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Equivariant Euler characteristics of discriminants of reflection groups

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ABSTRACT

Let G be a finite, complex reflection group acting on a complex vector space V, and δ its discriminant polynomial. The fibres of δ admit commuting actions of G and a cyclic group. The virtual $G \times C_m$ character given by the Euler characteristic of a fibre is a refinement of the zeta function of the geometric monodromy, calculated in [8]. We show that this virtual character is unchanged by replacing δ by a slightly more general class of polynomials. We compute it explicitly, by studying the poset of normalizers of centralizers of regular elements in G, and the subspace arrangement given by the proper eigenspaces of elements of G. As a consequence we also compute orbifold Euler characteristics and find some new 'case-free' information about the discriminant.

1. SUMMARY

Let G be a finite reflection group acting on the vector space $V = \mathbf{C}^\ell$. Let $\mathcal{A}$ denote the set of reflecting hyperplanes of G. For each $H \in \mathcal{A}$, let $\alpha_H \in V^*$ be a linear functional with kernel H. The discriminant polynomial δ of G is defined to be

$$\delta = \prod_{H \in \mathcal{A}} \alpha_H^{e_H},$$

where e_H is the order of the subgroup of G that fixes H pointwise. δ is the G-

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invariant polynomial of smallest degree whose zero set is exactly the set of reflecting hyperplanes. Let $m = \deg \delta$.

The fibres of δ over $\mathbf{C}^*$ are diffeomorphic, by a theorem of Milnor [16]; let $F = \delta^{-1}(1)$, the Milnor fibre of δ. The action of G on V restricts to an action on F. At the same time, a cyclic group C_m acts on F, generated by a geometric monodromy map $h : F \to F$ defined by $h(x) = e^{2\pi i/m}x$.

The actions of G and C_m commute. Let $\Gamma = G \times C_m$. Then $H_\bullet(F, \mathbf{C})$ is a finite-dimensional representation of Γ. In this paper, we consider the Euler characteristic of F, valued in the character ring of Γ. That is, we define a virtual character χ_Γ by

$$\chi_\Gamma(F)(g) = \sum_{p \geq 0} (-1)^p \mathrm{Tr}\,(g, H_p(F, \mathbf{C})). \tag{1.1}$$

This is a refinement of the usual zeta function of the monodromy of the Milnor fibre, which Denef and Loeser [8] have calculated for all reflection groups. Their technique uses Springer's theory of regular elements [20]: $g \in G$ is called a *regular element* of G iff it has an eigenvector that is not contained in any reflecting hyperplane.

In particular, Springer [20] has shown that the centralizer of a (noncentral) regular element in G acts as a reflection group on a (proper) subspace of V. Using this idea and its elaboration in [8,15], we find a recursive formula for the Euler characteristic χ_Γ (Theorem 3.13).

For a fixed reflection group G, let $M_G = M = V - \cup_{H \in \mathcal{A}} H$ be the hyperplane complement, and U its image in $\mathbf{P}(V)$. These spaces have been studied extensively in the context of hyperplane arrangement theory; see, for example, [14]. On the other hand, let $\mathcal{E}$ denote the set of all maximal eigenspaces E of elements of G for which $E \subsetneq V$. Let

$$M^\circ = V - \bigcup_{E \in \mathcal{E}} E, \tag{1.2}$$

and U° the image of M° in $\mathbf{P}(V)$. U° is the complement of a projective subspace arrangement, in the sense of Björner [1], and to the authors' knowledge has not been studied directly before.

$G/Z(G)$ acts freely on U°. We find a formula for the Euler characteristic of the orbit space in terms of degrees, codegrees, and regular numbers (4.22), and we calculate it for each irreducible G (Theorem 3.15). We show that this determines χ_Γ for each G (Theorem 3.13).

2. Springer's theory of regular elements

In this section, we recall the theory of regular elements and set up our notation. We refer to [17] for background on reflection groups and hyperplane arrangements and to [20] for background on the theory of regular elements.

Let $V = \mathbf{C}^\ell$ and let G be a finite reflection group acting on V. We will denote by $\mathbf{C}[V]$ the algebra of polynomial functions on V. The *degrees* $d_1, \ldots, d_\ell$ of G are the degrees of any set of homogeneous polynomials which generate the G-

invariant polynomial ring $\mathbf{C}[V]^G$. The order of G and of its centre $Z(G)$ are determined in terms of its degrees:

$$(2.3) \qquad |G| = \prod_{i=1}^{\ell} d_i \qquad |Z(G)| = \gcd\{d_i\}$$

A vector $v \in V$ is called *regular* if it is not contained in a reflection hyperplane of G. An element $g \in G$ is called *regular* if it has a regular eigenvector. Let $g \in G$ be regular of order d. Let v be a regular eigenvector with corresponding eigenvalue ξ and let $V(g, \xi)$ denote the ξ-eigenspace of g. We will refer to (g, ξ) as a *regular* (*d*-)*pair*.

With this notation, we have:

Theorem 2.4 (Springer [20]).

(a) *The root of unity ξ has order d.*

(b) *$V(g, \xi)$ has dimension $a(d) = |\{i : d|d_i\}|$.*

(c) *The centralizer $C_G(g)$ is a reflection group in $V(g, \xi)$ whose degrees are $\{d_i : d|d_i\}$ and whose order is $\prod_{d|d_i} d_i$.*

The orders of the regular elements of G are called the *regular numbers* of G. Let $\mathcal{R}$ denote the poset of regular numbers, ordered by divisibility.

The group $G \leq GL(V)$ also acts naturally on the algebra of polynomial vector fields on V, $\mathbf{C}[V] \otimes V$. The module $(\mathbf{C}[V] \otimes V)^G$ is free over $\mathbf{C}[V]^G$. Following [2], the *codegrees* $d_1^*, \ldots, d_\ell^*$ are defined to be the degrees of a homogeneous basis, with the convention that derivations have degree -1. By a theorem of Orlik and Solomon [19]

$$(2.5) \qquad \sum_{i=1}^{l} \dim H^i(U, \mathbf{C}) t^i = \prod_{i=2}^{\ell} (1 + (d_i^* + 1)t)$$

Using a case-based argument, Denef and Loeser [8, Theorem 2.8] proved that, for a regular d-pair (g, ξ), the codegrees of $C_G(g)$ acting on $V(g, \xi)$ are

$$(2.6) \qquad \{d_i^* : d|d_i^*\}$$

Lehrer and Springer [15, Theorem C] later reproved this result in a case-free way.

