Weakly Pancyclic Graphs

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A graph is called weakly pancyclic if it contains cycles of all lengths between its girth and circumference. A substantial result of Hägkvist, Faudree, and Schelp (1981) states that a Hamiltonian non-bipartite graph of order \( n \) and size at least \( \lfloor (n-1)/2 \rfloor + 2 \) contains cycles of every length \( l \), \( 3 \leq l \leq n \). From this, Brandt (1997) deduced that every non-bipartite graph of the stated order and size is weakly pancyclic. He conjectured the much stronger assertion that it suffices to demand that the size be at least \( \lfloor n^2/4 \rfloor - n + 5 \). We almost prove this conjecture by establishing that every graph of order \( n \) and size at least \( \lfloor n^2/4 \rfloor - n + 59 \) is weakly pancyclic or bipartite.

1. INTRODUCTION

Starting with Dirac's classical theorem [9] on Hamiltonian graphs, proved in 1952, much research has been done on conditions implying that a graph contains long cycles and cycles of specified length. For example, Dirac's theorem was extended by Erdős and Gallai [12], Dirac [10], Ore [16], Pósa [17], and Moon and Moser [15]; recent extensions concerning long cycles are due to Egawa and Miyamoto [11], Bollobás and Hägkvist [3], and Bollobás and Brightwell [2]. A discussion of the classical material may be found in Chapter 3 of [1]; the excellent survey of Bondy [5] covers more recent developments also.

Bondy [4] proved that every Hamiltonian graph of order \( n \) and size exceeding \( n^2/4 \) is pancyclic, that is, it contains cycles of length \( l \) for every \( l \), \( 3 \leq l \leq n \). His elegant proof is very short. His theorem was strengthened...
by Häggkvist, Faudree, and Schelp [14], who showed that a Hamiltonian graph of order $n$ and size exceeding $(n-1)^2/4 + 1$ is pancyclic or is bipartite. Although the number of edges involved here is only about $n/2$ less than in Bondy’s theorem, the proof is of substantial length. In [6], Brandt removed the special role of Hamiltonian graphs by defining a graph $G$ to be weakly pancyclic if it contains cycles of every length $l$, $g(G) \leq l \leq c(G)$, where $g(G)$ the girth of $G$, and $c(G)$, the circumference, are the shortest and longest cycle lengths in $G$. Extending the theorem of Häggkvist, Faudree, and Schelp, he proved that a non-bipartite graph of order $n$ and size greater than $(n-1)^2/4 + 1$ is weakly pancyclic (a non-bipartite graph of order $n$ and size $\lfloor (n-1)^2/4 \rfloor + 2$ contains a triangle).

Brandt conjectured a considerable strengthening of this result, namely that the bound $(n-1)^2/4 + 1 = n^2/4 - n/2 + 5/4$ can be replaced by $(n-1)(n-3)/4 + 4 = n^2/4 - n + 19/4$. Examples showing that Brandt’s conjecture cannot itself be strengthened are (for $k \geq 3$) complete bipartite graphs $K_{k-1,k+1}$ or $K_{k-1,k}$ together with a $C_4$ having two adjacent vertices identified with vertices in the larger class. The graph $K_{k,k}$ together with a triangle intersecting it in one vertex is also an extremal example. (Further information about weakly pancyclic graphs is available in [68].)

In this paper we come close to proving this conjecture.

**Theorem 1.** Let $G$ be a non-bipartite graph of order $n$ and size at least $\lfloor n^2/4 \rfloor - n + 59$. Then $G$ contains a cycle of length $l$ for every $l$, $4 \leq l \leq c(G)$.

In other words, we prove Brandt’s conjecture for graphs of size at least $n^2/4 - n + A$, for some constant $A$. In fact, our proof essentially gives that a minimal counterexample to Brandt’s conjecture is small, say of order at most 132, which more or less implies the previous assertion for some constant $A$. The constant we have stated in the theorem, namely $A = 59$, is chosen to allow us to assume $n \geq 115$ by Brandt’s theorem; use of this theorem could be avoided by the insertion of a larger constant or a little more work.

**2. PROOF OF THEOREM**

Our proof is based on considering the properties of a minimal counterexample $G$ to the theorem. We shall develop these properties to the point of showing that $G$ does not exist. Now it is clear that $\delta(G) \geq 2$. It is also true that $e(G) = \lfloor n^2/4 \rfloor - n + 59$. Otherwise, we could choose a shortest cycle, a longest cycle, and an odd cycle in $G$ and (as $e(G) \geq 3n$) we
could then choose an edge \( e \) not contained in any of these three cycles, in which case \( G \) could be replaced by \( G - e \).

