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Harmonic analysis of the space of $S_a \times S_b \times S_c$-invariant vectors in the irreducible representations of the symmetric group

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Abstract

In this paper, we find an orthogonal basis for the $S_a \times S_b \times S_c$-invariant vectors in the irreducible representations $S^{(\alpha,\beta,\gamma)}$ of the symmetric group. The basis chosen is part of a Gel'fand basis (or adapted basis) coming from the chain of subgroups $S_{a+b+c} > S_{a+b} \times S_c > S_a \times S_b \times S_c$. This is a generalization and a completion of the work of Dunkl [Pacific J. Math. 92 (1981) 57–71], who considered the $S_a \times S_b \times S_c$-invariant vectors in $S^{(N-k,k)}$.

Keywords: Hahn polynomials; Representation theory of the symmetric group; Radon transform; K-invariant vector

1. Introduction

Let G be a finite group and K a subgroup of G. The pair (G, K) is said to be a (finite) Gel'fand pair [2] if the permutation representation of G on the homogeneous space G/K decomposes without multiplicity, or equivalently, if the convolution algebra of bi-K-invariant functions defined on G is commutative. Many examples of finite Gel'fand

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pairs, where G is a Weyl group or a Chevalley group over a finite field, were studied by P. Delsarte, C. Dunkl and D. Stanton; see the surveys [7,15] or the book [11]. The most studied pair was $(S_N, S_{N-m} \times S_m)$, where S_h denotes the symmetric group on h letters: see for example the papers [1,4–6,15], and [3] for a probabilistic application. In [7] Dunkl suggested to study the nonGel'fand pair $(S_N, S_a \times S_b \times S_c)$, where $N = a + b + c$. In [8] Dunkl constructed a basis for the space of $S_a \times S_b \times S_c$-invariant vectors in the irreducible representations of S_N canonically associated to partitions of N in two parts. In the present we determine a basis for the space of $S_a \times S_b \times S_c$ -invariant vectors in the irreducible representations associated to partitions of N in three parts. This completes the work of Dunkl: no other irreducible representation contains nontrivial $S_a \times S_b \times S_c$-invariant vectors.

Let $S^{(\alpha,\beta,\gamma)}$ the irreducible representations associated to the partition (α, β, γ) of N. To reach our goal, we will use the following fact: if V is an irreducible representation of $S_{a+b} \times S_c$, then the dimension of the space of $S_a \times S_b \times S_c$-invariant vectors in V is $\leqslant 1$ and if it is $= 1$ and V appears in $S^{(\alpha,\beta,\gamma)}\downarrow_{S_{a+b}\times S_c}$ then the multiplicity of V in such restriction is equal to 1. For every such a V, we will write a nontrivial $S_a \times S_b \times S_c$-invariant vector in $V \subseteq S^{(\alpha,\beta,\gamma)}$; therefore, the set of all these vectors will form the desired orthogonal basis.

The plan of this paper is the following. In Section 2, we recall some basic properties of Hahn polynomials; all our invariant vectors will be expressed by mean of this family of orthogonal polynomials. In Section 3, we introduce a set of "Radon transforms" d_{ij} (compare with [2,16]) on the "flag manifolds" $S_N/(S_{a_1} \times S_{a_2} \times \cdots \times S_{a_m})$, where $a_1 + a_2 + \cdots + a_m = N$, and some "Laplace operators", for which we give the spectral analysis. We also recall the fundamental characterization of the irreducible representations of the symmetric group as the intersections of the kernels of a set of Radon transforms; such characterization is due to James [10]; see also [13]. In Section 4, the results of the preceding section are interpreted in terms of induced representations. In Section 5, we compute a basis for the space of $S_{N-m} \times S_m$-invariant functions defined on $S_N/(S_a \times S_b \times S_c)$. In Section 6, we study the effect of the operators d_{ij} on the vectors of this basis. In Section 7, we will write an explicit $S_{a+b} \times S_c$-equivariant isomorphism from a subspace of right $S_{a+b-\beta-\gamma} \times S_\beta \times S_\gamma$-invariant functions on S_{a+b} onto the space of S_c-invariant vectors in $S^{(\alpha,\beta,\gamma)}$. Using the results of the preceding sections, we will find the $S_a \times S_b$-invariant vectors in these spaces, obtaining a basis of $S_a \times S_b \times S_c$-invariant vectors in $S^{(\alpha,\beta,\gamma)}$. The main result of the paper is stated in Theorem 7.9

In the present paper, we use several facts from the representation theory of the symmetric group. Most of these facts might be deduced from our computations of K-invariant vectors, in particular from the discussions of our finite difference equations, but this would require longer proof and case by case arguments. Conversely, we will use the representation theory of the symmetric group to shorten many of the proofs of this paper. In other words, if we know that for some values of the parameters an irreducible representation does not contain nontrivial K-invariant vectors then we can avoid the discussion of the finite difference equation that determines such vectors (but the discussion of the finite difference equation would give us the same results of the representation theory).

The present paper is strictly connected with [12], where we solved a problem in [2] on a diffusion model on $S_{nm}/(S_m \times \cdots \times S_m)$. The results in this paper should be a first step towards the harmonic analysis of more complicated diffusion models. The harmonic analysis of invariant functions, in particular on the symmetric group, may also be applied to some statistical problems of Diaconis; see [14].

2. Hahn polynomials

Following [5] and [8], we introduce a family of renormalized Hahn polynomials. For m, a, b, c, x nonnegative integers satisfying: $c \leqslant a + b$, $m \leqslant \min\{a, b, c, a + b - c\}$ and $\max\{c - b, 0\} \leqslant x \leqslant \min\{a, c\}$, we define:

$$E_m(a, b, c, x) = \sum_{j=\max\{0,x-c+m\}}^{\min\{m,x\}} (-1)^j \binom{m}{j} (b - m + 1)_j (-x)_j (a - m + 1)_{m-j} (x - c)_{m-j}.$$

The following is a list of some of their properties ([5] and [8]).

Finite difference equations:

$$(c - x)E_m(a, b, c - 1, x) + xE_m(a, b, c - 1, x - 1) = (c - m)E_m(a, b, c, x), \tag{1}$$

$$(a - x)E_m(a, b, c + 1, x + 1) + (x + b - c)E_m(a, b, c + 1, x) = (a + b - c - m)E_m(a, b, c, x), \tag{2}$$

$$(c - x)(a - x)E_m(a, b, c, x + 1) + x(x + b - c)E_m(a, b, c, x - 1) = \big[(c - x)(a - x) + x(x + b - c) - m(a + b + 1 - m)\big]E_m(a, b, c, x). \tag{3}$$

Symmetry relations:

$$E_m(a, b, c, x) = (-1)^m E_m(b, a, c, c - x), \tag{4}$$

$$E_m(a, b, c, x) = E_m(c, a + b - c, a, x). \tag{5}$$

From the transformation formula (3.8 in [5]) $\sum_{x=\max\{0,c-d+y\}}^{\min\{c,y\}} \binom{y}{x}\binom{d-y}{c-x} E_m(a, b, c, x) = \binom{d-m}{c-m} E_m(a, b, d, y)$ and (5) it follows that

$$\sum_{x=\max\{0,a-d+y\}}^{\min\{a,y\}} \binom{y}{x} \binom{d-y}{a-x} E_m(a, b, c, x) = \binom{d-m}{a-m} E_m(d, a + b - d, c, y). \tag{6}$$

Particular values:

$$E_m(m, b, c, x) = E_m(c, m + b - c, m, x) = (-1)^{m-x} m!(b - c + 1)_x (c - m + 1)_{m-x}. \tag{7}$$

3. Representation theory of the symmetric group and Radon transforms

Let N be a positive integer. A composition of N is an ordered sequence of nonnegative integers $a = (a_1, a_2, \ldots, a_h)$ such that $a_1 + a_2 + \cdots + a_h = N$. A partition λ of N is a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ of N such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_h \geqslant 1$. We recall that there exists a canonical one to one correspondence between the set of all partitions of N and the set of all irreducible representations of the symmetric group S_N [9,10,16]. If λ is a partition of N, the irreducible representation canonically associated to λ is denoted by S^λ. The homogeneous space $\Omega_a = S_N/(S_{a_1} \times S_{a_2} \times \cdots \times S_{a_h})$ will be identified with the set of all "flags" $(A_1, A_2, \ldots, A_h)$ such that for $i = 1, \ldots, h$, A_i is an a_i-subset of $\{1, 2, \ldots, N\}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$ (so $A_1 \cup A_2 \cup \cdots \cup A_h = \{1, 2, \ldots, N\}$). We will denote by M^a the permutation module of all complex valued functions defined on Ω_a. The space M^a will be endowed with the natural scalar product $\langle f_1, f_2\rangle = \sum_{\omega\in\Omega_a} f_1(\omega)\overline{f_2(\omega)}$ for $f_1, f_2 \in M^a$. The Dirac function at a point $(A_1, A_2, \ldots, A_h) \in \Omega_a$ will be denoted simply by $(A_1, A_2, \ldots, A_h)$; that is, M^a will be identified with the set of all linear formal combinations of points of Ω_a. Now we define the "Radon transforms" d_{ij} and the operators Δ_{ij}. If $(A_1, A_2, \ldots, A_h)$ is a flag, $|A_j| > 0$ and $i \neq j$ then we set

$$d_{ij}(A_1, A_2, \ldots, A_h) = \sum_{x\in A_j} \bigl(A_1, \ldots, A_i \cup \{x\}, \ldots, A_j \setminus \{x\}, \ldots, A_h\bigr).$$

That is, if $(A_1, A_2, \ldots, A_h) \in \Omega_{(a_1,a_2,\ldots,a_h)}$ then the d_{ij}-image of $(A_1, A_2, \ldots, A_h)$ is the characteristic function of all the $(B_1, B_2, \ldots, B_h) \in \Omega_{(a_1,a_2,\ldots,a_i+1,\ldots,a_j-1,\ldots,a_h)}$ such that: $B_j \subset A_j$, $A_i \subset B_i$ and $B_k = A_k$ for $k \neq i, j$. Clearly, d_{ij} intertwines the permutation modules $M^{(a_1,a_2,\ldots,a_h)}$ and $M^{(a_1,a_2,\ldots,a_i+1,\ldots,a_j-1,\ldots,a_h)}$ and d_{ij} is the adjoint of d_{ji}.

If $(A_1, A_2, \ldots, A_h)$ is a flag, $i \neq j$ and $|A_i|, |A_j| > 0$ then we set

$$\Delta_{ij}(A_1, A_2, \ldots, A_h) = \sum_{\substack{x\in A_j\\ y\in A_i}} \bigl(A_1, \ldots, \bigl(A_i \setminus \{y\}\bigr) \cup \{x\}, \ldots, \bigl(A_j \setminus \{x\}\bigr) \cup \{y\}, \ldots, A_h\bigr).$$

Thus the Δ_{ij}-image of $(A_1, A_2, \ldots, A_h) \in \Omega_{(a_1,a_2,\ldots,a_h)}$ is the characteristic function of all the $(B_1, B_2, \ldots, B_h) \in \Omega_{(a_1,a_2,\ldots,a_h)}$ such that: $|A_i \cap B_i| = a_i - 1$, $|A_j \cap B_j| = a_j - 1$ and $B_k = A_k$ for $k \neq i, j$. Δ_{ij} intertwines $M^{(a_1,a_2,\ldots,a_h)}$ with itself and is selfadjoint. In the following lemma, we collect some basic properties of the operators d_{ij} and Δ_{ij}. We recall that the Pochhammer symbol $(a)_i$ is defined by $(a)_0 = 1$ and $(a)_i = a(a + 1) \times (a + 2)\cdots(a + i - 1)$ for $i = 1, 2, 3, \ldots$.