3. EULER CHARACTERISTICS

Following [2], for each G-orbit of hyperplanes $C \in \mathcal{A}/G$, set

$$\delta_C = \prod_{H \in C} \alpha_H^{e_C},$$

where e_C is defined to be the common value of e_H for all $H \in C$. noting that e_H is constant for all $H \in C$. Consider any homogeneous, G-invariant polynomial $f \in \mathbf{C}[V]^G$ with zero locus equal to $\bigcup_{H \in \mathcal{A}} H$. Then f has the form

(3.7) $$f = \prod_{C \in \mathcal{A}/G} \delta_C^{a_C}$$

for some positive integers a_C; in particular, the discriminant is obtained by choosing all $a_C = 1$. We shall call such f *unreduced discriminant polynomials.*

Denote the degree of f as above by m. Then $f(gv) = f(\zeta v) = \zeta^m f(v)$ for any regular element g with eigenvalue v, so the order of ζ must divide m. That is, all regular numbers $d \in \mathcal{R}$ divide m.

Let $F = f^{-1}(1)$. Note that the projectivization map from M to U restricted to F shows that F is a cyclic m-fold cover of U.

Let $P = \pi_1(M, 1)$, the pure Artin braid group corresponding to the group G. Since F is homotopy-equivalent to an infinite cyclic cover of M, we have $H_\bullet(F, \mathbf{C}) \cong H_\bullet(M, \mathbf{C}[x, x^{-1}])$, where $\mathbf{C}[x, x^{-1}] \cong \mathbf{C} \uparrow_{\pi_1(F)}^{P}$ as a P-module; see [9, Section 2.1] and [10]. Explicitly, P has a set of generators $\{\gamma_H : H \in \mathcal{A}\}$ for which γ_H acts by multiplication by $x^{a_C e_C}$.

The complement M is known to be a $K(P, 1)$ space for all irreducible reflection groups with the possible exception of G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, and G_{34}; see [2, (2.11)] for references. Thus we have $H_\bullet(F, \mathbf{C}) \cong H_\bullet(P, \mathbf{C}[x, x^{-1}])$ except perhaps in these cases. Let $B = \pi_1(M/G, 1)$, the braid group. Then $H_\bullet(P, \mathbf{C}[x^{\pm 1}])^G \cong H_\bullet(B, \mathbf{C}[x^{\pm 1}])$ (over $\mathbf{C}$). For all real reflection groups, this is computed explicitly in [7].

The Lefschetz zeta function of F is defined to be

$$Z(F) = \prod_{p \geq 0} \det(1 - h_* t|_{H_p(F,\mathbf{C})})^{(-1)^p},$$

where h^* denotes a preferred generator of C_m acting in homology. Since a complex representation of C_m is determined by the characteristic polynomial of a generator of the group, the zeta function can be seen as the restriction to C_m of the Euler characteristic $\chi_\Gamma(F)$ defined in (1.1), written multiplicatively. We will identify $C_m = \langle h^* \rangle$ with the cyclic group of m elements in $\mathbf{C}^*$. For convenience, we will take the convention that $\alpha \in C_m$ acts on F by multiplication by α^{-1}.

For a reflection group G and integers $d|m$, define

$$I_d(G) = I_d = 1 \uparrow_{C_d}^{G \times C_m}$$

whenever C_d is a cyclic subgroup of order d generated by a regular pair (g, ζ) of order d.

Lemma 3.8. *The cyclic groups generated by any two regular pairs of the same order are conjugate in Γ. In particular, the definition of $I_d(G)$ above does not depend on the choice of regular pair (g, ζ). The normalizer in Γ of the cyclic subgroup generated by $(g, \xi) \in \Gamma$ is $C_G(g) \times C_m$.*

Proof. Let (g, ζ) and (g', ξ) be two regular pairs of order d, generating cyclic groups K and K', respectively. Since any two primitive roots of unity of the same order generate the same cyclic subgroup of $\mathbf{C}^*$, $\xi = \zeta^k$ for some k. Then (g^k, ξ) is also a regular pair of order d, so g^k and g' are conjugate, by [20, 4.2]. It follows that the subgroups K and K' are conjugate. $I_d(G)$ is well defined since the permutation characters induced from K and K' are the same [6, 10.12]. Since $(h, \alpha) \in N_\Gamma\langle(g, \xi)\rangle$ iff $(hgh^{-1}, \alpha) = (g^k, \xi^k)$ for some k iff $h \in C_G(g)$, $\alpha \in C_m$, we have that $N_\Gamma(\langle(g, \xi)\rangle) = C_G(g) \times C_m$. □

The following theorem appeared independently as [8, Theorem 2.5] and [14, Corollary 5.8].

Lemma 3.9. *If (g, ζ) is a regular pair and V is the ζ-eigenspace of g, then the centralizer $C(g)$ acts as a reflection group on V. Its reflecting hyperplanes are $\{H \cap V : H \in \mathcal{A}\}$.*

Definition 3.10. For a regular d-pair (g, ζ), we define $U(g, \zeta)$ as the projective hyperplane complement for $C_G(g)$ acting on $V(g, \zeta)$ where (g, ζ) is a regular d-pair.

From Lemma 3.8, $C_G(g)$ is conjugate to $C_G(g')$ if (g, ζ) and (g', ξ) are both regular d-pairs. This means that they are isomorphic as reflection groups, and we will refer to them, up to isomorphism, as $G(d)$, as in [15].

By the lemma above, then, $U(g, \zeta)$ and $U(g', \xi)$ are diffeomorphic; we will refer to them as $U_G(d)$.