Notice that \( G \) certainly contains a 4-cycle. This follows either from the standard estimate for the extremal function \( \text{ex}(n, C_4) \sim n^{3/2}/2 \) or more straightforwardly from the observation that \( G \) must contain three vertices of degree at least \( \frac{1}{2}(n - 5) \).

We say a vertex \( x \) has low degree if \( d(x) \leq \lfloor n^2/4 \rfloor - n - \lfloor (n - 1)^2/4 \rfloor + (n - 1) \). Clearly if \( x \) has low degree then \( G - x \) has enough edges to satisfy the requirement of the theorem; it is therefore bipartite or weakly pancyclic. The apparently special bipartite case occupies a considerable part of the proof. A vertex not of low degree is of high degree. Notice that \( x \) has low degree if \( n = 2k + 2 \) and \( d(x) \leq k \) or if \( n = 2k + 1 \) and \( d(x) \leq k - 1 \); that is, if \( d(x) \leq n/2 - 1 \).

As remarked above, we know that \(|G| = n \geq 115\). The one place in the proof where we certainly need to assume that \( n \) is not small is Lemma 12 in Section 5. However, given that we are ignoring small \( n \) in that lemma, we have made occasional use of the assumption elsewhere in places where the proof can thereby be shortened.

It might reasonably be suggested that those lemmas which do not rely on \( n \) being large (principally in Sections 3 and 4) should have been stated and proved in such a way that they might be used in a full proof of Brandt’s conjecture. We spent some thought and effort on this possibility but decided it was not feasible. Apart from the obvious fact that a full proof might follow a completely different line of attack, it has been our experience in proving the lemmas below that many of them follow an unpleasant pattern; a simple argument will cover almost all the cases but one or two special cases require many times more work. This phenomenon was already present in the proof of Häggkvist, Faudree and Schelp [14]. In this paper the special cases would dominate entirely were we not to settle occasionally for something weaker than best possible.

A further, and more important, reason for not stating our lemmas in more generality is as follows. Brandt’s conjecture concerns graphs of size at least \( \lceil n^2/4 \rceil - n + 5 \). A proof by induction would require defining a vertex \( x \) to be of low degree when \( d(x) \leq k - 1 \) and \( n = 2k + 2 \) or when \( d(x) \leq k \) and \( n = 2k + 1 \). This would have a dramatic effect on the details of the proof. Moreover it would shift the weight of the proof toward working on the case \( n = 2k + 2 \), whereas (especially in Lemma 11) it has helped us greatly to have the odd and even cases treated more equitably.

A broad outline of our proof is as follows. We shall show that our minimal counterexample \( G \) has circumference at least \( n - 1 \). We then show that if \( G \) has an \((n - 1)\)-cycle it is weakly pancyclic. Finally we show that if \( G \) is Hamiltonian it has an \((n - 1)\)-cycle. These three properties show that \( G \) does not exist.
3. THE CIRCUMFERENCE OF A MINIMAL COUNTEREXAMPLE

We show first that the circumference cannot be too small.

**Lemma 2.** $c(G) > n - 2 \sqrt{n}$.

**Proof.** Theorem 5.2 on p. 149 in [1] states that if $c(G) = c$ and $e(G) > (c/4)(2n - c)$ then $G$ is weakly pancyclic (and has girth 3). Hence

$$(c/4)(2n - c) > n^2/4 - n,$$

implying the assertion. $lacksquare$

**Lemma 3.** If $c(G) \leq n - 2$ and $C$ is a longest cycle in $G$, then there is at most one vertex of high degree outside $C$.

**Proof.** Suppose that $x$ and $y$ are two vertices of high degree in $G - C$. Neither $x$ nor $y$ has more than $c/2$ neighbours in $C$. Since both $x$ and $y$ have degree exceeding $n/2 - 1$ it must be that either $xy$ is an edge of $G$ or else $x$ and $y$ have a common neighbour in $G - C$. It is now easily checked that between $\{x, y\}$ and any three consecutive vertices of $C$ there can be at most two edges. Each of $x$ and $y$ has no more than $n - c - 1$ neighbours outside $C$, so

$$n - 2 < d(x) + d(y) \leq 2c/3 + 2(n - c - 1),$$

which means $c < 3n/4$. But this contradicts Lemma 2 because $n \geq 64$. $lacksquare$

We now come to the first time that we need to consider graphs with a vertex $x$ such that $G - x$ is bipartite.

