_3}{4} \right) \oint \frac{1}{y} \, dx + \left( \delta_2 + \frac{\delta_3}{2} \right) \oint \frac{x^2}{y} \, dx \]

\[ + \delta_4 \oint y \, dx + \delta_5 \oint y^3 \, dx + \delta_6 \oint x^2 y \, dx \]

\[ = \left( \delta_2 + \frac{\delta_3}{2} \right) b_1 + \delta_3 b_2 + B_0 \delta_4 + B_0 \delta_5 + B_2 \delta_6. \]

Substituting \( \delta_1, \delta_2, \delta_3 \) into the above equality, we obtain

\[ \sigma_{11}(0) = \sigma_{12}(0) = Y_1 \delta_4 + Y_2 \delta_5 + Y_3 \delta_6. \]

By computing, we know \( Y_1 \approx 0.546349 > 0 \). Therefore the implicit function theorem implies that a unique function \( \phi_4(\delta_4, \delta_5, \delta_6) = -\frac{Y_2}{Y_1} \delta_5 - \frac{Y_3}{Y_1} \delta_6 + O(\varepsilon) \) exists such that for \( \varepsilon > 0 \) small

\[ \sigma_{11} \geq 0 \quad (\varepsilon < 0) \quad \text{if and only if} \quad \delta_4 \geq \phi_4(\varepsilon < \phi_4). \]

In the same way, under \( \delta_i = \phi_i, \ i = 1, 2, 3, 4, \) we can obtain \( \sigma_1 = Z_1 \delta_5 + Z_2 \delta_6 + O(\varepsilon) \). This completes the proof. \( \square \)

From Lemma 2.2, we know \( Z_1 \approx -0.066157 < 0 \). By the implicit function theorem again there exists a unique function

\[ \phi_5(\varepsilon, \delta_6) = -\frac{Z_2}{Z_1} \delta_6 + O(\varepsilon) \quad \text{if and only if} \quad \delta_5 \leq \phi_5(\varepsilon > \phi_5). \]

If denotes by \( R_{1i} \) the first order saddle value at the saddle points \( S_{i\varepsilon} \) of the system (1.1) \( \varepsilon, i = 1, 2, \) then by [9], we have

**Lemma 2.3.** Assume \( \delta_i = \phi_i, \ i = 1, \ldots, 5, \) then

\[ R_{11} = R_{21} = \left( -\frac{7}{9} \delta_1 - \frac{23}{90} \delta_2 - \frac{83}{180} \delta_3 + \frac{3}{10} \delta_4 + \frac{3}{20} \delta_6 \right) \varepsilon + O(\varepsilon^2). \]

**Proof.** If let \( T_i \) be an reversible matrix such that \( \det T_i = 1, \) \( T_i D_i T_i^{-1} = \text{diag}(\lambda_{i1}, \lambda_{i2}), \) where

\[ D_i = \frac{\partial (f, g)}{\partial (x, y)}(S_i), \ i = 1, 2, \) and \( T_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \) then

\[ T_2 \begin{pmatrix} a_{10} + \frac{3 a_{10}}{2} + \frac{5 a_{20}}{4} \varepsilon + O(\varepsilon^2) \\ \frac{3}{2} + O(\varepsilon^2) \end{pmatrix} \begin{pmatrix} 1 + O(\varepsilon^2) \\ b_{01} + \frac{b_{11}}{2} + \frac{b_{21}}{4} \varepsilon + O(\varepsilon^2) \end{pmatrix} = \begin{pmatrix} \lambda_{21} & 0 \\ 0 & \lambda_{22} \end{pmatrix} T_2 \]

and \( ad - bc = 1. \) Thus, we can obtain the following equations
\[ a \lambda_{21} = \frac{3b}{2} + a \left( \frac{3}{2}a_{10} + \frac{5}{4}a_{50} \right) \varepsilon + O(\varepsilon^2),\]
\[ b \lambda_{21} = a + b \left( \frac{b_{01} + \frac{b_{21}}{2} + \frac{b_{41}}{4}}{2} \right) \varepsilon + O(\varepsilon^2),\]
\[ c \lambda_{22} = \frac{3d}{2} + c \left( \frac{3}{2}a_{10} + \frac{5}{4}a_{50} \right) \varepsilon + O(\varepsilon^2),\]
\[ d \lambda_{22} = c + d \left( \frac{b_{01} + \frac{b_{21}}{2} + \frac{b_{41}}{4}}{2} \right) \varepsilon + O(\varepsilon^2).\]