Definition 3.11. Define a poset $\mathcal{D} = \mathcal{D}_G$ by

$$\mathcal{D} = \{d : d = |Z(C_G(g))| \text{ for a regular element g}\},$$

ordered by divisibility. That is, $\mathcal{D}$ is the set of orders of regular elements g that are maximal with respect to the property of having a given centralizer. For elements $d \in \mathcal{R}_G$, define $\lceil d \rceil$ to be the least multiple of d in $\mathcal{D}$.

Note that $\{G(d) : d \in \mathcal{D}\}$ forms a complete set of representatives of the isomorphism classes of centralizers of regular elements. Observe that $\lceil d \rceil = |Z(G(d))| = \gcd\{d_i : d | d_i\}$.

Recall $M^\circ \subseteq M$ from (1.2), and $U^\circ \subseteq U$. By construction,

Proposition 3.12. *$G/Z(G)$ acts freely on U°.*

We can now state our main result.

Theorem 3.13. *Let G be a reflection group, f an unreduced discriminant polynomial* (3.7) *of degree m, and F its Milnor fibre. Then*

$$\chi_\Gamma(F) = \sum_{d \in \mathcal{D}} a_d I_d,$$

where the integers a_d are given by

$$a_d = \chi(U(d)^\circ / G(d)) \tag{3.14}$$

A case analysis gives a more refined description. First, $\chi_\Gamma(F)$ is zero unless G is irreducible, since $\chi(U) = 0$ in this case: this appears first in the language of matroids in [5]. Denef and Loeser [8, 2.9] show that the centralizers of regular elements in irreducible G are themselves irreducible. With this in mind, it is enough to calculate $a_z = \chi(U^\circ/G)$, where $z = |Z(G)|$, for each irreducible G. We obtain:

Theorem 3.15. *For an irreducible reflection group G of rank n, $\chi(U^\circ/G) = (-1)^{n-1}$ if G is in the list below. Otherwise, $\chi(U^\circ/G) = 0$.*

(a) *Irreducibles of rank ≤ 2;*

(b) *Irreducibles of the form $G(d\ell, e\ell, \ell)$, except $G(3,3,3)$;*

(c) *$G(d\ell, e\ell, 2\ell)$ where $e\ell$ is odd;*

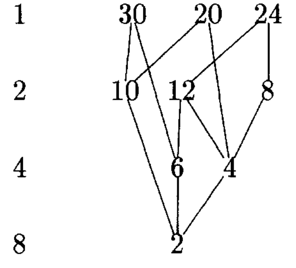
(d) *Exceptionals G_{29} and G_{34}.*

Corollary 3.16. *For a Milnor fibre of a given reflection group as above, χ_Γ is a linear combination of at most six permutation characters I_d, with coefficients ± 1.*

The value of χ_Γ for each irreducible reflection group is tabulated in Section 6.

We also observe empirically that, like the zeta function, χ_Γ continues to be a braid diagram invariant, in the sense of Broué, Malle, and Rouquier [2].

Example 3.17. Let G be the irreducible reflection group of type E_8. The degrees are $[2, 8, 12, 14, 18, 20, 24, 30]$ and the poset $\mathcal{D}$ is:



where the numbers on the left of the diagram are the ranks of each of the centralizer subgroups corresponding to the elements of $\mathcal{D}$ in that row. By the Shephard-Todd classification, the centralizers of elements of order 4 and 6 are, respectively, G_{31} and G_{32}. Applying Theorems 3.13 and 3.15 shows that

$$\chi_{E_8 \times C_m} = I_{30} + I_{24} + I_{20} - I_{12} - I_{10} - I_8.$$

The rest of this paper is as follows. Theorem 3.13 is proven in Section 4. Theorem 3.15 is proven in Section 6. We are unable to provide a case-free proof of Theorem 3.15 in general, although in Section 5 we do so for rank 2 irreducibles.

4. PROOF OF THEOREM 3.13

Throughout this section, fix a particular reflection group G acting on $V = \mathbf{C}^\ell$. We will use the notation of Section 3.

For a regular element $g \in G$ of order d, let $i_G(d)$ be the order of its centralizer, and let $u_G(d)$ be the Euler characteristic of $U(d)$. Then $i(d) = \prod_{d|d_i} d_i$ (Theorem 2.4). By evaluating (2.5) at $t = -1$ and using (2.6), we see $u(d) = \prod_{d|d_i^*, i>1}(-d_i^*)$.

If (g, ζ) is a regular pair of order d, then by [20, 3.4], there is a regular pair (h, ξ) of order $\lceil d \rceil$ for which $C(g) = C(h)$ (and, hence, $U(g, \zeta) = U(h, \xi)$). It follows that $i(d) = i(\lceil d \rceil)$ and $u(d) = u(\lceil d \rceil)$.

For $(g, \zeta) \in G \times C_m$, we have that

$$
\begin{aligned}
\chi_\Gamma(F)(g,\zeta) &= \chi(F^{(g,\zeta)}) = \deg(\delta) \cdot \chi(U(g,\zeta)) \\
&= \begin{cases} m \cdot u(\lceil d \rceil) & \text{if } (g,\zeta) \text{ is regular of order } d; \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
\tag{4.18}
$$

where the first equality follows from a refinement of the Hopf-Lefschetz fixed point theorem [3] and the second equality is a consequence of the fact that F is a cyclic m-fold cover of U.

Lemma 4.19. *There exist rationals $\{a_d : d \in \mathcal{D}\}$ for which*

$$\chi_\Gamma = \sum_{d \in \mathcal{D}} a_d I_d. \tag{4.20}$$

Proof. We claim that, if I_d is induced from a regular pair (g, ζ) of order d, then

$$I_d(h,\xi) = \begin{cases} \frac{m}{d}|C(h)| & \text{if } (h,\xi) \text{ is regular of order } d', d'|d; \\ 0 & \text{otherwise.} \end{cases} \tag{4.21}$$

This follows by directly evaluating the induced character and using Lemma 3.8: $I_d(h, \xi)$ is nonzero only if (h, ξ) is conjugate to a power of (g, ζ), the generator of C_d, which is equivalent to (h, ξ) being a regular pair of an order dividing that of g.

Define an equivalence relation $\sim$ on $G \times C_m$ by setting $(g, \zeta) \sim (h, \xi)$ iff either: neither is a regular pair; or, both are regular pairs, and $\lceil g \rceil = \lceil h \rceil$.