Lemma 3.1. *Suppose that i, j, k are three distinct numbers and that f belongs to M^a. Then*:

(1) $d_{ij}d_{ji}f = a_i f + \Delta_{ij} f$;
(2) *if* $1 \leqslant q \leqslant a_i$ *then* $d_{ij}(d_{ji})^q f = (d_{ji})^q d_{ij} f + q(a_i - a_j - q + 1)(d_{ji})^{q-1} f$;
(3) *if* $1 \leqslant p \leqslant q \leqslant a_i$ *and* $d_{ij} f = 0$ *then* $(d_{ij})^p(d_{ji})^q f = (q - p + 1)_p(a_i - a_j - q + 1)_p(d_{ji})^{q-p} f$;

(4) $d_{ij}d_{kj} = d_{kj}d_{ij}$ *and* $d_{kj}d_{ki} = d_{ki}d_{kj}$;
(5) $(d_{ij})^q d_{jk} = q d_{ik}(d_{ij})^{q-1} + d_{jk}(d_{ij})^q$.

Proof. (1) If $(A_1, A_2, \dots, A_h) \in \Omega_a$ then

$$\begin{aligned} d_{ij}d_{ji}&(A_1, A_2, \dots, A_h) \\ &= d_{ij} \sum_{y \in A_i} \bigl(A_1, \dots, A_i \setminus \{y\}, \dots, A_j \cup \{y\}, \dots, A_h\bigr) \\ &= a_i(A_1, A_2, \dots, A_h) \\ &\quad + \sum_{x \in A_j} \sum_{y \in A_i} \bigl(A_1, \dots, \bigl(A_i \setminus \{y\}\bigr) \cup \{x\}, \dots, \bigl(A_j \cup \{y\}\bigr) \setminus \{x\}, \dots, A_h\bigr) \\ &= a_i(A_1, A_2, \dots, A_h) + \Delta_{ij}(A_1, A_2, \dots, A_h). \end{aligned}$$

(2) For $q = 1$ the identity may be obtained subtracting $d_{ji}d_{ij} f = a_j f + \Delta_{ij} f$ from (1); the general case follows by induction on q.

(3) The case $p = 1$ is a consequence of (2); the general case follows again by induction.

(4) These identities are obvious.

(5) If $(A_1, A_2, \dots, A_h) \in \Omega_a$ then

$$\begin{aligned} d_{ij}d_{jk}&(A_1, A_2, \dots, A_h) \\ &= d_{ij} \sum_{x \in A_k} \bigl(A_1, \dots, A_j \cup \{x\}, \dots, A_k \setminus \{x\}, \dots, A_h\bigr) \\ &= \sum_{x \in A_k} \bigl(A_1, \dots, A_i \cup \{x\}, \dots, A_k \setminus \{x\}, \dots, A_h\bigr) \\ &\quad + \sum_{x \in A_k} \sum_{y \in A_j} \bigl(A_1, \dots, A_i \cup \{y\}, \dots, \bigl(A_j \cup \{x\}\bigr) \setminus \{y\}, \dots, A_k \setminus \{x\}, \dots, A_h\bigr) \\ &= d_{ik}(A_1, A_2, \dots, A_h) + d_{jk}d_{ij}(A_1, A_2, \dots, A_h). \end{aligned}$$

The case $q > 1$ follows by induction. □

In the following corollary, we begin to investigate the case of a three parts composition (a, b, c).

Corollary 3.2.

(1) *If* $0 \leqslant h \leqslant \min\{a, b\}$ *then* $(d_{21})^{b-h}$ *is injective from* $M^{(a+b-h,h,c)} \cap \operatorname{Ker} d_{12}$ *to* $M^{(a,b,c)}$;
(2) $(d_{21})^{b-h}[M^{(a+b-h,h,c)} \cap \operatorname{Ker} d_{12}]$ *is an eigenspace of* Δ_{12} *and the corresponding eigenvalue is*: $ab - h(a + b - h + 1)$;

(3) $$M^{(a,b,c)} = \bigoplus_{h=0}^{\min\{a,b\}} (d_{21})^{b-h}\big[M^{(a+b-h,h,c)} \cap \operatorname{Ker} d_{12}\big]$$

is a decomposition of $M^{(a,b,c)}$ into invariant mutually orthogonal subspaces;

(4) *if $p \geqslant \max\{1, b-a+1\}$ then*

$$M^{(a,b,c)} \cap \operatorname{Ker}(d_{12})^p = \bigoplus_{h=\min\{0,b-p+1\}}^{\min\{a,b\}} (d_{21})^{b-h}\big[M^{(a+b-h,h,c)} \cap \operatorname{Ker} d_{12}\big].$$

Proof. (1) is a consequence of Lemma 3.1(3): if $p = q = b-h$ then $\|(d_{21})^{b-h} f\|^2 = \langle f, (d_{12})^{b-h}(d_{21})^{b-h} f\rangle = (b-h)!(a-h+1)_{b-h}\|f\|^2$.

(2) is a consequence of Lemma 3.1(1) and (2): if $f \in M^{(a+b-h,h,c)}$ and $d_{12} f = 0$ then

$$\Delta_{12}(d_{21})^{b-h} f = d_{12}d_{21}(d_{21})^{b-h} f - a(d_{21})^{b-h} f = \big[ab - h(a+b-h+1)\big](d_{21})^{b-h} f.$$

Moreover, in the interval $0 \leqslant h \leqslant \min\{a,b\}$, the function $h \to [ab - h(a+b-h+1)]$ is decreasing. Therefore we have obtained $\min\{a,b\}+1$ distinct eigenvalues and the eigenspaces are orthogonal.

For the moment, suppose that $0 \leqslant b \leqslant a$. The orthogonal decomposition $M^{(a,b,c)} = (d_{21}M^{(a+1,b-1,c)}) \oplus (M^{(a,b,c)} \cap \operatorname{Ker} d_{12})$ is an immediate consequence of the fact that d_{21} is the adjoint of d_{12}. Iterating this decomposition, one obtain easily (3). (4) is a consequence of (1), (3) and of Lemma 3.1(3). Finally, the case $b > a$ follows from the isomorphism $M^{(a,b,c)} = M^{(b,a,c)}$. □

The case $c = 0$, that corresponds to $l = 0$ in Theorem 2.8 of [4] (d_{12} and d_{21} correspond to d and d^* in [4,5,8]), gives the well-known decomposition of $M^{(a,b)}$ into its irreducible constituents:

$$M^{(a,b)} = \bigoplus_{h=0}^{\min\{a,b\}} (d_{21})^{b-h}\big[M^{(a+b-h,h)} \cap \operatorname{Ker} d_{12}\big]. \tag{8}$$

Now $M^{(a+b-h,h)} \cap \operatorname{Ker} d_{12}$ is the irreducible representation of S_{a+b} denoted by $S^{(a+b-h,h)}$. Indeed, all the representations of the symmetric group may be characterized as intersections of kernels of the operators $d_{i,i+1}$ (and this is one of the main ingredients in our computations of invariant vectors):

Theorem 3.3. *If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ is a partition of N then $S^\lambda = M^\lambda \cap (\bigcap_{i=1}^{h-1} \operatorname{Ker} d_{i,i+1})$.*

This was proved by James in [10], in the context of a characteristic free approach to the representation theory of the symmetric group ($d_{i,i+1}$ corresponds to $\psi_{i,a_{i+1}-1}$ in [10, p. 67]. See also [16]. An elementary proof, in the case of ordinary representations, may be found in [13].

4. A class of induced representations and harmonic analysis of the operators Δ_{ij}

We begin this section recalling the definition of induced representation [9]. Let G be a finite group, $K \subset G$ a subgroup, V a representation of G and W an K-invariant subspace of V. Suppose that S is a system of representatives for the set of left cosets G/K, that is $G = \bigcup_{s\in S} sK$ with disjoint union. We say that V is induced by W if we have the following direct sum decomposition: $V = \bigoplus_{s\in S} sW$. The standard notation is $V = \mathrm{Ind}_K^G W$. Now we introduce a notation: if X is a finite set and a and λ are respectively a composition and a partition of $|X|$, then $M^a(X)$ and $S^\lambda(X)$ will denote the usual spaces M^a and S^λ constructed using the space of complex valued functions defined on X. If (a,b,c) is a three parts composition of a fixed positive integer N, then the homogeneous space $\Omega_{(a,b,c)}$ may be seen as the set of all pairs (A,B) such that A and B are respectively an a-subset and a b-subset of $\{1,2,\dots,N\}$ and $A \cap B = \emptyset$. It follows that

$$M^{(a,b,c)} = \bigoplus_{\substack{X\subseteq\{1,2,\dots,N\}\\ |X|=a+b}} M^{(a,b)}(X). \tag{9}$$

This decomposition tells us that the representation of S_N on $M^{(a,b,c)}$ is induced from the representation $M^{(a,b)} \otimes S^{(c)}$ of $S_{a+b} \times S_c$. Moreover, the decomposition (9) is stable under the action of d_{12}: if X is an $a+b$-subset of $\{1,2,\dots,N\}$ then the d_{12}-image of $M^{(a,b)}(X)$ is contained in $M^{(a+1,b-1)}(X)$. This proves that if $a \geqslant b$ then

$$M^{(a,b,c)} \cap \mathrm{Ker}\, d_{12} = \bigoplus_{\substack{X\subseteq\{1,2,\dots,N\}\\ |X|=a+b}} S^{(a,b)}(X). \tag{10}$$

This decomposition proves the case $h = 0$ in the following proposition; the case $h > 0$ may be proved similarly.

Proposition 4.1. *In the permutation module*

$$M^{(a,b,c)} = \mathrm{Ind}_{S_{a+b}\times S_c}^{S_{a+b+c}} \big[M^{(a,b)} \otimes S^{(c)}\big]$$

the subspace $(d_{21})^{b-h}[M^{(a+b-h,h,c)} \cap \mathrm{Ker}\, d_{12}]$ *corresponds to*

$$\mathrm{Ind}_{S_{a+b}\times S_c}^{S_{a+b+c}} \big[S^{(a+b-h,h)} \otimes S^{(c)}\big], \quad 0 \leqslant h \leqslant \min\{a,b\}.$$

In the following theorem we state two particular cases respectively of the Young's Rule and of the Littlewood–Richardson Rule [10,16].

Theorem 4.2. *Let* (α,β,γ) *and* (a,b,c) *be respectively a partition and a composition of* N. *We allow the cases* $\gamma = 0$ *and* $\beta = \gamma = 0$.