Therefore, we have
\[ T_2 \left( \begin{array}{c} \frac{1}{\sqrt{6}} + \frac{b_{01} + \frac{b_{21}}{2} + \frac{b_{41}}{4} - a_{10} - \frac{1}{2}a_{30} - \frac{5}{4}a_{50}}{12} \varepsilon + O(\varepsilon^2) \\ \frac{1}{2} + \frac{\sqrt{6}(a_{10} + \frac{3}{2}a_{30} + \frac{5}{4}a_{50} - b_{01} - \frac{1}{2}b_{21} - \frac{1}{4}b_{41})}{12} \varepsilon + O(\varepsilon^2) \end{array} \right), \]

Now making a linear transformation of the form
\[ \left( \begin{array}{c} u \\ v \end{array} \right) = T_2 \left( \begin{array}{c} x - x_2 \\ y - y_2 \end{array} \right), \]

where
\[ S_2 \varepsilon = (x_2, y_2) = \left( \frac{\sqrt{2}}{2} + O(\varepsilon^2), -\frac{\sqrt{2}(a_{10} + \frac{3}{2}a_{30} + \frac{5}{4}a_{50})}{2} \varepsilon + O(\varepsilon^2) \right). \]

For \((u, v)\) near the origin, we obtain from (1.1)\(_\varepsilon\),
\[ \dot{u} = Y + \frac{\sqrt{6}}{3} \left( -X^5 + \frac{5}{2}X^3 - X \right) + \left[ -\frac{\sqrt{2}}{2} A + a_{10}X + a_{30}X^3 + a_{12}XY^2 + a_{14}XY^4 \\
+ a_{50}X^5 + a_{32}X^3Y^2 \right] \varepsilon + O(\varepsilon^2) \]
\[ \equiv \lambda_{21} \left[ u + \sum_{k=2}^{3} \sum_{j+l=k} m_{jl}u^jv^l + O(|u, v|^4) \right], \]
\[ \dot{v} = -\frac{\sqrt{6}}{4} Y + \frac{1}{2} \left( -X^5 + \frac{5}{2}X^3 - X \right) + \left[ -\frac{\sqrt{6}}{4} \left( -\frac{\sqrt{2}}{2} A + a_{10}X + a_{30}X^3 + a_{12}XY^2 \\
+ a_{14}XY^4 + a_{50}X^5 + a_{32}X^3Y^2 \right) + \frac{2}{3} \left( -5X^4N + \frac{15}{2}X^2N - N + b_{01}Y + b_{21}X^2Y + b_{03}Y^3 \\
+ b_{05}Y^3 + b_{41}X^4Y + b_{23}X^2Y^3 \right) + \frac{1}{2} \left( b_{01} + \frac{b_{21}}{2} + \frac{1}{4}b_{41} - a_{10} - \frac{3}{2}a_{30} - \frac{5}{4}a_{50} \right) \right] \varepsilon + O(\varepsilon^2) \]
\[ \equiv -\lambda_{22} \left[ -v + \sum_{k=2}^{3} \sum_{j+l=k} n_{jl}u^jv^l + O(|u, v|^4) \right], \]
where

\[ X = \frac{\sqrt{2}}{2} + \frac{u}{2} - \frac{\sqrt{6}}{3}v, \quad Y = \frac{\sqrt{6}}{4}u + v, \quad A = a_{10} + \frac{a_{30}}{2} + \frac{a_{50}}{4}, \]

\[ N = \left( \frac{\sqrt{6}}{12}u + \frac{3}{4}v \right) \left( a_{10} + \frac{3}{2}a_{30} + \frac{5}{4}a_{50} - b_{01} - \frac{1}{2}b_{21} - \frac{1}{4}b_{41} \right). \]

According to [9] the first saddle value of (1.1) \( \varepsilon \) at \( S_{i\varepsilon} \) is given by

\[ R_{i1} = m_{21} + n_{12} - m_{20}m_{11} + n_{02}n_{11}, \quad i = 1, 2. \quad (2.4) \]