The equivalence classes of $\sim$ are unions of conjugacy classes. Moreover, (4.21) shows that each I_d is constant on classes of $\sim$.

The characters I_d span the $\mathbf{Q}$-vector space of functions that are both constant on $\sim$ classes and zero on the nonregular equivalence class, since their values on the regular pairs form a triangular matrix, by (4.21).

By (4.18), χ_Γ is such a class function, which completes the proof. □

4.1. coefficients a_d from (co)degrees

By evaluating (4.20) on a regular pair of order $d \in \mathcal{D}$, we obtain

$$m \cdot u(d) = \sum_{k \in \mathcal{D}: d|k} a_k \frac{m}{k} i(d),$$

whence by Möbius inversion,

$$a_d = d \sum_{k:d|k} \mu(d,k) \frac{u(k)}{i(k)},$$

(4.22)
$$= d \sum_{k:d|k} \mu(d,k) \prod_{\substack{i:k|d_i^* \\ d_i^* \neq 0}} d_i^* \prod_{i:k|d_i} d_i^{-1}$$

where μ is the Möbius function of the poset $\mathcal{D}$.

Remark 4.23. Any rational character of a finite group can be expressed as a rational linear combination of induced permutation characters from cyclic subgroups by the Artin-Brauer induction theorem [6, 15.2]. The coefficients can be determined and are non-zero only if the character is non-zero on that cyclic subgroup. In our situation, however, the cyclic subgroups $C_d, d \in \mathcal{D}$ are representatives of isomorphism classes (hence a subset of the set) of cyclic subgroups of Γ with $\chi_\Gamma(F)$ non-zero. Our argument above is then a direct way to compute the coefficients $a_d, d \in \mathcal{D}$ for our special case.

4.2. induction from a centralizer subgroup.

Given a regular element $g_0 \in G$ of order $e \in \mathcal{D}$, let $H = C_G(g_0)$. From [20, 4.2], and Definition 3.11 we observe:

Lemma 4.24. *The maximal regular numbers of H are* $\mathcal{D}_H = \{d \in \mathcal{D}_G : e|d\}$.

By Lemma 4.19,

$$\chi_{H\times C_m} = \sum_{d\in\mathcal{D}_H} a'_d I_d(H), \quad \text{and}$$

$$\chi_{G\times C_m} = \sum_{d\in\mathcal{D}} a_d I_d$$

for some coefficients $\{a'd\}$, and $\{a_d\}$.

Theorem 4.25. *If G and H are as above, and $m'|m$, then*

$$\chi_{G\times C_m} = \chi_{H\times C_{m'}} \uparrow_{H\times C_{m'}}^{G\times C_m} + \sum_{d\in\mathcal{D}:e\nmid d} a_d I_d.$$

Consequently, in the notation above, $a_d = a'_d$ for all $d \in \mathcal{D}_H$.

Proof. We need to show that the values a_d given by equation (4.22) for $d \in \mathcal{D}_H$ are the same in H as they are in G. Specifically, we need to show for multiples k of e, that $i_G(k) = i_H(k)$, and that $u_G(k) = u_H(k)$.

So suppose that $g \in H$ is a regular element of order k, where $e|k$. Then g_0 is conjugate to a power of g, so $C_G(g) \subseteq C_G(g_0) = H$; that is, $C_G(g) = C_H(g)$, so $i_G(k) = i_H(k)$.

Now assume, without loss of generality, that $g = g_0^r$ for some r, and choose ζ so that (g,ζ) is a regular pair. Then the ζ eigenspace of g in V_G is contained in V_H, so $U_G(g,\zeta) = U_H(g,\zeta)$ (Lemma 3.9.) Then $u_G(k) = u_H(k)$ as well. □

4.3. interpretation of coefficients a_d.

For $d \in \mathcal{D}$, let $F_d = F \cap V_d$, where

$$V_d = \bigcap_{d \nmid d_i} f_i^{-1}(0),$$

the variety of eigenvectors of elements of G having eigenvalue a primitive dth root of unity; see [20, 3.2]. In particular, $F_z = F$, where z is the order of the centre of G. Let

$$F_d^\circ = F_d - \bigcup_{\substack{d|d' \\ d' \neq d}} F_{d'}.$$

Proposition 4.26. *For any $d \in \mathcal{D}$,*

$$\chi_{G \times C_m}(F_d) = \sum_{\substack{d' \in \mathcal{D} \\ d|d'}} \chi_{G \times C_m}(F_{d'}^\circ).$$

Proof. The finite collection $\{F_{d'} : d|d'\}$ of closed subsets of F_d, is closed under intersections and contains F_d as an element. So in the terminology of [11, 2], this is an Eulerian collection. In addition, by [11, 2.5], the equivariant Euler characteristic is an additive function. Then the result follows directly from [11, 2.2]. □

To complete the proof of Theorem 3.13, it remains to show that the coefficients a_d given by Lemma 4.19 satisfy $a_d = \chi(U(g, \zeta)^\circ / C_G(g))$ for regular pairs (g, ζ) of order $d \in \mathcal{D}$.

Let $z = |Z(G)|$. From Proposition 3.12, the quotient map $F_z^\circ \to U^\circ/G$ is a covering with deck transformation group $G \times C_m/\langle (g, \zeta) \rangle$, where (g, ζ) is any regular z-pair. It follows that

$$\begin{aligned} \chi_{G \times C_m}(F^\circ) &= \chi(U^\circ/G) \cdot 1 \uparrow_{C_z}^{G \times C_m}, \\ &= \chi(U^\circ/G) \cdot I_z. \end{aligned}$$

By Theorem 4.25, then, for each $d \in \mathcal{D}$,

$$\chi_\Gamma(F_d^\circ) = \chi(U(d)^\circ/G(d)) I_d,$$

Now, using Proposition 4.26 with $d = z$, we have

$$\begin{aligned} \chi_\Gamma(F) &= \sum_{d \in \mathcal{D}} \chi_\Gamma(F_d^\circ) \\ &= \sum_{d \in \mathcal{D}} a_d I_d; \end{aligned}$$

equating the coefficient of I_d for each d gives the characterization of the values a_d that we claimed.