(1) *The irreducible representation* $S^{(\alpha,\beta,\gamma)}$ *is contained in the permutation module* $M^{(a,b,c)}$ *if and only if* $\alpha \geqslant \max\{a,b,c\}$ *and* $\gamma \leqslant \min\{a,b,c\}$ *and its multiplicity is equal to* $m_{(\alpha,\beta,\gamma)} = \min\{\alpha-a, \alpha-b, \alpha-c, \alpha-\beta, \beta-\gamma, a-\gamma, b-\gamma, c-\gamma\}+1$. *Moreover,*

$$M^{(a,b,c)} = \bigoplus m_{(\alpha,\beta,\gamma)} S^{(\alpha,\beta,\gamma)}.$$

(2) *Let* $a \geqslant b$. *The multiplicity of* $S^{(\alpha,\beta,\gamma)}$ *in* $\mathrm{Ind}_{S_{a+b}\times S_c}^{S_{a+b+c}}[S^{(a,b)} \otimes S^{(c)}]$ *is equal to* 1 *if* $\gamma \leqslant b \leqslant \beta \leqslant a \leqslant \alpha$, *otherwise is zero. In particular,* $\mathrm{Ind}_{S_{a+b}\times S_c}^{S_{a+b+c}}[S^{(a,b)} \otimes S^{(c)}]$ *decomposes without multiplicity.*

In the following theorem, we state a particular case of a result proved in [12,13].

Theorem 4.3. *Let* (α,β,γ) *and* (a,b,c) *be respectively a partition and a composition of* N *and suppose that the conditions in* (1) *of Theorem* 4.2 *are satisfied. Then the* $S^{(\alpha,\beta,\gamma)}$*-isotypic component of* $M^{(a,b,c)}$ *is an eigenspace of* $\Delta_{12}+\Delta_{13}+\Delta_{23}$ *and the corresponding eigenvalue is*

$$\frac{1}{2}\left[\alpha^2+\beta^2+\gamma^2-2\beta-4\gamma-a^2-b^2-c^2\right].$$

Let V the direct sum of all $S^{(\alpha,\beta,\gamma)}$-isotypic subspaces of $M^{(a,b,c)}$ with $\gamma = 0$. Corollary 3.2 and Theorem 4.3 yield a characterization of a decomposition of V into mutually orthogonal irreducible subrepresentations. See [5, Theorem 2.3], for the case $c = 0$.

Corollary 4.4. *Suppose that* $f \in V$, $0 \leqslant h \leqslant \min\{a,b\}$ *and* $h \leqslant k \leqslant a+b-h \leqslant N-k$. *Then* f *belongs to the subrepresentation of* $(d_{21})^{b-h}[M^{(a+b-h,h,c)} \cap \operatorname{Ker} d_{12}]$ *isomorphic to* $S^{(N-k,k)}$ *if and only if* $\Delta_{12} f = [ab - h(a+b-h+1)]f$ *and* $[\Delta_{13}+\Delta_{23}]f = [c(a+b-h)-(k-h)(N-k-h+1)]f$.

Proof. The only if part is a consequence of Corollary 3.2(2) and of Theorem 4.3. Note that this part is true not only for V but also for the whole $M^{(a,b,c)}$. Now we prove the if part. As noted during its proof, the decomposition in Corollary 3.2(3) gives $\min\{a,b\}+1$ distinct eigenvalues of Δ_{12}. That is, the eigenvalues determine the subspaces in the decomposition. Again, this point is true for the whole $M^{(a,b,c)}$. Finally, for a fixed h, the function $k \to [c(a+b-h)-(k-h)(N-k-h+1)]$ is decreasing for $0 \leqslant k \leqslant (N+1)/2$. Thus the eigenvalues of $\Delta_{13}+\Delta_{23}$ separate the subrepresentations in $V \cap (d_{21})^{b-h}[M^{(a+b-h,h,c)} \cap \operatorname{Ker} d_{12}]$. This fact is true because we have restricted to V. □

5. An orthogonal basis for the $S_{N-m} \times S_m$-invariant functions in the permutation module $M^{(a,b,c)}$

We introduce some notations. If Ω is a finite set, $L(\Omega)$ will denote the space of all complex valued functions defined on Ω. If Ω_1 and Ω_2 are two finite sets and $\xi_i \in L(\Omega_i)$,

$i=1,2$, then product $(\xi_1\xi_2)(\omega_1,\omega_2)=\xi_1(\omega_1)\xi_2(\omega_2)$, defined for $(\omega_1,\omega_2)\in\Omega_1\times\Omega_2$, corresponds to the tensor product $\xi_1\otimes\xi_2\in L(\Omega_1)\otimes L(\Omega_2)=L(\Omega_1\times\Omega_2)$.

If X is a finite set and u,v are two integers satisfying the conditions $0\leqslant u$, $0\leqslant v$ and $u+v\leqslant |X|$, then we will denote by $\sigma_{u,v}(X)$ the function in $M^{(|X|-u-v,u,v)}(X)=L(\Omega_{(|X|-u-v,u,v)}(X))$ which is constant and identically equal to 1. If Y,Z are two disjoint subsets of $\{1,2,\dots,N\}$ such that $|Y|=m$, $|Z|=N-m$, and the integers x,y satisfy the conditions $0\leqslant x\leqslant b$, $0\leqslant y\leqslant c$, $m-a\leqslant x+y\leqslant m$, $N=a+b+c$, then the (tensor) product $\sigma_{x,y}(Y)\sigma_{b-x,c-y}(Z)$ is the characteristic function of the set $\{(A,B,C)\in\Omega_{(a,b,c)}(Y\cup Z)\colon\ |B\cap Y|=x \text{ and } |C\cap Y|=y\}\equiv\Omega_{(m-x-y,x,y)}(Y)\times\Omega_{(a-m+x+y,b-x,c-y)}(Z)$.

Clearly, if $S_{N-m}\times S_m$ is the stabilizer of the pair (Y,Z) then the functions $\sigma_{x,y}(Y)\sigma_{b-x,c-y}(Z)$ form a basis for the $S_{N-m}\times S_m$-invariant functions in $M^{(a,b,c)}$ (they are the characteristic functions of the orbits).

A more symmetric, but also more cumbersome, notation for $\sigma_{x,y}(Y)\sigma_{b-x,c-y}(Z)$ would make easy to write general formulas for the action of d_{ij} and Δ_{ij} on $\sigma_{x,y}(Y)\sigma_{b-x,c-y}(Z)$. In what follows, apart for an example at the beginning of the proof of the next lemma, we will leave to the reader the elementary task to derive such identities when they are used (we will give only the identities or their immediate consequences; their proofs are based on the repeated application of the fact that the number of u-subsets of a finite set D is equal to $\binom{|D|}{u}$).

Definition 5.1. For $0\leqslant m\leqslant N$, $0\leqslant k\leqslant \min\{a+b,a+c,b+c,N-m,m\}$ and $\max\{0,k-c\}\leqslant h\leqslant\min\{k,a+b-k,a,b\}$ we define

$$\Psi(a,b,c,m,k,h)=\sum_{y=\max\{0,m-a-b+h\}}^{\min\{c,m-h\}}E_{k-h}(c,a+b-2h,m-h,y)\times\sum_{x=\max\{0,m-y-a\}}^{\min\{b,m-y\}}E_h(b,a,m-y,x)\sigma_{x,y}(Y)\sigma_{b-x,c-y}(Z).$$

In the following lemma, we collect some properties of the functions $\Psi(a,b,c,m,k,h)$.

Lemma 5.2.

(1) $\Delta_{12}\Psi(a,b,c,m,k,h)=[ab-h(a+b-h+1)]\Psi(a,b,c,m,k,h)$;

(2) $[\Delta_{13}+\Delta_{23}]\Psi(a,b,c,m,k,h)=[c(a+b-h)-(k-h)(N-k-h+1)]\Psi(a,b,c,m,k,h)$;

(3) $\frac{1}{(t-b)!}(d_{21})^{t-b}\Psi(a,b,c,m,k,h)=\binom{t-h}{b-h}\Psi(a+b-t,t,c,m,k,h)$;

(4) *if* $\xi: M^{(a,b,c)}\to M^{(b,a,c)}$ *is the natural isomorphism* $\xi(A,B,C)=(B,A,C)$ *then* $\xi\Psi(a,b,c,m,k,h)=(-1)^h\Psi(b,a,c,m,k,h)$;

(5) $\frac{1}{(t-a)!}(d_{12})^{t-a}\Psi(a,b,c,m,k,h)=\binom{t-h}{a-h}\Psi(t,a+b-t,c,m,k,h)$.

Proof. First note that

$$\begin{aligned}
&\Delta_{12}\big(\sigma_{x,y}(Y)\sigma_{b-x,c-y}(Z)\big)\\
&\quad=\big[x(m-x-y)+(b-x)(a-m+x+y)\big]\sigma_{x,y}(Y)\sigma_{b-x,c-y}(Z)\\
&\qquad+(x+1)(a-m+x+y+1)\sigma_{x+1,y}(Y)\sigma_{b-x-1,c-y}(Z)\\
&\qquad+(b-x+1)(m-x-y+1)\sigma_{x-1,y}(Y)\sigma_{b-x+1,c-y}(Z). \qquad (11)
\end{aligned}$$

For instance, given $(A', B', C') \in \Omega_{(a,b,c)}$ such that $|B' \cap Y| = x+1$ and $|C' \cap Y| = y$ the number of $(A, B, C) \in \Omega_{(a,b,c)}$ such that $|B \cap Y| = x$, $|C \cap Y| = y$, $|B' \cap B| = b-1$ and $C = C'$ is equal to

$$\begin{aligned}
&\binom{|B'\cap Y|}{|B\cap Y|}\binom{|A'\cap Z|}{|A\cap Z|}\\
&\quad=\binom{x+1}{x}\binom{a-m+x+y+1}{a-m+x+y}=(x+1)(a-m+x+y+1).
\end{aligned}$$

From (11) it follows that the coefficient of $\sigma_{x,y}(Y)\sigma_{b-x,c-y}(Z)$ in $\Delta_{12}\Psi(a,b,c,m,k,h)$ is equal to

$$\begin{aligned}
&\big\{\big[x(m-x-y)+(b-x)(a-m+x+y)\big]E_h(b,a,m-y,x)\\
&\quad+x(a-m+x+y)E_h(b,a,m-y,x-1)\\
&\quad+(b-x)(m-x-y)E_h(b,a,m-y,x+1)\big\}E_{k-h}(c,a+b-2h,m-h,y).
\end{aligned}$$

Applying (3) to the expression in curly braces, it becomes

$$\big[ab-h(a+b-h+1)\big]E_h(b,a,m-y,x)E_{k-h}(c,a+b-2h,m-h,y-1),$$

and this proves (1).