**Lemma 4.** Let $H$ be a graph of order $2k + 2$ consisting of a Hamiltonian bipartite graph with vertex classes $V_1$ and $V_2$, $|V_1| = |V_2| = k$, together with an edge $uv$ such that the neighbourhoods $U$ and $W$ of $u$ and $w$ satisfy $U \subseteq V_1$, $W \subseteq V_1$ and $|U|, |W| \geq 1$. Suppose that $H$ has $k^2 + 4$ edges. Then $H$ has a $(2k + 1)$-cycle.

**Proof.** We proceed by induction on $k$. For $k = 2$ there is a unique graph satisfying the conditions of the lemma, and it is easily checked that it has a 5-cycle.

For $k \geq 3$, let $y \in V_2$ and let $x, z$ be the vertices in $V_1$ next to $y$ in the $2k$-cycle. Remove $x, z$ and $y$ from the graph and form a new graph $H_y$ by joining a new vertex $t$ to the two vertices of the $2k$-cycle adjacent to $x$ and $z$, and also to any common neighbours of $x$ and $z$ in $V_2$. Join $u$ to $t$ if $u$ was joined to $y$ and do the same for $w$. 
Observe that if \( e(H_y) \geq (k - 1)^2 + 4 \) then \( H_y \) satisfies the conditions of the lemma, so \( H_y \) has a \((2k - 1)\)-cycle. But this cycle must contain \( t \), and so corresponds in \( H \) to a \((2k + 1)\)-cycle passing through \( xyz \).

Therefore we are done unless \( e(H_y) \leq (k - 1)^2 + 3 \) for every \( y \in V_2 \). Now

\[
e(H) - e(H_y) \leq d(y) + k - 1 + \delta(y) + \varepsilon(y),
\]

where \( \delta(y) = 1 \) if \( \{ x, z \} \subset U \) and \( \delta(y) = 0 \) otherwise, \( \varepsilon(y) \) being defined similarly in terms of \( W \). Thus, for each \( y \in V_2 \),

\[
d(y) + k - 1 + \delta(y) + \varepsilon(y) \geq 2k,
\]

so the number of edges in the bipartite graph satisfies

\[
k^2 + 3 - |U| - |W| = \sum_{y \in V_2} d(y) \geq k^2 + k - \sum_{y \in V_2} \delta(y) - \sum_{y \in V_2} \varepsilon(y)
\]

\[
\geq k^2 + k - (|U| - 1) - (|W| - 1),
\]

giving \( k \leq 1 \), a contradiction.

We can now prove the main lemma of this section.

**Lemma 5.** If \( G \) is a minimal counterexample to Theorem 1, then \( c(G) \geq n - 1 \).

**Proof.** Suppose the lemma is false, and let \( C \) be a longest cycle with \(|C| < n - 2\). Let \( x \notin C \) be a vertex of low degree (there is at least one such by Lemma 3). Since \( G - x \) is not a counterexample, \( G - x \) must be bipartite. In particular, \(|C| \) is even.

Suppose there is another vertex \( y \notin C \cup \{x\} \) of low degree. Since \( G - y \) is not a counterexample \( G - y \) is also bipartite. Now \( G - x - y \) is a bipartite graph of order \( n - 2 \) with at least

\[
\lfloor n^2/4 \rfloor - n + 59 - 2(n/2 - 1) > (\lfloor n/2 \rfloor - 2)(\lfloor n/2 \rfloor - 2) + 1
\]

edges, so either \( G - x - y \) is connected or it has two components, one of which is an isolated vertex \( v \). Now, in the latter case, \( v \) must be joined to both \( x \) and \( y \) since \( \delta(G) \geq 2 \), and \( v \notin C \). Moreover \( x \) is joined to only one colour class in \( G - \{x, y, v\} \), as too is \( y \). If \( x \) and \( y \) are joined to the same colour class then \( xy \in G \) else \( G \) would be bipartite; but then, by the minimality of \( G \) it must be that \( G - v \) is weakly pancyclic, and so \( G \) must be also, a contradiction. Therefore \( x \) and \( y \) must be joined to opposite colour classes in \( G - x - y - v \). Construct \( G' \) from \( G - x - y - z \) by joining a new vertex \( w \) to the neighbours of \( x \) and \( y \). Then \( e(G') \geq e(G) - 3 \) so \( G' \) is weakly pancyclic. But \( C \subset G' \) so \( G' \) has a \((|C| - 1)\)-cycle, which must contain \( w \), implying that \( G \) contains a \((|C| + 1)\)-cycle, a contradiction.
We conclude that if \( x, y \in G - C \) are of low degree then \( G - x - y \) is connected. But then \( x \) and \( y \) must be joined to the same vertex class of \( G - x - y \) and \( xy \) must be an edge, or else \( G \) itself would be bipartite. It follows that \( y \) is the unique neighbour of \( x \) in its vertex class of \( G - x \).