For finite groups acting on $\mathbf{C}^2$, a stronger version of Theorem 3.15(a) is obtained from the theory of du Val singularities, for which we refer to [12].

Theorem 5.37. *Let G be an irreducible finite subgroup of $U_2(\mathbf{C})$. Then*

$$\chi(U_G^\circ/G) = -1.$$

Proof. Note that a finite subgroup G of $GL_\ell(\mathbf{C})$ can always be embedded in $U_\ell(\mathbf{C})$. Let $Z := Z(U_\ell(\mathbf{C})) = \{\alpha I : \alpha\bar{\alpha} = 1\}$. Let $H := (G \cdot Z) \cap SU_\ell(\mathbf{C})$; then $Z(H) = H \cap SU_\ell(\mathbf{C})$. It is easy to check that the map $G/Z(G) \to H/Z(H)$ defined by $gZ(G) \mapsto g(\det(g))^{1/\ell})Z(H)$ is an isomorphism; therefore G and H are both central extensions of the same group $\bar{H}$. Then G and H act on $\mathbf{C}^\ell$ and $\bar{H}$ acts on $\mathbf{P}^{\ell-1}$. Proper eigenspaces for the action of G on $\mathbf{C}^\ell$ are the same as those for H. Moreover, if v is an eigenvector for an element $h \in H$, its image $[v] \in \mathbf{P}^{\ell-1}$ is a fixed point of $hZ(H) \in \bar{H}$.

Now, let $\ell = 2$. Note that for $G \leq U_2(\mathbf{C})$, there exists a reflection group $G' \leq U_2(\mathbf{C})$ with $\bar{H} = G/Z(G) \cong G'/Z(G')$. Then $\mathbf{P}^1/\bar{H} \cong \mathbf{P}^1/G \cong \mathbf{P}^1/G' \cong \mathbf{P}^1$ where the last isomorphism follows from the Shephard Todd Chevalley theorem.

Since $PSU_2(\mathbf{C}) \cong SO_3$, $\bar{H}$ is isomorphic to a finite subgroup of SO_3. The finite subgroups of SO_3 are known: they are the groups of symmetry of the regular polyhedra: cyclic, dihedral, tetrahedral (A_4), octahedral (S_4) and icosahedral (A_5). Since G was assumed to act irreducibly, $\bar{H}$ is not cyclic. For the remaining finite subgroups of SO_3, Klein [13] showed that there are exactly three orbits in S^2 of points with nontrivial stabilizers (corresponding to vertices, barycenters of edges of the regular polyhedron, and barycenters of faces.) Since $\mathbf{P}^1/G \cong \mathbf{P}^1$, it follows that $U_G^\circ/G = \mathbf{P}^1 - \{p_0, p_1, p_2\}$, where the p_i's are the 3 'special' orbits under the action of H on $\mathbf{P}^1$. Thus $\chi(U_G^\circ/G) = 2 - 3 = -1$. □

6. PROOF OF THEOREM 3.15

The Shephard-Todd classification of irreducible (complex) reflection groups consists of one infinite family and 34 exceptional groups labelled as $G_4, \ldots, G_{37}$. The tables in Figures 1,2,3 give χ_Γ for the exceptional groups; these values are readily calculated from (4.22).

For any reflection group G, put $c(G) = \chi(U^\circ/G)$. From Theorem 3.13, this equals the coefficient of $I_{|Z(G)|}$ in χ_Γ. By Theorem 4.25, for $d \in \mathcal{D}$, the coefficient a_d of I_d is $c(G(d))$.

Theorem 3.15 claims that for exceptional irreducible G, $c(G) = 0$ unless G has rank $\ell = 2$ or G is one of G_{29} or G_{34}, in which case $c(G) = (-1)^{\ell-1}$. Our proof in ranks $\ell > 2$ is by inspection, after having computed χ_Γ for each group.

The rest of this section is devoted to proving Theorem 3.15 for the infinite family of reflection groups $G(r, p, \ell)$.

6.1. (co)degrees and regular numbers.

For $r, p \in \mathbf{N}$ with $p|r$ and rank $\ell \geq 2$, $G(r,p,\ell)$ is a group of order $r^\ell/p\ell!$. It is the semidirect product of the symmetric group S_ℓ acting by permutations on the standard basis $\{e_i : 1 \leq i \leq \ell\}$, and the group of diagonal maps $e_i \mapsto \theta_i e_i$, where $\theta_i^r = 1$ and $(\theta_1 \cdots \theta_\ell)^q = 1$ where $q = r/p$. The group acts irreducibly on $\mathbf{C}^\ell$ iff $r > 1$ and $(r,p,\ell) \neq (2,2,2)$. $G(r,1,\ell)$ is the full monomial group $C_r^\ell \rtimes S_\ell$ and that the Weyl groups $A_{\ell-1}, B_\ell, D_\ell, G_2$ and the dihedral groups $I_2(\ell)$ equal respectively $G(1,1,\ell), G(2,1,\ell), G(2,2,\ell), G(6,6,2)$ and $G(\ell,\ell,2)$. The codegrees are calculated in [18], and the regular numbers appear in [4]. The degrees of $G(r,p,\ell)$ are

$$(6.28)\qquad \begin{cases} r, 2r, \ldots, (\ell-1)r, \ell q, & p|r, r > 1 \\ 2, 3 \ldots, (\ell-1), \ell, & p = r = 1. \end{cases}$$

the order of the center z is $q\gcd(r,\ell)$. The codegrees are

$$(6.29)\qquad \begin{cases} 0, r, 2r, \ldots, (\ell-1)r, & p < r \\ 0, r, 2r, \ldots, (\ell-2)r, (\ell-1)r - \ell & p = r > 1, \ell > 1 \\ 0, 1, 2, \ldots, (\ell-2), & p = r = 1, \ell > 2 \end{cases}$$

Note that the adjustments in the degrees and codegrees of $G(1,1,\ell) = S_\ell$ are made so that S_ℓ acts irreducibly.