Analogously, it is easy to check that the coefficient of $\sigma_{x,y}(Z)\sigma_{b-x,c-y}(Y)$ in $[\Delta_{13} + \Delta_{23}]\Psi(a,b,c,m,k,h)$ is equal to

$$\begin{aligned}
&\big[y(m-x-y)+(c-y)(a-m+x+y)+xy+(b-x)(c-y)\big]\\
&\qquad\times E_h(b,a,m-y,x)E_{k-h}(c,a+b-2h,m-h,y)\\
&\quad+(c-y)\big[xE_h(b,a,m-y-1,x-1)+(m-x-y)E_h(b,a,m-y-1,x)\big]\\
&\qquad\times E_{k-h}(c,a+b-2h,m-h,y+1)\\
&\quad+y\big[(b-x)E_h(b,a,m-y+1,x+1)+(a-m+x+y)E_h(b,a,m-y+1,x)\big]\\
&\qquad\times E_{k-h}(c,a+b-2h,m-h,y-1).
\end{aligned}$$

Applying Eqs. (1) and (2) to the last two expressions in square brackets, it becomes

$$\big\{\big[y(m-x-y)+(c-y)(a-m+x+y)+xy+(b-x)(c-y)\big]$$
$$\times E_{k-h}(c,a+b-2h,m-h,y)$$
$$+(c-y)(m-y-h)E_{k-h}(c,a+b-2h,m-h,y+1)$$
$$+y(a+b-m-h+y)E_{k-h}(c,a+b-2h,m-h,y-1)\big\}E_h(b,a,m-y,x).$$

An application of (3) to the expression in curly brackets changes it into

$$\big[c(a+b-h)-(k-h)(N-k-h+1)\big]$$
$$\times E_h(b,a,m-y,x)E_{k-h}(c,a+b-2h,m-h,y),$$

and this proves (2).

To prove (3), first note that

$$\frac{1}{(t-b)!}(d_{21})^{t-b}\sigma_{x,y}(Y)\sigma_{b-x,c-y}(Z)$$
$$=\sum_{z=\max\{x,t-a-b+m-y\}}^{\min\{m-y,t-b+x\}}\binom{z}{x}\binom{t-z}{b-x}\sigma_{z,y}(Y)\sigma_{t-z,c-y}(Z).$$

Therefore, the coefficient of $\sigma_{z,y}(Y)\sigma_{t-z,c-y}(Z)$ in $\frac{1}{(t-b)!}(d_{21})^{t-b}\Psi(a,b,c,m,k,h)$ is equal to

$$\sum_{x=\max\{0,z-t+b\}}^{\min\{b,z\}}\binom{z}{x}\binom{t-z}{b-x}E_h(b,a,m-y,x)E_{k-h}(c,a+b-2h,m-h,y);$$

an application of (6) changes it into $\binom{t-h}{b-h}E_h(t,a+b-t,m-y,z)E_{k-h}(c,a+b-2h,$ $m-h,y)$ and this establishes (3).

(4) follows from the symmetry relation (4) applied to $E_h(b,a,m-y,x)$.

Finally, (5) is a consequence of (3) and (4) (clearly $d_{12}\xi=\xi d_{21}$); equivalently, it may be deduced from (3) in this lemma and Lemma 3.1(3). □

Remark 5.3. Two applications of the orthogonality relations for the Hahn polynomials [5, p. 631] yield the following expression for the norm of $\Psi(a,b,c,m,k,h)$:

$$\big\|\Psi(a,b,c,m,k,h)\big\|^2=\binom{a+b}{b}\binom{a+b}{h}^{-1}\frac{a+b-h+1}{a+b-2h+1}$$
$$\times(b-h+1)_h(a-h+1)_h\frac{(N-m)!m!}{(N-m-h)!(m-h)!}$$
$$\times\binom{N-2h}{c}\binom{N-2h}{k-h}^{-1}\frac{N-h-k+1}{N-2k+1}(N-m-k+1)_{k-h}$$
$$\times(m-k+1)_{k-h}(c-k+h+1)_{k-h}(a+b-h-k+1)_{k-h}.$$

Theorem 5.4.

(1) *The space of* $S_{N-m} \times S_m$*-invariant vectors in the subrepresentation of* $(d_{21})^{b-h} \times [M^{(a+b-h,h,c)} \cap \operatorname{Ker} d_{12}]$ *isomorphic to* $S^{(N-k,k)}$ *is spanned by* $\Psi(a,b,c,m,k,h)$.
(2) *The set* $\{\Psi(a,b,c,m,k,h)\colon\ 0 \leqslant k \leqslant \min\{a+b, a+c, b+c, N-m, m\}$ *and* $\max\{0, k-c\} \leqslant h \leqslant \min\{k, a+b-k, a, b\}$ *is an orthogonal basis for the* $S_{N-m} \times S_m$*-invariant functions in* $M^{(a,b,c)}$.

Proof. From the Young's Rule and Frobenius reciprocity, it follows that the dimension of the space of $S_{N-m} \times S_m$-invariant vectors in an irreducible representation $S^{(\alpha,\beta,\gamma)} \subseteq M^{(a,b,c)}$ is $\leqslant 1$ and is equal to one if and only if $\gamma = 0$ and $\alpha \geqslant \max\{N-m, m\}$. Therefore (1) is a consequence of Lemma 5.2(1),(2) and of Corollary 4.4.

If we denote by $(S^{(N-k,k)})_h$ the subrepresentation of $(d_{21})^{b-h}[M^{(a+b-h,h,c)} \cap \operatorname{Ker} d_{12}]$ isomorphic to $S^{(N-k,k)}$ then from Corollary 3.2(3), Proposition 4.1 and Theorem 4.2 it follows that

$$\bigoplus_{k=0}^{\min\{a+b,a+c,b+c,N-m,m\}} \bigoplus_{\max\{0,k-c\}}^{\min\{k,a+b-k,a,b\}} \left(S^{(N-k,k)}\right)_h$$

is an orthogonal decomposition of the direct sum of all the irreducible subrepresentations of $M^{(a,b,c)}$ containing nontrivial $S_{N-m} \times S_m$-invariant vectors. Therefore (2) is a consequence of (1). □

Remark 5.5. The $S_{N-m} \times S_m - S_a \times S_b \times S_c$-invariant functions $\Psi(a,b,c,m,k,h)$ might be obtained from the $S_a \times S_b \times S_c - S_{N-m} \times S_m$-invariant functions in [8] by mean of the transformation $g \to g^{-1}$. Another way to derive the intertwining functions in [8], and therefore the functions $\Psi(a,b,c,m,k,h)$, will be sketched in Remark 7.6.

6. The action of d_{13} and d_{23} on $\Psi(a,b,c,m,k,h)$

In this section, we compute the action of d_{13} and d_{23} on the invariant vectors $\Psi(a,b,c,m,k,h)$. We begin with a particular case.

Lemma 6.1. $d_{13}\Psi(a+b-h,h,c,m,k,h) = (a+b-h-k+1)\Psi(a+b-h+1, h, c-1, m, k, h)$.

Proof. The condition $b = h$ yields two simplifications. First, from the identity $d_{12}d_{13} = d_{13}d_{12}$ (see Lemma 3.1(4)) it follows that the d_{13}-image of $M^{(a+b-h,h,c)} \cap \operatorname{Ker} d_{12}$ is contained in $M^{(a+b-h+1,h,c-1)} \cap \operatorname{Ker} d_{12}$. Since d_{13} is an intertwining operator, the d_{13}-image of the subrepresentation of $M^{(a+b-h,h,c)} \cap \operatorname{Ker} d_{12}$ isomorphic to $S^{(N-k,k)}$ is (contained in) the subrepresentation of $M^{(a+b-h+1,h,c-1)} \cap \operatorname{Ker} d_{12}$ isomorphic to $S^{(N-k,k)}$. Therefore $d_{13}\Psi(a+b-h,h,c,m,k,h)$ is a multiple of $\Psi(a+b-h+1,h,c-1,m,k,h)$. Moreover, in the case $b = h$ we may apply (7) to the coefficients $E_h(h, a+b-h, m-y, x)$ in $\Psi(a+b-h,h,c,m,k,h)$.

The coefficient of $\sigma_{xy}(Y)\sigma_{h-x,c-1-y}(Z)$ in $d_{13}\Psi(a+b-h,h,c,m,k,h)$ is equal to:

$$\begin{aligned}&(m-x-y)E_h(h,a+b-h,m-y-1,x)E_{k-h}(c,a+b-2h,m-h,y+1)\\&\quad+(a+b-h-m+x+y+1)E_h(h,a+b-h,m-y,x)\\&\qquad\times E_{k-h}(c,a+b-2h,m-h,y).\end{aligned}\tag{12}$$

Applying (7) to transform the coefficients $E_h(h,a+b-h,m-y-1,x)$ and $E_h(h,a+b-h,m-y,x)$ into a multiple of $E_h(h,a+b-h+1,m-y,x)$, (12) becomes:

$$\begin{aligned}&E_h(h,a+b-h+1,m-y,x)\big[(m-h-y)E_{k-h}(c,a+b-2h,m-h,y+1)\\&\quad+(a+b-m-h+y+1)E_{k-h}(c,a+b-2h,m-h,y)\big].\end{aligned}\tag{13}$$

The symmetry relation (5) changes (2) into $(c-x)E_m(a+1,b-1,c,x+1)+(x+b-c)E_m(a+1,b-1,c,x)=(b-m)E_m(a,b,c,x)$. Applying this to the expression in square brackets, (13) becomes:

$$(a+b-h-k+1)E_h(h,a+b-h+1,m-y,x)E_{k-h}(c-1,a+b-2h+1,m-h,y)$$

and this proves the lemma. □

Lemma 6.2. *There exists a constant $\lambda(a,b,c,m,k,h)$, that satisfies the identity $\lambda(a,b,c,m,k,h)=\lambda(a+b-h,h,c,m,k,h)$ for $b\geqslant h$, such that*:

$$\begin{aligned}d_{13}\Psi(a,b,c,m,k,h)&=(a-h+1)\frac{a+b-k-h+1}{a+b-2h+1}\Psi(a+1,b,c-1,m,k,h)\\&\quad-\lambda(a,b,c,m,k,h)\Psi(a+1,b,c-1,m,k,h+1),\\d_{23}\Psi(a,b,c,m,k,h)&=(b-h+1)\frac{a+b-k-h+1}{a+b-2h+1}\Psi(a,b+1,c-1,m,k,h)\\&\quad+\lambda(a,b,c,m,k,h)\Psi(a,b+1,c-1,m,k,h+1).\end{aligned}$$

Proof. From $(d_{12})^2d_{23}=2d_{13}d_{12}+d_{23}(d_{12})^2$ (set $q=2$ in Lemma 3.1(5)) it follows that the d_{23}-image of $M^{(a+b-h,h,c)}\cap\operatorname{Ker}d_{12}$ is contained in

$$\begin{aligned}&M^{(a+b-h,h+1,c-1)}\cap\operatorname{Ker}(d_{12})^2\\&\quad=\big[M^{(a+b-h,h+1,c-1)}\cap\operatorname{Ker}d_{12}\big]\oplus\big[d_{21}\big(M^{(a+b-h+1,h,c-1)}\cap\operatorname{Ker}d_{12}\big)\big]\end{aligned}$$