The straightforward computing give

\[
m_{21} = \frac{5\sqrt{6}}{12} + \left[ -\frac{5(a_{10} + b_{01})}{12} + \frac{(a_{12} + 3b_{03})}{4} + \frac{3(a_{32} + b_{23})}{8} \right. \\
- \frac{3(3a_{30} + b_{21})}{8} - \frac{29(5a_{50} + b_{41})}{48} \] \( \varepsilon + O(\varepsilon^2), \)

\[
m_{20} = \frac{5\sqrt{2}}{24} + \frac{1}{48\sqrt{3}} \left[ (18a_{12} + 36a_{30} + 9a_{32} + 60a_{50} - 20b_{01} + 14b_{21} + 19b_{41})\varepsilon \right] \\
+ O(\varepsilon^2), \)

\[
m_{11} = -\frac{5\sqrt{3}}{9} + \frac{1}{18\sqrt{2}} \left[ (20a_{10} + 18a_{12} - 6a_{30} + 9a_{32} - 35a_{50})\varepsilon \right] + O(\varepsilon^2), \)

\[
n_{02} = \frac{5\sqrt{3}}{18} - \frac{1}{36\sqrt{2}} \left[ (18a_{12} + 36a_{30} + 9a_{32} + 60a_{50} - 20b_{01} + 14b_{21} + 19b_{41})\varepsilon \right] \\
+ O(\varepsilon^2), \)

\[
n_{11} = -\frac{5\sqrt{2}}{12} - \frac{1}{24\sqrt{3}} \left[ (20a_{10} + 18a_{12} - 6a_{30} + 9a_{32} - 35a_{50})\varepsilon \right] + O(\varepsilon^2), \)

\[
n_{12} = -\frac{5\sqrt{3}}{12} + \left[ -\frac{5(a_{10} + b_{01})}{12} + \frac{(a_{12} + 3b_{03})}{4} + \frac{3(a_{32} + b_{23})}{8} \right. \\
- \frac{3(3a_{30} + b_{21})}{8} - \frac{29(5a_{50} + b_{41})}{48} \] \( \varepsilon + O(\varepsilon^2). \)

Substituting \( m_{21}, m_{10}, m_{11}, n_{02}, n_{11}, n_{12} \) into (2.4), we have

\[ R_{21} = \left( -\frac{7}{9}\delta_1 - \frac{23}{90}\delta_2 - \frac{83}{180}\delta_3 + \frac{3}{10}\delta_4 + \frac{3}{20}\delta_6 \right)\varepsilon + O(\varepsilon^2). \]

Using the same arguments, we obtain

\[ R_{11} = R_{21} = \left( -\frac{7}{9}\delta_1 - \frac{23}{90}\delta_2 - \frac{83}{180}\delta_3 + \frac{3}{10}\delta_4 + \frac{3}{20}\delta_6 \right)\varepsilon + O(\varepsilon^2). \]

This is the end of Lemma 2.3. \( \square \)

By [7,9,10], we know if \( r_{10} = r_{20} = 1, \sigma_1 = 0, \) and then \( \sigma_2 = R_{11} + R_{21} = 2R_{11}, \) where \( \sigma_2 \) is defined as [9]. Hence, we have \( \sigma_2 = 2R_{11} > 0 \) for (1.1) \( \varepsilon \) when \( \delta_i = \phi_i, i = 1, \ldots, 5. \) Further we have the following rule to discriminate the stability of \( L_i^* \) (\( i = 1, 2 \)), \( L^* \) and the double figure eight loop \( \Gamma^* \).
Lemma 2.4. For $\varepsilon > 0$ small, the homoclinic loops $L^*_1$ and $L^*_2$ are stable inside (resp., unstable) if $\sigma_0(\varepsilon, \delta) < 0$ (resp., $> 0$) or $\sigma_0 = 0$ and $\sigma_1 = \sigma_1 < 0$ (resp., $\sigma_1 = \sigma_1 > 0$) or $\sigma_0 = \sigma_1 = \sigma_1 = 0$ and $R_{11} = R_{21} > 0$ (resp., $R_{11} = R_{21} < 0$).