Remark 6.30. *Since the degrees and codegrees of $G(d)$ are those of G which are divisible by d, we see for $G = G(r,p,\ell)$ and $e = (\gcd(d,r))^{-1}d$, the degrees of $G(d)$ are*

$$(6.31)\qquad \begin{cases} er, 2er, \ldots, \lfloor\frac{\ell-1}{e}\rfloor er, & d \nmid \ell q \\ er, 2er, \ldots, \lfloor\frac{\ell-1}{e}\rfloor er, \ell q, & d|lq \\ 2, 3, \ldots, \ell, & d = r = 1, \end{cases}$$

as noted in [15]. Lehrer and Springer prove indirectly in [15, 5.2] that $G(d) = G(r',p',\ell')$ for $G = G(r,p,\ell)$ and $d \in \mathcal{R}$. In the proposition below we make the determination of $G(d)$ explicit to help in our computation of $c(G)$.

Proposition 6.32.

(a) *For $G = G(r,p,\ell)$, $q = r/p > 1$, we have $\mathcal{R} = \{d : d|\ell q\}$ and $\mathcal{D} = \{kq : t|k|\ell\}$ where $t = \gcd(p,\ell)$. Then for $kq \in \mathcal{D}$, $G(kq) = G(kr/t, p, \ell t/k)$.*

(b) *For $G = G(r,r,\ell)$, $\ell > 1$, we have $\mathcal{R} = \{d : d|\ell q\} \cup \{d : d|(\ell-1)r\}$ and*

$$\mathcal{D} = \{kr : k|\ell - 1\} \cup \{d : z|d|\ell\}$$

where $z = \gcd(r,\ell)$. For $k|\ell-1, kr \neq z$, we have $G(kr) = G(kr, 1, (\ell-1)/k)$ and for $z|d|\ell$, we have $G(d) = G(dr/z, r, \ell z/d)$.

Proof. In each case, let $z = |Z(G)|$, equal to the greatest common divisor of the degrees. In the first case, $z = \gcd(pq, \ell q) = tq$, where $t = \gcd(p,\ell)$. In the second case, this is just $z = \gcd(r,\ell)$.

Case (a): For $G = G(r,p,\ell)$ with $q > 1$, the only maximal regular degree of G is ℓq [4, 2.11]. Thus the set of regular numbers is $\mathcal{R} = \{d : d|\ell q\}$. If $d \in \mathcal{D}$, then d is a gcd of a subset of the degrees. So $qt|d$, since $z = qt$ is the gcd of all the degrees. To show that

$$\mathcal{D} = \{kq : \gcd(p,\ell)|k|\ell\}$$

we need to show that $G(d)$ are distinct for distinct d in this set. Observe that if $\gcd(p,\ell)|k|\ell$, then $\gcd(p,k) = \gcd(p,\ell)$. But as is implicitly shown in the proof of [15, 5.2] we have for $t|k|\ell$,

$$G(kq) = G(kr/t, p, \ell t/k)$$

So we have shown that $\mathcal{D}$ and $G(d), d \in \mathcal{D}$ are described as in the claim.
Case (b): For $G = G(r,r,\ell)$, with $\ell > 1$, the maximal regular degrees are $(\ell - 1)r$ and ℓ [4, 2.11]. So the set of regular numbers is

$$\mathcal{R} = \{d : d|\ell\} \cup \{d : d|(\ell - 1)r\}$$

For $d \in \mathcal{D}$, $z = \gcd(r,\ell)$, and $z|d$ since d is the gcd of a subset of the degrees. If $d \in \mathcal{D}$ does not divide ℓ, then $d|(\ell - 1)r$ and d is a gcd of a subset of $\{r, \ldots, (\ell - 1)r\}$ so that $r|d$. We have shown that $\mathcal{D} \subseteq T_1 \cup T_2$, where

$$T_1 = \{d : z|d|\ell\} \quad \text{and} \quad T_2 = \{kr : k|\ell - 1\}. \tag{6.33}$$

Note that these sets intersect iff $\gcd(r,\ell) = r$, iff $r|\ell$. To show the inclusion is an equality, we have to show that $G(d)$ are distinct for distinct $d \in T_1 \cup T_2$. If $d \in T_1$, we have $\gcd(r,d) = \gcd(r,\ell) = z$, so that by the proof of [15, 5.2], $G(d) = G(dr/z, r, \ell z/d)$.

On the other hand, if $d = kr$ where $k|\ell - 1$ but $d \neq z$, we have by the proof of [15, Proposition 5.2] that $G(kr) = G(kr, 1, (\ell - 1)/k)$. These groups have distinct parameters for each $d \in T_1 \cup T_2$, so $\mathcal{D} = T_1 \cup T_2$ as claimed. □

6.2. Proof of Theorem 3.15 for $G(r,p,\ell)$.

It remains to show that $c(G) = 0$ for irreducible $G = G(r,p,\ell)$, except for the parameters $(de\ell, e\ell, \ell) \neq (3,3,3)$, and for $(de\ell, e\ell, 2\ell)$, where $e\ell$ is odd. For these exceptions, we show $c(G) = (-1)^{\ell-1}$. It will be convenient to let $S(m,k) = 1/m \sum_{d|m} \mu(d)(-1)^{md/k-1}$, where μ is the (number-theoretic) Möbius function on $\mathbf{N}$.

Lemma 6.34.

$$S(m,k) = \begin{cases} (-1)^{km-1}, & m = 1, \text{ or } m = 2 \text{ and } k \text{ odd} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Recall that $\sum_{d|m} \mu(d) = \delta_{m,1}$. Consider the case where k is odd and m even. Write $m = 2^s t$ for t odd. Then

$$S(m,k) = \frac{1}{m}\left(-\sum_{d|m,\, m/d \text{ even}} \mu(d) + \sum_{d|m,\, m/d \text{ odd}} \mu(d)\right)$$
$$= \frac{1}{m}\left(-\sum_{d|m/2} \mu(d) - \sum_{d|m,\, 2^s|d} \mu(2^s)\mu(\frac{d}{2^s})\right)$$
$$= -\frac{2}{m}\delta_{m,2} = -\delta_{m,2}.$$

The remaining cases are similar. □

At the same time, we will calculate χ_Γ.