(see Corollary 3.2(4) for this decomposition). Thus the d_{23}-image of the subspace of $M^{(a+b-h,h,c)}$ isomorphic to $S^{(N-k,k)}$ is contained in direct sum of the subspaces of $M^{(a+b-h,h+1,c-1)}\cap\operatorname{Ker}d_{12}$ and $d_{21}[M^{(a+b-h+1,h,c-1)}\cap\operatorname{Ker}d_{12}]$ isomorphic to $S^{(N-k,k)}$. It follows that $d_{23}\Psi(a+b-h,h,c,m,k,h)$ is a linear combination of $\Psi(a+b-h,h+1,c-1,m,k,h+1)$ and $\Psi(a+b-h,h+1,c-1,m,k,h)$

$$
\begin{aligned}
&d_{23}\Psi(a+b-h,h,c,m,k,h)\\
&\quad=\lambda(a+b-h,h,c,m,k,h)\Psi(a+b-h,h+1,c-1,m,k,h+1)\\
&\qquad+\mu(a+b-h,h,c,m,k,h)\Psi(a+b-h,h+1,c-1,m,k,h). \qquad (14)
\end{aligned}
$$

The constant $\mu(a+b-h,h,c,m,k,h)$ can be derived easily: if we apply to both the left-hand and the right-hand side of (14) the operator d_{12} and use the identity $d_{12}d_{23}=d_{13}+d_{23}d_{12}$ (Lemma 3.1) in the left member, we obtain:

$$
\begin{aligned}
&d_{13}\Psi(a+b-h,h,c,m,k,h)\\
&\quad=\mu(a+b-h,h,c,m,k,h)d_{12}\Psi(a+b-h,h+1,c-1,m,k,h). \qquad (15)
\end{aligned}
$$

But from Lemmas 5.2(3) and 3.1(3) it follows that $d_{12}\Psi(a+b-h,h+1,c-1,m,k,h)=d_{12}d_{21}\Psi(a+b-h+1,h,c-1,m,k,h)=(a+b-2h+1)\Psi(a+b-h+1,h,c-1,m,k,h)$ (indeed $\Psi(a+b-h+1,h,c-1,m,k,h)\in\operatorname{Ker} d_{12}$). Applying this to (15) and using Lemma 6.1 we obtain:

$$
\begin{aligned}
&(a+b-h-k+1)\Psi(a+b-h+1,h,c-1,m,k,h)\\
&\quad=(a+b-2h+1)\mu(a+b-h,h,c,m,k,h)\Psi(a+b-h+1,h,c-1,m,k,h)
\end{aligned}
$$

thus

$$\mu(a+b-h,h,c,m,k,h)=\frac{a+b-h-k+1}{a+b-2h+1}.$$

To get the formula for $d_{23}\Psi(a,b,c,m,k,h)$, apply to both the left-hand and the right-hand side of (14) the operator $\frac{1}{(b-h)!}(d_{21})^{b-h}$ and use Lemma 5.2(3) (we recall that d_{21} and d_{23} commute). The formula for $d_{13}\Psi(a,b,c,m,k,h)$ follows easily from Lemma 5.2(4). Indeed, $d_{13}\xi=\xi d_{23}$. □

Now we compute a particular value of $\lambda(a,b,c,m,k,h)$.

Lemma 6.3. $\lambda(N-k,h,k-h,m,k,h)=k-h$.

Proof. First note that $d_{23}\Psi(N-k,h,k-h,m,k,h)$ is a multiple of $\Psi(N-k,h+1,k-h-1,m,k,h+1)$, because $\Psi(N-k,h+1,k-h-1,m,k,h)$ does not exist; the $S_{N-m}\times S_m$-invariant vector $\Psi(a,b,c,m,k,h)$ exists only if $c\geqslant k-h$; this corresponds to the condition $a\leqslant\alpha$ in (2), Theorem 4.2. The coefficient of $\sigma_{x,y}(Y)\sigma_{h+1-x,k-h-1-y}(Z)$ in $d_{23}\Psi(N-k,h,k-h,m,k,h)$ is equal to:

$$
\begin{aligned}
&xE_{k-h}(k-h,N-h-k,m-h,y+1)E_h(h,N-k,m-y-1,x-1)\\
&\quad+(h-x+1)E_{k-h}(k-h,N-h-k,m-h,y)E_h(h,N-k,m-y,x). \qquad (16)
\end{aligned}
$$

Now we can transform all the coefficients $E_{\dots}(\dots)$ in (16) using (7). By simple calculations, one can prove that (16) is equal to:

$$(k-h)E_{k-h-1}(k-h-1, N-k-h-1, m-h-1, y)E_{h+1}(h+1, N-k, m-y, x),$$

thus $d_{23}\Psi(N-k, h, k-h, m, k, h) = (k-h)\Psi(N-k, h+1, k-h-1, m, k, h+1)$. □

Theorem 6.4. *The functions* $\Psi(a, b, c, m, k, h)$ *satisfy the following identities*:

$$\begin{aligned} d_{13}\Psi(a, b, c, m, k, h) &= (a-h+1)\frac{a+b-k-h+1}{a+b-2h+1}\Psi(a+1, b, c-1, m, k, h) \\ &\quad - \frac{(k-h)(N-k-h+1)}{a+b-2h+1}\Psi(a+1, b, c-1, m, k, h+1), \\ d_{23}\Psi(a, b, c, m, k, h) &= (b-h+1)\frac{a+b-k-h+1}{a+b-2h+1}\Psi(a, b+1, c-1, m, k, h) \\ &\quad + \frac{(k-h)(N-k-h+1)}{a+b-2h+1}\Psi(a, b+1, c-1, m, k, h+1). \end{aligned}$$

Proof. From Lemma 3.1(4), we know that d_{13} and d_{23} commute; if we equate the coefficients of $\Psi(a+1, b+1, c-2, m, k, h+1)$ in $d_{13}d_{23}\Psi(a, b, c, m, k, h)$ and $d_{23}d_{13}\Psi(a, b, c, m, k, h)$, computed by using Lemma 6.2, we obtain (we recall that $\lambda(a, b, c, m, k, h) = \lambda(a+b-h, h, c, m, k, h)$):

$$\lambda(a+b-h, h, c, m, k, h) = \frac{a+b-2h+2}{a+b-2h+1}\lambda(a+b-h+1, h, c-1, m, k, h). \quad (17)$$

By $c-k+h$ applications of (17) we obtain:

$$\lambda(a+b-h, h, c, m, k, h) = \frac{N-k-h+1}{a+b-2h+1}\lambda(N-k, h, k-h, m, k, h);$$

and we may finish with Lemma 6.3. □

Remark 6.5. Clearly, for some values of the parameters, the identities of Theorem 6.4 degenerate into identities containing a unique term on the right-hand side: it happens when one of the functions Ψ on the right-hand side does not exist; for example, see Lemmas 6.1 or 6.3.

7. An orthogonal basis for the space of $S_a \times S_b \times S_c$-invariant vectors in the irreducible representations $S^{(\alpha,\beta,\gamma)}$

Fix three disjoint subsets A, B and C and define $S_a \times S_b \times S_c$ and $S_{a+b} \times S_c$ as the stabilizer respectively of (A, B, C) and $(A \cup B, C)$. If (α, β, γ) is a three part partition of $N = a+b+c$, then Theorem 3.3 ensures us that $S^{(\alpha,\beta,\gamma)} = M^{(\alpha,\beta,\gamma)} \cap \operatorname{Ker} d_{12} \cap \operatorname{Ker} d_{23}$

is the irreducible representation canonically associated to (α,β,γ). It is not hard to see that the characteristic functions of the orbits of $S_a \times S_b \times S_c$ on $\Omega_{(\alpha,\beta,\gamma)}$ are given by the products: $\sigma_{xy}(A)\sigma_{uv}(B)\sigma_{\beta-u-x,\gamma-v-y}(C)$, where σ is defined as at the beginning of Section 5.

Thus the $S_a \times S_b \times S_c$-invariant vectors in $S^{(\alpha,\beta,\gamma)}$ might be computed by solving the following system:

$$\begin{cases} d_{12}\sum_{x,y,u,v} f(x,y,u,v)\sigma_{xy}(A)\sigma_{uv}(B)\sigma_{\beta-u-x,\gamma-v-y}(C)=0, \\ d_{23}\sum_{x,y,u,v} f(x,y,u,v)\sigma_{xy}(A)\sigma_{uv}(B)\sigma_{\beta-u-x,\gamma-v-y}(C)=0 \end{cases} \tag{18}$$

where both the sums are over the set $D=\{(x,y,u,v)\colon\ x,y,u,v\geqslant 0, x+y\leqslant a, u+v\leqslant b, x+u\leqslant\beta, y+v\leqslant\gamma, x+y+u+v\geqslant\beta+\gamma-c\}$. The system (18) may be easily translated into the following system of finite difference equations for the coefficients $f(x,y,u,v)$:

$$\begin{cases} (a-x-y)f(x+1,y,u,v)+(b-u-v)f(x,y,u+1,v) \\ \quad +(\alpha-a-b+x+y+u+v+1)f(x,y,u,v)=0, \\ xf(x-1,y+1,u,v)+uf(x,y,u-1,v+1)+(\beta-x-u+1)f(x,y,u,v)=0. \end{cases} \tag{19}$$

First of all, we want to sketch how elementary solutions of (19) may be found using the techniques in [8, p. 60]. Indeed, both the equations are of the same type of (2.1) in [8] when restricted, respectively, to a plane $y=$ constant, $v=$ constant and to a plane $x+y=$ constant, $u+v=$ constant. Therefore we may use (2.2) in [8]: if we apply it to the second equation in (19), we get an expression of the value $f(x,y,u,v)$ in terms of the values on the set $D_1=\{(x+y-k,k,u+v-\gamma+k,\gamma-k)\colon\ \max\{\gamma-u-v,y\}\leqslant k\leqslant \min\{x+y,\gamma-v\}\}$; then we may apply again (2.2) of [8], now to the first equation in (19), obtaining an expression of the values on a point in D_1 in terms of the value on the set $D_2=\{(h,k,\beta-h,\gamma-k)\colon \max\{\beta+\gamma-b-k,x+y-k\}\leqslant h\leqslant\min\{a-k,\beta+\gamma-u-v-k\}\}$. Finally, on D_2 the second equation in (19), written for $x+u=\beta+1$ and $y+v=\gamma-1$, yields the following recurrence relation:

$$f(h,k,\beta-h,\gamma-k)=\frac{(-h)_{\gamma-k}}{(\beta+1-h)_{\gamma-k}}=f(h+k-\gamma,\gamma,\beta+\gamma-h-k,0).$$

After some elementary calculations (and setting $j=h+k$) the final result is:

$$f(x,y,u,v)=\sum_{j=\max\{\beta+\gamma-b,x+y\}}^{\min\{a,\beta+\gamma-u-v\}} f(j-\gamma,\gamma,\beta+\gamma-j,0)f_j(x,y,u,v) \tag{20}$$

where

$$f_j(x,y,u,v)=\binom{\beta+\gamma-x-y-u-v}{j-x-y}$$

$$\times \frac{(x+y-a)_{j-x-y}(u+v-b)_{\beta+\gamma-u-v-j}}{(x+y+u+v-\beta-\gamma)_{\gamma-y-v}(-c)_{\beta-x-y-u-v+\gamma}(-1)^{\beta-x}}$$
$$\times \sum_{k=\max\{y,\gamma-u-v\}}^{\min\{x+y,\gamma-v\}} \binom{\gamma-y-v}{k-y} \frac{(-x)_{k-y}(-u)_{\gamma-k-v}(j-\gamma+1)_{\gamma-k}}{(\beta+1+k-j)_{\gamma-k}(-1)^{\gamma-k}}.$$