Lemma 2.5. For $\varepsilon > 0$ small, the heteroclinic loop $L^*$ is stable inside (resp., unstable) if $r_1(\varepsilon, \delta) = 0$ (resp., $r_1(\varepsilon, \delta) < 0$) or $r_1(\varepsilon, \delta) = 1$ and $\sigma_1 < 0$ (resp., $\sigma_1 > 0$) or $r_1(\varepsilon, \delta) = 1$ and $\sigma_1 = 0$ and $\sigma_2 > 0$ (resp., $\sigma_2 < 0$).

Lemma 2.6. For $\varepsilon > 0$ small, the double figure eight loop $\Gamma^*$ is stable outside (resp., unstable) if $r_1(\varepsilon, \delta) = 0$ (resp., $r_1(\varepsilon, \delta) = 1$) or $r_1(\varepsilon, \delta) = 1$ and $\sigma_1 < 0$ (resp., $\sigma_1 > 0$) or $r_1(\varepsilon, \delta) = 1$ and $\sigma_1 = 0$ and $\sigma_2 > 0$ (resp., $\sigma_2 < 0$).

3. Proof of the main results

In the following, we will find a larger limit cycles which surrounds all five singular points. By [11,14] we need to consider the relative position of separatrices near the double figure eight loop $\Gamma^*$ and the behavior of orbits near a large periodic orbit $L^*_h$. The first order Melnikov function of $L^*_h$ is

$$M^*(h) = \oint_{L^*_h} Q \, dx - P \, dy, \quad h > \frac{11}{96}.$$ 

Here $L^*_h$: $H(x, y) = h$ surrounds all singular points of system (1.1) for $h > \frac{11}{96}$ and $\varepsilon > 0$ small. From Lemmas 2.1 and 2.2, when $\delta_i = \phi_i$, $i = 1, \ldots, 5$, we can obtain

$$M^*(\frac{13}{96}) = \delta_1 \oint_{L^*_h} y \, dx + \delta_2 \oint_{L^*_h} x^2 y \, dx + \frac{1}{4} \delta_4 \oint_{L^*_h} y^3 \, dx + \frac{1}{5} \delta_5 \oint_{L^*_h} y^5 \, dx$$

$$+ \delta_3 \oint_{L^*_h} x^4 \, dx + \frac{1}{3} \delta_6 \oint_{L^*_h} x^2 y^3 \, dx + O(\varepsilon).$$

Using Mathematica 4.0, for $h = \frac{13}{96}$, we have

$$\oint_{L^*_h} y \, dx = 2 \int_{x_1}^{x_2} \sqrt{\frac{13}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \, dx \approx 3.12974,$$

$$\oint_{L^*_h} x^2 y \, dx = 2 \int_{x_1}^{x_2} x^2 \sqrt{\frac{13}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \, dx \approx 3.48546,$$

$$\oint_{L^*_h} x^4 \, dx = 2 \int_{x_1}^{x_2} x^4 \sqrt{\frac{13}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \, dx \approx 6.22568,$$

$$\oint_{L^*_h} y^3 \, dx = 2 \int_{x_1}^{x_2} \left(\sqrt{\frac{13}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2}\right)^3 \, dx \approx 1.006234.$$
\[ \oint_{L^*} x^2 y^3 \, dx = 2 \int_{x_1}^{x_2} x^2 \left( \sqrt{\frac{13}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \right)^3 \, dx \doteq 1.47457, \]

\[ \oint_{L^*_h} y^5 \, dx = 2 \int_{x_1}^{x_2} \left( \sqrt{\frac{13}{48} - \frac{x^6}{3} + \frac{5x^4}{4} - x^2} \right)^5 \, dx \doteq 0.429204, \]

where

\[ x_1 = -x_2 = -\frac{1}{4} \sqrt{60 + (53568 - 3456\sqrt{58})^{1/3} + 12(31 + 2\sqrt{58})^{1/3}}. \]

Hence, \( M^*(h) = -0.00226\delta_6 + O(\varepsilon) \). By the Poincaré–Bendixson theorem one large cycle \( \Gamma_1 \) exists which surrounds five singular points since the double figure eight loop \( \Gamma^* \) is unstable for \( \sigma_2 > 0 \) (see Fig. 4).