Proposition 6.35. *Let $\Gamma = G \times C_m$ and $m = \deg(\delta_G)$. Then*

(a) *For $G = G(r,p,\ell)$ and $q = r/p > 1$,*

$$\chi_\Gamma = \begin{cases} I_{\ell q}, & \ell \text{ odd}; \\ -I_{\ell q}, & \ell \text{ and } p \text{ even}; \\ I_{\ell q} - I_{q\ell/2}, & \ell \text{ even, } p \text{ odd}. \end{cases}$$

(b) *For $G = G(r,r,\ell)$, and $\ell > 1$,*

$$\chi_\Gamma = \begin{cases} I_{(\ell-1)r} - I_{r(\ell-1)/2} + I_\ell, & \ell \text{ odd}; \\ I_{(\ell-1)r} - I_\ell, & r, \ell \text{ even}; \\ I_{(\ell-1)r} + I_\ell - I_{\ell/2} & \ell \text{ even, } r \text{ odd}. \end{cases}$$

Proof. We will handle C_r as the special case $G(r,1,1)$ of (a) and S_ℓ as the special case $G(1,1,\ell)$ of (b). Recall that $n = \mathrm{rk}\,(G(r,p,\ell)) = \ell$ unless $r = p = 1$ when $n = \mathrm{rk}\,(G(1,1,\ell)) = \ell - 1$ since $G(1,1,\ell) = S_\ell$ acts irreducibly on an $\ell - 1$ dimensional space.

(a) For $G = G(r,p,\ell)$, $p < r$, we have

$$\mathcal{D} = \{kq : t|k|\ell\}$$

where $t = \gcd(p,\ell)$ and for $t|k|\ell$, $G(kq) = G(kr/t, p, \ell t/k)$. So

$$c(G) = a_{tq} = tq \sum_{tq|kq} \mu(tq, kq) \frac{u(kq)}{i(kq)}$$
$$= \frac{tq}{\ell q} \sum_{t|k|\ell} \mu(k/t)(-1)^{\ell t/k - 1}$$
$$= S(\ell/t, t)$$
$$= \begin{cases} (-1)^{\ell-1} & \ell = t \text{ or } \ell = 2t, t \text{ odd } (\ell|p \text{ or } \frac{\ell}{2}|p, p \text{ odd.}) \\ 0 & \text{otherwise.} \end{cases}$$

For $G(kq) = G(kr/t, p, \ell t/k)$ with $t|k|\ell$ we have

$$c(G(kq)) = a_{kq} = \begin{cases} (-1)^{\ell t/k - 1}, & k = \ell \text{ or } k = \ell/2, p \text{ odd} \\ 0, & \text{otherwise.} \end{cases}$$

This shows that

$$\chi_\Gamma = \begin{cases} I_{\ell q}, & \ell \text{ odd;} \\ -I_{\ell q}, & p, \ell \text{ even;} \\ I_{\ell q} - I_{q\ell/2}, & \ell \text{ even, } p \text{ odd,} \end{cases}$$

as required.

(b) For $G = G(r,r,\ell)$, $\ell > 1$, by Proposition 6.32, $\mathcal{D} = T_1 \cup T_2$, defined in (6.33). For $d \in T_1$, we have $G(d) = G(dr/z, r, \ell z/d)$, and for $d \in T_2$, $G(d) = G(kr, 1, (\ell-1)/k)$. Since $z = \gcd(r, \ell)$, we have $z \in T_1$, and $z \in T_2$ if and only if $z = r$.

$$\begin{aligned} c(G) = a_z &= z \sum_{d \in \mathcal{D}} \mu(z,d) \frac{u(d)}{i(d)} \\ &= z\frac{u(z)}{i(z)} + z \sum_{d \in T_1 - \{z\}} \mu(z,d)\frac{u(d)}{i(d)} + z \sum_{kr \in T_2 - \{z\}} \mu(z,kr)\frac{u(kr)}{i(kr)} \\ &= (-1)^{\ell-1}\frac{z((\ell-1)r-\ell)}{\ell(\ell-1)r} + \frac{z}{\ell}\sum_{1 \neq \frac{d}{z} | \frac{\ell}{z}} \mu(\frac{d}{z})(-1)^{\frac{\ell/z}{d/z}z-1} + \\ &\quad + \frac{z}{(\ell-1)r}\sum_{k|\ell-1, kr \neq z} \mu(z,kr)(-1)^{(\ell-1)/k-1} \\ &= (-1)^{\ell-1}\frac{z}{(\ell-1)r} + S(\ell/z, z) + \frac{z}{(\ell-1)r}\sum_{k|\ell-1, kr\neq z} \mu(z,kr)(-1)^{(\ell-1)/k-1} \end{aligned}$$

Note that if $z \neq r$ then $\mu(z,r) = -1$ and $\mu(z,kr) = 0$ for all $1 \neq k|\ell-1$ whereas if $z = r$ then $\mu(z,kr) = \mu(kr/z) = \mu(k)$. So we have

$$c(G) = S(\ell/z, z) + \delta_{zr}S(\ell-1, 1)$$

This means that $c(G) = (-1)^{n-1}$ where $n = \operatorname{rk}(G)$ iff

$$\begin{cases} \ell | r, & (r,\ell) \neq (2,2),(3,3) \\ \frac{\ell}{2} | r & r \text{ odd} \\ (r,\ell) = (1,3) & \end{cases}$$

Otherwise $c(G) = 0$. Note that $c(G(2,2,2)) = 0$ agrees with the statement since $G(2,2,2)$ is reducible. Also observe that if $G = G(r,p,\ell)$ is a rank 2 irreducible, we obtain $c(G) = -1$ as was predicted in Section 5. This includes the case of the rank 2 irreducible $G(1,1,3) = S_3$.