The representation formula (20) tells us that f is determined by its values on the set $D_3 = \{(j-\gamma, \gamma, \beta+\gamma-j, 0)\colon \max\{0, \beta+\gamma-b\} \leqslant j \leqslant \min\{a, \beta+\gamma\}\} \equiv D \cap \{x+u=\beta, y=\gamma, v=0\}$. Under the assumption $c \geqslant \beta \geqslant \max\{a,b\}$, that corresponds to the condition '$c \geqslant r$' in [8, p. 60], the multiplicity $m_{\alpha,\beta,\gamma}$ in Theorem 4.3 is equal to $\alpha - c + 1$ and coincides with the cardinality of D_3. This means that now the functions $\{f_j\colon \beta+\gamma-b \leqslant j \leqslant a\}$ form a basis for the solutions of (19) (and note also that now the sum in (20) is always over all D_3). However, as in [8], our main result will be a solution of (19) by mean of ideas from representation theory. We will study the restriction of $S^{(\alpha,\beta,\gamma)}$ to $S_{a+b} \times S_c$. For every subrepresentation of the form $S^{(a+b-k,k)} \otimes S^{(c)}$ we will write a nontrivial $S_a \times S_b \times S_c$-invariant vector belonging to it. In this way we will obtain an orthogonal basis for the solution that has a nice group theoretical interpretation.

To start our computations, first observe that the orbits of $S_{a+b} \times S_c$ on $\Omega_{(\alpha,\beta,\gamma)}$ are given by the subsets

$$\begin{aligned} \Gamma_{uv} &= \bigl\{(A', B', C') \in \Omega_{(\alpha,\beta,\gamma)}\colon \bigl|B' \cap (A \cup B)\bigr| = u,\ \bigl|C' \cap (A \cup B)\bigr| = v\bigr\} \\ &= \Omega_{(a+b-u-v,u,v)}(A \cup B) \times \Omega_{(c-\beta-\gamma+u+v,\beta-u,\gamma-v)}(C) \end{aligned}$$

for $0 \leqslant u \leqslant \beta$, $0 \leqslant v \leqslant \gamma$ and $\beta+\gamma-c \leqslant u+v \leqslant a+b$, and that the space $L(\Gamma_{uv})$ of all complex valued functions defined on Γ_{uv} may be written as a tensor product:

$$L(\Gamma_{uv}) = M^{(a+b-u-v,u,v)}(A \cup B) \otimes M^{(c-\beta-\gamma+u+v,\beta-u,\gamma-v)}(C).$$

Therefore, the restriction of $M^{(\alpha,\beta,\gamma)}$ to the subgroup $S_{a+b} \times S_c$ may be decomposed as follows:

$$\begin{aligned} &M^{(\alpha,\beta,\gamma)}{\downarrow}(S_{a+b} \times S_c) \\ &\quad = \bigoplus_{\substack{0 \leqslant u \leqslant \beta,\ 0 \leqslant v \leqslant \gamma \\ \beta+\gamma-c \leqslant u+v \leqslant a+b}} M^{(a+b-u-v,u,v)}(A \cup B) \otimes M^{(c-\beta-\gamma+u+v,\beta-u,\gamma-v)}(C). \end{aligned}$$

Clearly, the tensor product $M^{(a+b-u-v,u,v)}(A \cup B) \otimes M^{(c-\beta-\gamma+u+v,\beta-u,\gamma-v)}(C)$ is spanned by the products $p_{uv} q_{\beta-u,\gamma-v}$, where

$$p_{uv} \in M^{(a+b-u-v,u,v)}(A \cup B) \quad \text{and} \quad q_{\beta-u,\gamma-v} \in M^{(c-\beta-\gamma+u+v,\beta-u,\gamma-v)}(C).$$

Therefore the direct sum of all the subrepresentations of $M^{(\alpha,\beta,\gamma)}{\downarrow}(S_{a+b} \times S_c)$ of the form $S^{(a+b-k,k)} \otimes S^{(c)}$ is contained in:

$$\bigoplus_{\substack{0\leqslant u\leqslant\beta,\,0\leqslant v\leqslant\gamma\\ \beta+\gamma-c\leqslant u+v\leqslant a+b}} \left[M^{(a+b-u-v,u,v)}(A\cup B)\otimes S^{(c)}\right]$$
$$= \bigoplus_{\substack{0\leqslant u\leqslant\beta,\,0\leqslant v\leqslant\gamma\\ \beta+\gamma-c\leqslant u+v\leqslant a+b}} \left\{p_{uv}\sigma_{\beta-u,\gamma-v}(C):\ p_{uv}\in M^{(a+b-u-v,u,v)}(A\cup B)\right\} \tag{21}$$

which is the sum of all the subrepresentations of $M^{(\alpha,\beta,\gamma)}\downarrow(S_{a+b}\times S_c)$ that are trivial on S_c. (An irreducible subrepresentation of $S^{(\alpha,\beta,\gamma)}\downarrow(S_{a+b}\times S_c)$ contains nontrivial $S_a\times S_b\times S_c$-invariant vectors if and only if it is of the form $S^{(a+b-k,k)}\otimes S^{(c)}$ and $0\leqslant k\leqslant \min\{a,b\}$.)

In what follows, we suppose that the conditions imposed by the Young's Rule (Theorem 4.2), namely $\max\{a,b,c\}\leqslant\alpha$ and $\gamma\leqslant\min\{a,b,c\}$, are verified. By Frobenius reciprocity, $S^{(\alpha,\beta,\gamma)}$ contains nontrivial $S_a\times S_b\times S_c$-invariant vectors if and only if these conditions are satisfied. From Theorem 3.3 and (21) it follows that the subrepresentations of $S^{(\alpha,\beta,\gamma)}\downarrow(S_{a+b}\times S_c)$ that are trivial on S_c may be characterized by solving the system:

$$\begin{cases} d_{12}\sum_{v=0}^{\gamma}\sum_{u=\max\{0,\beta+\gamma-c-v\}}^{\beta} p_{uv}\sigma_{\beta-u,\gamma-v}(C)=0,\\ d_{23}\sum_{v=0}^{\gamma}\sum_{u=\max\{0,\beta+\gamma-c-v\}}^{\beta} p_{uv}\sigma_{\beta-u,\gamma-v}(C)=0 \end{cases} \tag{22}$$

where the vectors $p_{uv}\in M^{(a+b-u-v,u,v)}(A\cup B)$ are unknown (the conditions on u and v come from the conditions in (21) simplified by the Young's Rule: $c\leqslant\alpha$ and $v\leqslant\gamma\Rightarrow\beta\leqslant a+b-v$).

To solve (22), first note that $d_{12}(p_{uv}q_{\beta-u,\gamma-v})=(d_{12}p_{uv})q_{\beta-u,\gamma-v}+p_{uv}(d_{12}q_{\beta-u,\gamma-v})$ (verify this identity on the product of two Dirac functions). It follows that

$$\begin{aligned} d_{12}\big(p_{uv}\sigma_{\beta-u,\gamma-v}(C)\big) &= (d_{12}p_{uv})\sigma_{\beta-u,\gamma-v}(C)\\ &\quad + (c-\beta-\gamma+u+v+1)p_{uv}\sigma_{\beta-u-1,\gamma-v}(C). \end{aligned} \tag{23}$$

Therefore, the first equation in (22) may be studied for every fixed v, i.e., it is equivalent to

$$d_{12}\sum_{u=\max\{0,\beta+\gamma-c-v\}}^{\beta} p_{uv}\sigma_{\beta-u,\gamma-v}(C)=0 \tag{24}$$

for $0\leqslant v\leqslant\gamma$.

Lemma 7.1. *For* $0\leqslant v\leqslant\gamma$ *the solutions of* (24) *are given by*

$$p_{uv}=\frac{1}{(\gamma-c-v)_{\beta-u}}(d_{12})^{\beta-u}p_{\beta v},\quad u=\max\{0,\beta+\gamma-c-v\},\dots,\beta, \tag{25}$$

where $p_{\beta v}$ *belongs to* $M^{(a+b-v-\beta,\beta,v)}(A \cup B)$ *and satisfies the condition*:

$$(d_{12})^{c+v-\gamma+1} p_{\beta v} = 0 \tag{26}$$

(*which is trivial when* $v \geqslant \beta + \gamma - c$).

Proof. Set $m = \max\{0, \beta + \gamma - c - v\}$. We have to examine two cases. If $m = 0$ then from (23) it follows that (24) may be written as:

$$\sum_{u=1}^{\beta} (d_{12} p_{uv}) \sigma_{\beta-u,\gamma-v}(C) + \sum_{u=0}^{\beta-1} p_{uv}(c - \beta - \gamma + u + v + 1) \sigma_{\beta-u-1,\gamma-v}(C) = 0$$

thus it is equivalent to the recurrence relation

$$d_{12} p_{uv} + (c - \beta - \gamma + u + v) p_{u-1,v} = 0 \tag{27}$$

for $u = 1, 2, \ldots, \beta$. (27) is solved by (25). In this case, every $p_{\beta v}$ in $M^{(a+b-\beta-v,\beta,v)}(A \cup B)$ gives rise to a solution of (24) (i.e., (26) is trivial).

Analogously, it is easy to show that if $0 < m = \beta + \gamma - c - v$ then (24) is equivalent to the recurrence relation (27) for $u = \beta + \gamma - c - v + 1, \ldots, \beta$, together with the supplementary condition $d_{12} p_{\beta+\gamma-c-v,v} = 0$ which, by (25), is satisfied if and only if $(d_{12})^{c+v-\gamma+1} p_{\beta v} = 0$. □

For the second equation in (22), observe that

$$d_{23}\big(p_{uv}\sigma_{\beta-u,\gamma-v}(C)\big) = (d_{23} p_{uv})\sigma_{\beta-u,\gamma-v}(C) + (\beta - u + 1) p_{uv} \sigma_{\beta-u+1,\gamma-v-1}(C).$$

Thus such equation may be restricted to the 'straight lines' $u + v = k$, for k a constant satisfying the conditions $\max\{0, \beta + \gamma - c\} \leqslant k \leqslant \beta + \gamma$. This gives:

$$d_{23} \sum_{u=\max\{0,k-\gamma\}}^{\min\{k,\beta\}} p_{u,k-u} \sigma_{\beta-u,\gamma-k+u}(C) = 0. \tag{28}$$

(To obtain the condition $\max\{0, k - \gamma\} \leqslant u \leqslant \min\{k, \beta\}$, compute the intersection of the line $u + v = k$ with $u = 0$, $u = \beta$, $v = 0$, $v = \gamma$, according to $0 \leqslant k < \gamma$, $\gamma \leqslant k \leqslant \beta$ or $\beta < k \leqslant \beta + \gamma$.) Equation (28) is only slightly different from (24); we give its solution in the following lemma ((24) and (28) would have been the same equation if, before writing (22), we had not imposed the conditions of the Young's Rule).