Now we are in a position to prove our main result. From the above analysis, we know the single homoclinic loop \( L_i^* \) (\( i = 1, 2 \)) and the heteroclinic loop \( L^* \) are stable, and the double figure eight loop is unstable, when \( R_{11} > 0 \). Keep \( (\varepsilon, \delta_6) \) fixed and let \( \delta_5 < \phi_5 \) and \( 0 < \phi_5 - \delta_5 \ll \varepsilon \). Thus \( L^* \) has changed their stability from stable into unstable, and hence one small stable limit cycle \( L_{31} \) has appeared inside \( L^* \). Keep \( \delta_5 \) fixed and let \( \delta_4 \) satisfy \( 0 < \delta_4 - \phi_4 < \phi_5 - \delta_5 \ll \varepsilon \), thus \( L_1^* \) and \( L_2^* \) have changed their stability from stable into unstable, and hence two small stable limit cycles \( L_{11} \) and \( L_{21} \) have appeared with \( L_{11} \subset L_1^* \) and \( L_{21} \subset L_2^* \), see Fig. 5.
Now keep $\delta_4$ fixed and let $\delta_3$ satisfy $0 < \delta_3 - \phi_3 \ll \delta_4 - \phi_4 \ll \phi_5 - \delta_5 \ll \varepsilon$. Again the stability of $L_1^*$ and $L_2^*$ changed, and then another two small unstable limit cycles $L_{12}$ and $L_{22}$ are born out with $L_{11} \subset L_{12} \subset L_1^*$ and $L_{21} \subset L_{22} \subset L_2^*$. Simultaneously the stability of heteroclinic loop $L_3^*$ has got changed again, and then a small unstable limit cycle $L_{32}$ has appeared with $L_{31} \subset L_{32} \subset L_3^*$. Also the double eight figure loop $\Gamma^*$ has changed its stability from unstable into stable, and hence a large unstable limit cycle $\Gamma_2$ has appeared outside $\Gamma^*$ with $\Gamma_2 \subset \Gamma_1$, see Fig. 6.

Keep $\delta_3$ fixed, if we change $\delta_2$ by $0 < \delta_2 - \phi_2 \ll \delta_3 - \phi_3 \ll \delta_4 - \phi_4 \ll \phi_5 - \delta_5 \ll \varepsilon$. And then $L_3^*$ and $L_4^*$ have broken, and hence generated another one small stable limit cycle $L_{33}$ with $L_{31} \subset L_{32} \subset L_{33} \subset L_*$. Finally keep $\delta_2$ fixed and let $\delta_1$ satisfy $0 < \phi_1 - \delta_1 \ll \delta_2 - \phi_2 \ll \delta_3 - \phi_3 \ll \delta_4 - \phi_4 \ll \phi_5 - \delta_5 \ll \varepsilon$ so that $L_1^*$ and $L_2^*$ have broken and two small stable limit cycles $L_{13}$ and $L_{23}$ have appeared with $L_{11} \subset L_{12} \subset L_{13} \subset L_1^*$ and $L_{21} \subset L_{22} \subset L_{23} \subset L_2^*$, see Fig. 7.

Under $\delta_i = \phi_i$, $i = 1, \ldots, 5$, we have

\[
\text{div}(O(0, 0)) = \delta_1 \varepsilon + O(\varepsilon^2) \doteq -0.0226 \delta_6 \varepsilon + O(\varepsilon^2) < 0,
\]

\[
\text{div}(O_{1\varepsilon}) = \text{div}(O_{2\varepsilon}) = (\delta_1 + 2 \delta_2 + 4 \delta_3) \varepsilon + O(\varepsilon^2) \doteq -0.20055 \delta_6 \varepsilon + O(\varepsilon^2) < 0.
\]

Hence, the singular points $O$, $O_{1\varepsilon}$ and $O_{2\varepsilon}$ are stable. Notice that $L_{11}$, $L_{21}$ and $L_{31}$ are also stable. By the Poincaré–Bendixson theorem, we know there are three small unstable limit cycles $L_{10}$, $L_{20}$ and $L_{30}$ with $L_{10} \subset L_{11}$, $L_{20} \subset L_{21}$ and $L_{30} \subset L_{31}$ respectively. The proof of Fig. 1(a) is completed. Using the same arguments, we can obtain the second distribution.

This is the end of proof for the main result.
References

[1] N.N. Bautin, On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type, Mat. Sb. (N.S.) 30 (72) (1952) 181–196.