In particular there cannot be three vertices \( x, y, z \) in \( G - C \) of low degree, because then \( z \) would be the unique neighbour of \( x \) in the other vertex class of \( G - x \) implying \( d(x) = 2 \) and similarly \( d(y) = d(z) = 2 \) making \( \{ x, y, z \} \) a triangular component of \( G \). It follows from Lemma 3 that \( G - C \) has at most three vertices, and since \( |C| \) is even we see that \( |C| = n - 3 \) if \( n \) is odd and \( |C| = n - 2 \) if \( n \) is even. We treat these two cases separately.

Consider first the case \( n = 2k + 1 \), then \( |C| = 2k - 2 \). Let \( V_1 \) and \( V_2 \) be the vertex classes of \( G - x \) with \( |V_1| \geq |V_2| \). If \( |V_1| > |V_2| \) then \( |V_1| = k + 1 \), \( |V_2| = k - 1 \) and \( |V_1 \setminus C| = 2 \). Lemma 3 shows that \( V_1 \setminus C \) contains a vertex \( y \) with \( d(y) \leq k - 1 \), and by the discussion above \( y \) is the unique neighbour of \( x \) in \( V_1 \). The other vertex \( v \in V_1 \setminus C \) is therefore of high degree; but then \( d(v) \geq k \) so \( v \) must be joined to \( x \), a contradiction.

Therefore \( |V_1| = |V_2| = k \): let \( V_1 \setminus C = \{ y \} \) and \( V_2 \setminus C = \{ z \} \). If \( xy \) is not an edge then \( y \) is of high degree; that is, \( d(y) \geq k \) and so \( y \) is joined to all of \( V_2 \). Then some neighbour of \( x \) in \( V_1 \cap C \) could be replaced in the cycle \( C \) by \( y \), so we may assume that \( xy \) is an edge and likewise that \( xz \) is an edge. Form \( G' \) from \( G - x - y - z \) by adding a new vertex \( w \) joined to the neighbours of \( y \) and \( z \). Then \( e(G') = e(G) - d(x) \) and \( G' \) is not bipartite, so \( G' \) is weakly pancyclic. As before, the fact that \( G' \) contains a \((|C| - 1)\)-cycle implies that \( G \) contains a \((|C| + 1)\)-cycle, a contradiction.

So we turn finally to the second case: \( n = 2k + 2 \). Then \( |C| = 2k \), \( |V_1| = k + 1 \) and \( |V_2| = k \). Let \( V_1 \setminus C = \{ y \} \). If \( y \) has high degree then \( d(y) \geq k + 1 \) and (as before) \( y \) can replace any vertex of \( V_2 \), so either all vertices of \( V_1 \) have high degree, in which case \( G \) contains a \( K_{k+1,k+1} \) and is Hamiltonian, or we may assume that \( y \) is of low degree. But in the latter case the conditions of Lemma 4 apply, so \( G \) contains a \((2k + 1)\)-cycle, our final contradiction.

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4. GRAPHS CONTAINING AN \((n - 1)\)-CYCLE

In this section we shall derive a contradiction from the assumption that our minimal counterexample \( G \) contains an \((n - 1)\)-cycle. There are essentially two possibilities: either the remaining vertex \( x \) has low degree, and so \( G - x \) is bipartite, or \( x \) has high degree. The first possibility is a significant one, in readiness for which we catalogue a few straightforward properties of dense bipartite graphs. A strengthened version of Lemma 8 was proved by Entringer and Schmeichel [13] (they used \( k^2/2 \) edges instead of
LEMMA 6. Let \( H \) be a bipartite graph with vertex classes \( V_1, V_2 \) with \( |V_1| = |V_2| = k \). Let \( H \) have \( k^2 - k + 2 \) edges. Let \( u \in V_1, w \in V_2 \) and let \( 3 \leq l \leq k \). Then \( H \) contains a \( u - w \) path of length \( 2l - 1 \). In particular, \( H \) contains even cycles of all lengths \( 2l, 2 \leq l \leq k \).

Proof. We proceed by induction on \( k \). Note there are \( k - 2 \) edges missing from the complete bipartite graph (so we may assume \( k \geq 2 \)) and hence \( \delta(H) \geq 2 \). Let \( A = \Gamma(u) - \{w\} \) and \( B = \Gamma(w) - \{u\} \), so \( |A|, |B| \geq 1 \). Now \( |A| + |B| \geq 2(k - 1) - (k - 2) = k \), so \( |A|, |B| \geq k - 1 \geq k - 2 \). Thus there is an edge between \( A \) and \( B \), so the lemma holds with \( l = 2 \).