Lemma 7.2. *On the straight lines* $u + v = k$, $\max\{\gamma, \beta + \gamma - c\} \leqslant k \leqslant \beta + \gamma$, *the solutions of* (28) *are given by*:

$$p_{u,k-u} = \frac{1}{(k - \beta - \gamma)_{u-k+\gamma}} (d_{23})^{u-k+\gamma} p_{k-\gamma,\gamma}, \quad u = k - \gamma, \ldots, \min\{k, \beta\} \tag{29}$$

where $p_{k-\gamma,\gamma}$ *belongs to* $M^{(a+b-k,k-\gamma,\gamma)}(A \cup B)$ *and satisfies the condition*

$$(d_{23})^{\beta+\gamma-k+1} p_{k-\gamma,\gamma} = 0 \tag{30}$$

which is trivial in the case $\gamma \leqslant k \leqslant \beta$. *If* $k < \gamma$, *then on the line* $u + v = k$ *Eq.* (28) *has only the trivial solution.*

Proof. We have to examine three cases.

(1) If $k < \gamma$ then

$$d_{23} \sum_{u=0}^{k} p_{u,k-u} \sigma_{\beta-u,\gamma+u-k}(C) = \sum_{u=0}^{k-1} \big[(\beta-u) p_{u+1,k-u-1} + d_{23} p_{u,k-u}\big] \sigma_{\beta-u,\gamma-k+u}(C) + (\beta+1) p_{0k} \sigma_{\beta+1,\gamma-k-1}(C),$$

thus (28) is satisfied if and only if

$$p_{u+1,k-u-1} = -\frac{1}{\beta-u} d_{23} p_{u,k-u} \tag{31}$$

for $u = 0, 1, \ldots, k-1$ and $p_{0k} = 0$.

(2) If $\gamma \leqslant k \leqslant \beta$ then (28) is equivalent to (31) for $u = k-\gamma, \ldots, k-1$.

(3) If $\beta < k$ then (28) is equivalent to (31) for $u = k-\gamma, \ldots, \beta-1$, together with the condition $d_{23} p_{\beta,k-\beta} = 0$.

Then it is easy to complete the proof of the lemma (note that the condition $p_{0k} = 0$ in case (1) forces $p_{u,k-u} = 0$ for all values of u). □

Corollary 7.3. *The solutions of* (22) *are identically zero outside the domain*

$$D = \big\{(u,v)\colon\ 0 \leqslant v \leqslant \gamma \text{ and } \max\{\gamma-v, \beta+\gamma-c-v\} \leqslant u \leqslant \beta\big\}. \tag{32}$$

Proof. These conditions on u and v come from the conditions in (22) together with $u+v = k \geqslant \gamma$ from Lemma 7.2. □

Lemma 7.4. *If* $p_{\beta\gamma} \in M^{(a+b-\beta-\gamma,\beta,\gamma)}(A \cup B) \cap \operatorname{Ker} d_{23}$ *then the condition*

$$p_{uv} = \frac{1}{(u+v-\beta-\gamma)_{\gamma-v} (-c)_{\beta+\gamma-u-v}} (d_{23})^{\gamma-v} (d_{12})^{\beta+\gamma-u-v} p_{\beta\gamma}, \quad (u,v) \in D, \tag{33}$$

is equivalent to:

$$p_{uv} = \frac{1}{(-c)_{\beta+\gamma-u-v}} (d_{12})^{\beta-u} (d_{13})^{\gamma-v} p_{\beta\gamma}, \quad (u,v) \in D \tag{34}$$

(D *is the domain of Corollary* 7.3).

Proof. Indeed, from Lemma 3.1(5) it follows that if $d_{23}p_{\beta\gamma}=0$ then $d_{23}(d_{12})^q p_{\beta\gamma} = -q(d_{12})^{q-1}d_{13}p_{\beta\gamma}$. Using this identity repeatedly, one can change (33) into (34) and viceversa. □

Theorem 7.5. *The solutions of the system* (22) *are given by* (33) (*or equivalently by* (34)) *for* $p_{\beta\gamma}\in M^{(a+b-\beta-\gamma,\beta,\gamma)}(A\cup B)$ *satisfying the system*:

$$\begin{cases} (d_{12})^{c+1}p_{\beta\gamma}=0,\\ d_{23}p_{\beta\gamma}=0. \end{cases} \tag{35}$$

Proof. If $\{p_{uv}\colon (u,v)\in D\}$ is a solution of (22) then (33) is a consequence of (25) and (29), while the conditions in (35) follow from (26) and (30).

Conversely, suppose that $\{p_{uv}\colon (u,v)\in D\}$ satisfies (33) and (35) (and thus (34)). Then, as in the proof of Lemma 7.4, we have $d_{23}(d_{12})^q p_{\beta\gamma} = -q(d_{12})^{q-1}d_{13}p_{\beta\gamma}$. By repeatedly using this identity, we obtain that

$$\begin{aligned} (d_{23})^{\beta-u+1}p_{u\gamma} &= \frac{1}{(-c)_{\beta-u}}(d_{23})^{\beta-u+1}(d_{12})^{\beta-u}p_{\beta\gamma}\\ &= \frac{(-1)^{\beta-u}(\beta-u)!}{(-c)_{\beta-u}}(d_{13})^{\beta-u}d_{23}p_{\beta\gamma}=0 \end{aligned}$$

(d_{13} and d_{23} commute), thus (30) is verified. Analogously

$$\begin{aligned} (d_{12})^{c+v-\gamma+1}p_{\beta v} &= \frac{1}{(-c)_{\gamma-v}}(d_{12})^{c+v-\gamma+1}(d_{13})^{\gamma-v}p_{\beta\gamma}\\ &= \frac{1}{(-c)_{\gamma-v}}(d_{13})^{\gamma-v-1}(d_{12})^{c+v-\gamma+1}d_{13}p_{\beta\gamma}\\ &= \frac{-1}{(-c)_{\gamma-v}(c+v-\gamma+2)}(d_{13})^{\gamma-v-1}d_{23}(d_{12})^{c+v-\gamma+2}p_{\beta\gamma}\\ &= \cdots = \frac{(-1)^{\gamma-v}}{(-c)_{\gamma-v}(c+v-\gamma+2)_{\gamma-v}}(d_{23})^{\gamma-v}(d_{12})^{c+1}p_{\beta\gamma}=0 \end{aligned}$$

(d_{13} commutes with d_{12} and with d_{23}). Thus the (26) is verified. Moreover, (33) and (34) ensure us that p_{uv} verifies (25) and (29). Therefore $\{p_{uv}\colon (u,v)\in D\}$ is a solution of (22). □

Remark 7.6. Theorem 7.5 says that the map

$$\begin{aligned} p_{\beta\gamma}\to \sum_{(u,v)\in D}&\frac{1}{(u+v-\beta-\gamma)_{\gamma-v}(-c)_{\beta+\gamma-u-v}}\\ &\times\big[(d_{23})^{\gamma-v}(d_{12})^{\beta+\gamma-u-v}p_{\beta\gamma}\big]\sigma_{\beta-u,\gamma-v}(C) \end{aligned}$$

is an explicit $S_{a+b} \times S_c$ equivariant isomorphism from $M^{(a+b-\beta-\gamma,\beta,\gamma)}(A \cup B) \cap \operatorname{Ker}(d_{12})^{c+1} \cap \operatorname{Ker} d_{23}$ onto the S_c invariant vectors in $S^{(\alpha,\beta,\gamma)}$.

In the case $\gamma = 0$, this isomorphism yields another method to reconstruct the basis of Dunkl [8]. In this case we know explicitly the decomposition of $M^{(a+b-\beta,\beta)}(A \cup B) \cap \operatorname{Ker}(d_{12})^{c+1}$ into its irreducible constituents (from Corollary 3.2(4)) and the $S_a \times S_b$-invariant functions in these representations are well known. In the case $\gamma > 0$, the $S_a \times S_b$-invariant functions are less tractable and we need two more lemmas to derive the main result of this paper.

Lemma 7.7. *If* $0 \leqslant c+t-\gamma+1 \leqslant \beta$ *and* $\max\{\gamma, \beta+\gamma-c\} \leqslant k \leqslant \min\{\beta, a+b-\beta, a, b\}$ *then* $M^{(a+b-\beta-t,\beta,t)}(A \cup B) \cap \operatorname{Ker}(d_{12})^{c+t-\gamma+1} \cap \operatorname{Ker} d_{23}$ *contains a subrepresentation isomorphic to* $S^{(a+b-k,k)}$ *and in this subrepresentation the* $S_a \times S_b$*-invariant vectors are given by the multiples of*:

$$\sum_{h=k-t}^{\min\{k,a+b-\beta-t\}} \vartheta(t,k,h)\Psi(a+b-\beta-t,\beta,t,a,k,h)$$

where

$$\vartheta(t,k,h) = \frac{(-1)^{h-k+t}(a+b-t-2h+1)(k-h+1)_{h-k+t}(a+b-h-k+2)_{h-k+t}}{(a+b-t-h-k+1)_{h-k+t}(\beta-h+1)_{h-k+t}}. \tag{36}$$

Proof. From Theorem 5.4 we know that if $0 \leqslant k \leqslant \min\{a+b-t, a+b-\beta, \beta+t, a, b\}$ then an orthogonal basis for the $S_a \times S_b$-invariant vectors in the $S^{(a+b-k,k)}$-isotypic subspace of $M^{(a+b-t-\beta,\beta,t)}(A \cup B) \cap \operatorname{Ker}(d_{12})^{c+1+t-\gamma}$ is given by the functions $\Psi(a+b-\beta-t,\beta,t,a,k,h)$ for

$$\max\{k-t, \beta+\gamma-c-t\} \leqslant h \leqslant \min\{k, a+b-k-t, a+b-t-\beta, \beta\}. \tag{37}$$

The condition $\beta+\gamma-c-t \leqslant h$ comes from Corollary 3.2(4) and eliminates $h \geqslant 0$ (because $c+t-\gamma+1 \leqslant \beta$). Now we impose the conditions of the Littlewood–Richardson Rule 4.2: $\gamma \leqslant k \leqslant \beta \leqslant a+b-k \leqslant \alpha$. By Frobenius reciprocity, $S^{(\alpha,\beta,\gamma)}\downarrow S_{a+b} \times S_c$ contains a subrepresentation isomorphic to $S^{(a+b-k,k)} \otimes S^{(c)}$ if and only if these conditions are satisfied. We use them (in particular $\beta+\gamma-c \leqslant k \leqslant \beta$) to simplify (37), which becomes: $k-t \leqslant h \leqslant \min\{k, a+b-\beta-t\}$. Therefore, under these conditions, to prove that $M^{(a+b-\beta-t,\beta,t)}(A \cup B) \cap \operatorname{Ker}(d_{12})^{c+t-\gamma+1} \cap \operatorname{Ker} d_{23}$ contains a subrepresentation isomorphic to $S^{(a+b-k,k)}$ (computing in this subrepresentation a nontrivial $S_a \times S_b$-invariant vector) it suffices to solve the equation:

$$d_{23} \sum_{h=k-t}^{\min\{k,a+b-\beta-t\}} \vartheta(h)\Psi(a+b-\beta-t,\beta,t,a,k,h) = 0 \tag{38}$$

where $\vartheta(h)$ are unknown coefficients. Theorem 6.4 yields:

$$
\begin{aligned}
d_{23} & \sum_{h=k-t}^{\min\{k,a+b-\beta-t\}} \vartheta(h)\Psi(a+b-\beta-t,\beta,t,a,k,h) \\
&= \sum_{h=k-t+1}^{\min\{k,a+b-\beta-t\}} \vartheta(h)\frac{(a+b-t-h-k+1)(\beta-h+1)}{a+b-t-2h+1} \\
&\quad \times \Psi(a+b-\beta-t,\beta+1,t-1,a,k,h) \\
&\quad + \sum_{h=k-t}^{\min\{k,a+b-\beta-t\}-1} \vartheta(h)\frac{(k-h)(a+b-h-k+1)}{a+b-t-2h+1} \\
&\quad \times \Psi(a+b-\beta-t,\beta+1,t-1,a,k,h+1)
\end{aligned} \tag{39}
$$