To compute χ_Γ, it remains only to find a_d, for $d \in \mathcal{D}$. If $d \in T_1$, by part (a),

$$a_d = c(G(\frac{dr}{z}, r, \frac{\ell z}{d})) = \begin{cases} (-1)^{\frac{\ell z}{d}-1} & d = \ell \text{ , or } d = \ell/2, r \text{ odd;} \\ 0 & \text{otherwise.} \end{cases}$$

For $d \in T_2$,

$$a_d = c(G(kr, 1, \frac{\ell-1}{k})) = \begin{cases} 1 & d = (\ell-1)r; \\ -1 & d = r(\ell-1)/2, \ell \text{ odd;} \\ 0 & \text{otherwise.} \end{cases}$$

The expression for χ_Γ in (b) follows. □

#	χ_Γ	#	χ_Γ
3	I_r	13	$I_{12} - I_4$
4	$I_6 + I_4 - I_2$	14	$I_{24} - I_6$
5	$I_{12} - I_6$	15	$-I_{12}$
6	$I_{12} - I_4$	16	$I_{30} + I_{20} - I_{10}$
7	$-I_{12}$	17	$I_{60} - I_{20}$
8	$I_{12} + I_8 - I_4$	18	$I_{60} - I_{30}$
9	$I_{24} - I_8$	19	$-I_{60}$
10	$I_{24} - I_{12}$	20	$I_{30} + I_{12} - I_6$
11	$-I_{24}$	21	$I_{60} - I_{12}$
12	$I_8 + I_6 - I_2$	22	$I_{20} + I_{12} - I_4$

Figure 1. χ_Γ for G_3 and rank-2 exceptionals

#	χ_Γ	#	χ_Γ
23	$I_{10} + I_6$	28	$I_{12} + I_8 - I_6 - I_4$
24	$I_{14} + I_6$	29	$I_{20} - I_4$
25	$I_{12} + I_9 - I_6$	30	$I_{30} + I_{20} + I_{12} - I_{10} - I_6 - I_4$
26	I_{18}	31	$I_{24} + I_{20} - I_{12} - I_8$
27	I_{30}	32	$I_{30} + I_{24} - I_{12}$

Figure 2. χ_Γ for exceptionals of rank 3 and 4

#	rank	χ_Γ
33	5	$I_{18} + I_{10}$
34	6	$I_{42} - I_6$
35	6	$I_{12} + I_8 - I_6 - I_4$
36	7	$I_{18} + I_{14}$
37	8	$I_{30} + I_{24} + I_{20} - I_{12} - I_{10} - I_8$

Figure 3. χ_Γ for exceptionals of rank ≥ 5

7. ORBIFOLD EULER CHARACTERISTICS

The orbifold Euler characteristic of a space X under the action of a group G is defined to be

$$\sum_{[g]} \chi(X^g / C_G(g)),$$

where the sum is taken over all conjugacy classes of G.

The orbifold Euler characteristic of the Milnor fibre F under the action of a $G \times C_m$ can be expressed in terms of the integers $\{a_d\}$ from (3.14), and we include its calculation here as an example.

Lemma 7.36. *For any reflection group G,*

$$\chi(U/G) = \sum_{d \in \mathcal{D}} a_d.$$

Proof. We have a disjoint union $U = \bigcup_{d \in \mathcal{D}} G \cdot U(d)^\circ$. The claim follows by Proposition 3.12, additivity of the equivariant Euler characteristic (cf. [11, 2.5]), and the definition $a_d = \chi(U(d)^\circ)/G(d)$. □

Theorem 7.37. *Let G be a reflection group, f an unreduced discriminant polynomial of degree m, and F its Milnor fibre. The orbifold Euler characteristic of F with respect to* $\Gamma = G \times C_m$ *equals*

$$\sum_{d \in \mathcal{D}} d a_d,$$

in the notation of Section 3.

Proof. For a regular pair (g, ζ), we have $F^{(g,\zeta)}/C(g,\zeta) = U^g/C_G(g)$. Recall that if (g,ζ) is not a regular pair, then the set of fixed points is empty, so we need only consider a sum over conjugacy classes of regular pairs in $G \times C_m$.

Using Lemma 3.8, there are $\phi(d)$ conjugacy classes of regular pairs of order d, for each regular number d. So we have

$$\begin{aligned}
\sum_{[(g,\zeta)]} \chi(F^{(g,\zeta)}/C(g,\zeta)) &= \sum_{d \in \mathcal{R}} \phi(d)\chi(U^g/C_G(g)) \\
&= \sum_{d \in \mathcal{R}} \phi(d) a_{\lceil d \rceil} \\
&= \sum_{d \in \mathcal{D}} \sum_{d' | d} \phi(d') a_d \\
&= \sum_{d \in \mathcal{D}} d a_d,
\end{aligned}$$

where the second equality follows from Lemma 7.36 together with Lemma 3.9. □

Remark 7.38. By way of comparison, the ordinary or orbifold Euler characteristics of F/Γ are equal to the image of χ_Γ under homomorphisms from the character ring of Γ to $\mathbf{Z}$ that take I_d to 1, or to d, respectively.

8. CONCLUDING REMARKS

This investigation leaves the obvious open question of whether Theorem 3.15 could be proven in a more conceptual way. Our proof depends on knowing (co)degrees and regular numbers for each group, which are not reflected in the simplicity of the statement.

We also note that $\Gamma = G \times C_m$ is not the most general group for which these calculations make sense. In general one should replace G by $N(G)$, the normalizer of G in $U(V)$, and C_m by $\Lambda = C_m \rtimes \mathrm{Gal}(K_m/K)$, where K is the splitting field for $N(G)$ and K_m is the extension of K containing all mth roots of unity. Note that $\mathrm{Gal}(K_m/K)$ is a finite group of order dividing $\phi(m)$ which acts on $\mathbf{C}$ by inflation to $\mathrm{Gal}(\mathbf{C}/K)$, since K_m/K is a Galois extension. This action can be extended to a diagonal action on $\mathbf{C}^n$ which stabilizes $F = \delta^{-1}(1)$ since the coefficients of δ lie in K. Note that $\mathrm{Gal}(K_m/K) \cap C_m = 1$ and that $\mathrm{Gal}(\mathrm{K_m/K})$ normalizes C_m. The actions of $\mathrm{Gal}(K_m/K)$ and $N(G)$ on F commute by construction. It may be interesting to examine the Λ-module structure of the equivariant Euler characteristic χ_Λ. It is probable that an answer would involve Springer's twisted regular numbers.

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