(see also Remark 6.5). Equating to zero the coefficient of $\Psi(a+b-\beta-t,\beta+1,t-1,a,k,h)$ in the right-hand side of (39), we obtain the following recurrence relation for the coefficients $\vartheta(h)$:

$$
\vartheta(h)\frac{(a+b-t-h-k+1)(\beta-h+1)}{a+b-t-2h+1} + \vartheta(h-1)\frac{(k-h+1)(a+b-h-k+2)}{a+b-t-2h+3},
$$

$h=k-t+1,\ldots,\min\{k,a+b-\beta-t\}$, and this recurrence relation is solved by (36). □

Lemma 7.8. *Let $\vartheta(v,k,h)$ be as in Lemma* 7.7. *Then*:

$$
\begin{aligned}
d_{13} & \sum_{h=k-v}^{\min\{k,a+b-\beta-v\}} \vartheta(v,k,h)\Psi(a+b-\beta-v,\beta,v,a,k,h) \\
&= -\frac{(a+b+v-2k+1)v}{\beta-k+v} \\
&\quad \times \sum_{h=k-v+1}^{\min\{k,a+b-\beta-v+1\}} \vartheta(v-1,k,h)\Psi(a+b-\beta-v+1,\beta,v-1,a,k,h).
\end{aligned}
$$

Proof. From the first formula of Theorem 6.4, it follows that the coefficient of $\Psi(a+b-\beta-v+1,\beta,v-1,a,k,h)$ in $d_{13}\sum_{h=k-v}^{\min\{k,a+b-\beta-v\}}\vartheta(v,k,h)\Psi(a+b-\beta-v,\beta,v,a,k,h)$ is equal to

$$
\begin{aligned}
&\frac{(a+b-v-\beta-h+1)(a+b-v-h-k+1)}{a+b-v-2h+1}\vartheta(v,k,h) \\
&\quad - \frac{(k-h+1)(a+b-h-k+2)}{a+b-v-2h+3}\vartheta(v,k,h-1),
\end{aligned}
$$

and this, using the explicit formula for $\vartheta(v,k,h)$, may be easily transformed into $-\frac{(a+b+v-2k+1)v}{\beta-k+v}\vartheta(v-1,k,h)$. □

Theorem 7.9. *If* $\alpha \geqslant \max\{a,b,c\}$, $\gamma \leqslant \min\{a,b,c\}$ *and* $\max\{\gamma, \beta+\gamma-c\} \leqslant k \leqslant \min\{a,b,a+b-\beta,\beta\}$, *then the* $S_a \times S_b \times S_c$*-invariant vectors in the subrepresentation of* $S^{(\alpha,\beta,\gamma)}{\downarrow}S_{a+b} \times S_c$ *isomorphic to* $S^{(a+b-k,k)} \otimes S^{(c)}$ *are given by the multiples of*:

$$\sum_{v=0}^{\gamma}\sum_{u=k-v}^{\beta}\sum_{h=k-v}^{\min\{k,u,a+b-v-\beta\}} \frac{(-1)^{h-v}(a+b-h-k+2)_{h-k+\gamma}(k-h+1)_{h-k+\gamma}(a+b-v-2h+1)}{(c-\beta-\gamma+h+v+1)_{\beta+\gamma-h-v}(a+b-v-h-k+1)_{h-k+v}} \\ \times E_{\beta-h}(a+b-2h-v, c-\gamma+v, \beta-h, u-h) \\ \times \Psi(a+b-u-v,u,v,a,k,h)\sigma_{\beta-u,\gamma-v}(C).$$

Proof. From Theorem 7.5 and Lemma 7.7 it follows that if in (33) or in (34) we set

$$p_{\beta\gamma} = \sum_{h=k-\gamma}^{\min\{k,a+b-\beta-\gamma\}} \vartheta(\gamma,k,h)\Psi(a+b-\beta-\gamma,\beta,\gamma,a,k,h) \tag{40}$$

then we solve the system (22) obtaining a nontrivial $S_a \times S_b \times S_c$-invariant vector in the subrepresentation of $S^{(\alpha,\beta,\gamma)}{\downarrow}S_{a+b} \times S_c$ isomorphic to $S^{(a+b-k,k)} \otimes S^{(c)}$. If $p_{\beta\gamma}$ is given by (40), then Lemmas 7.8 and 5.2(5) yield

$$\begin{aligned} p_{uv} &= \frac{1}{(-c)_{\beta+\gamma-u-v}}(d_{12})^{\beta-u}(d_{13})^{\gamma-v} \\ &\quad \times \sum_{h=k-\gamma}^{\min\{k,a+b-\beta-\gamma\}} \vartheta(\gamma,k,h)\Psi(a+b-\beta-\gamma,\beta,\gamma,a,k,h) \\ &= \frac{(-1)^{\gamma-v}(a+b+v-2k+2)_{\gamma-v}(v+1)_{\gamma-v}}{(\beta-k+v+1)_{\gamma-v}(-c)_{\beta+\gamma-u-v}} \\ &\quad \times \sum_{h=k-v}^{\min\{k,a+b-\beta-v,u\}} \vartheta(v,k,h)(a+b-\beta-v-h+1)_{\beta-u} \\ &\quad \times \Psi(a+b-u-v,u,v,a,k,h) \end{aligned} \tag{41}$$

for $u+v \geqslant k$, and $p_{uv} = 0$ for $u+v < k$. Then the theorem follows from (7) and (36); in the final formula we have omitted the factor $(-1)^k/(\beta-k+\gamma)!$. □

In the following corollary, we want to restate Theorem 7.9 in the form of a result on four variables orthogonal polynomials. First of all we perform in (19) the change of variables $x \to x$, $y \to y$, $u \to u-x$ and $v \to v-y$; because of the hierarchy between the variables, our solutions are naturally expressed as linear combinations of the characteristic functions of the orbits parametrized in this way: $\sigma_{xy}(A)\sigma_{u-x,v-y}(B)\sigma_{\beta-u,\gamma-v}(C)$. After this change of variables, (19) becomes:

$$\begin{cases} (a-x-y)f(x+1,y,u+1,v)+(b-u-v+x+y)f(x,y,u+1,v) \\ \quad +(\alpha-a-b+u+v+1)f(x,y,u,v)=0, \\ xf(x-1,y+1,u-1,v+1)+(u-x)f(x,y,u-1,v+1) \\ \quad +(\beta-u+1)f(x,y,u,v)=0 \end{cases} \tag{42}$$

where f is defined on $D=\{(x,y,u,v)\colon\ 0\leqslant x\leqslant u\leqslant\beta, 0\leqslant y\leqslant v\leqslant\gamma, x+y\leqslant a, \beta+\gamma-c\leqslant u+v\leqslant b+x+y\}$.

Corollary 7.10. *Suppose that $\alpha\geqslant\max\{a,b,c\}$ and $\gamma\leqslant\min\{a,b,c\}$. For $\max\{\gamma,\beta+\gamma-c\}\leqslant k\leqslant\min\{a,b,a+b-\beta,\beta\}$ define the polynomial $\phi_k(x,y,u,v)$ by setting*

$$\begin{aligned} &\phi_k(x,y,u,v) \\ &\quad=\sum_{h=k-v}^{\min\{k,u,a+b-v-\beta,y-v+b,a-y\}}\frac{(-1)^{h-v}(a+b-h-k+2)_{h-k+\gamma}(k-h+1)_{h-k+\gamma}}{(c-\beta-\gamma+h+v+1)_{\beta+\gamma-h-v}} \\ &\qquad\times\frac{a+b-v-2h+1}{(a+b-v-h-k+1)_{h-k+v}}E_{\beta-h}(a+b-2h-v,c-\gamma+v,\beta-h,u-h) \\ &\qquad\times E_{k-h}(v,a+b-v-2h,a-h,y)E_h(u,a+b-u-v,a-y,x) \end{aligned}$$

for $0\leqslant v\leqslant\gamma$, $k-v\leqslant u\leqslant\beta$, $0\leqslant y\leqslant v$ and $\max\{0,u+v-b-y\}\leqslant x\leqslant\min\{u,a-y\}$, and $\phi_k(x,y,u,v)=0$ for the other values $(x,y,u,v)\in D$.

Then the set $\{\phi_k\colon\ \max\{\gamma,\beta+\gamma-c\}\leqslant k\leqslant\min\{a,b,a+b-\beta,\beta\}\}$ is a basis for the solutions of the system (42)*. This basis is orthogonal with respect to the weight*

$$\binom{a}{a-x-y,x,y}\binom{b}{b-u-v+x+y,u-x,v-y}\binom{c}{c-\beta-\gamma+u+v,\beta-u,\gamma-v}.$$

Proof. It follows immediately from the explicit formula for the Ψ functions in Definition 5.1. The weight is equal to $\|\sigma_{xy}(A)\sigma_{u-x,v-y}(B)\sigma_{\beta-u,\gamma-v}(C)\|^2$. □

We think that it is impossible to get a simpler expression for the polynomials in Corollary 7.10: the sum over h, that comes from Lemma 7.7, is a linear combination of independent vectors. The fact is that the vectors of our basis depend on the single parameter k but are made up of vectors that depend on the two parameters h,k. For the same reason, the norm of the polynomials in Corollary 7.10 (i.e., the norm of the vectors in Theorem 7.9) can be easily computed by mean of the formula in Remark 5.3 and of the orthogonality relations for the Hahn polynomials, but the final result is a very cumbersome double sum expression that seems not easy to simplify.

We end the paper showing how our polynomials specialize to Dunkl's two variables Hahn polynomials when $\gamma=0$. Under such condition, we also have $v=y=0$ and

$h = k$ and the polynomial in Corollary 7.10 becomes a multiple of $E_{\beta-k}(a + b - 2k, c, \beta - k, u - k)E_k(u, a + b - u, a, x)$; applying the symmetry relations (4), (5) and performing the change of variables $x \to x$, $u \to x + u$, one obtains easily the polynomials in [8, (3.11)(ii)].

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