Now let \( l \geq 3 \). Select \( a \in A, b \in B \). If \( e(G - u - w) \geq (k - 1)^2 - (k - 1) + 2 \) then by the induction hypothesis there is a \((2l - 3)\)-path from \( a \) to \( b \) in \( G - u - w \), giving us the \( u - w \) path we are after. On the other hand, if \( e(G - u - w) \leq (k - 1)^2 - (k - 1) + 1 \) then \( e(G) - e(G - u - w) \geq 2k - 1 \), so \( d(u) = d(w) = k \) and \( e(G) - e(G - u - w) = (k - 1)^2 - (k - 1) + 1 \). Adding one edge to \( G - u - w \) creates a \((2k - 2)\)-cycle (by the induction hypothesis). But then \( G - u - w \) has a Hamilton path, and the lemma follows at once from \( d(u) = d(w) = k \).

LEMMA 7. Let \( G \) be a non-bipartite graph of order \( 2k + 1 \) and size \( k^2 - k + 10 \), having a vertex \( x \) such that \( G - x \) is bipartite and any set of \( b \geq 2 \) vertices in one class of \( G - x \) has at least \( b + 1 \) neighbours in the other class. (This condition is trivially satisfied if \( G - x \) is Hamiltonian.) Then \( G \) contains a 5-cycle.

Proof. Suppose the lemma is false. Let \( U \) and \( W \) be the neighbourhoods of \( x \) in the vertex classes of \( G - x \), where \( |U| \geq |W| \geq 1 \).

We shall select a subset \( B \) of \( W \), and disjoint sets \( A \) and \( C \) in the opposite class, such that \( |B| \leq |A| = |C| = p \leq 3 \), and every vertex \( w \) in the class of \( B \) but not in \( B \) is such that either \( \Gamma(w) \cap A = \emptyset \) or \( \Gamma(w) \cap C = \emptyset \). In particular, every such \( w \) has at least \( p \) non-neighbours in the other class, so \( G - x \) is obtained from \( K_{k,k} \) by deleting at least \( p(k - p) \) edges.

The selection is as follows. If \( |W| \geq 6 \), let \( B \) consist of two vertices of \( W \), let \( A \) be a set of three neighbours of \( B \) and let \( C \) consist of three vertices of \( U \). There is no \( w \in W \setminus B \) joined to some vertices in \( A \) and to some vertices in \( C \) since otherwise \( G \) contains a 3-cycle \( xwabc \) with \( a \in A, b \in B \) and \( c \in C \). If \( |W| \leq 5 \) and \( |U| \geq 4 \) then let \( B = \{b\} \subset W \), let \( A \) consist of two neighbours of \( b \), and let \( C \) be made up of two vertices of \( U \). Finally, if \( |U| \leq 3 \), let \( B = \{b\} \subset W \), let \( C \subset U \) consist of one vertex, and let \( A = \{a\} \), where \( a \notin C \) is a neighbour of \( b \). As \( G \) has no 5-cycle it is true in these cases as well that no vertex \( w \in W \setminus B \) is joined to some vertices in \( A \) and to some in \( C \).
Observe that \( p(k - p) > k - 10 + |U| + |W| \) in each case, contradicting our assumption about \( G \).

**Lemma 8.** Let \( H \) be a Hamiltonian bipartite graph of order \( 2k \) and size more than \( k(k + 1)/2 \). Then \( H \) contains cycles of every even length \( 2l + 2, 1 \leq l \leq k - 1 \).

**Proof.** Label the cycle \( x_1 x_2 \cdots x_{2k-1} x_2 \). Suppose \( H \) has no cycle of length \( 2l + 2 \). Then \( H \) cannot contain both of \( x_1 x_i \) and \( x_{2k} x_{i+2} \) for even \( i \), nor both of \( x_1 x_{2k} \) and \( x_2 x_{i+1} \) for even \( i \). Therefore \( d(x_1) + d(x_{2k}) \leq k + 1 \). Applying this to every pair of vertices adjacent on the cycle, we derive \( e(H) \leq k(k + 1)/2 \), a contradiction.

When a graph of order \( n \) has an \((n-1)\)-cycle, in determining whether it is weakly pancyclic it is immaterial whether it is Hamiltonian, so the next lemma seems somewhat spurious. However, it will be useful in the proof of Lemma 10.

**Lemma 9.** Let \( G \) be a non-bipartite graph of order \( 2k + 1 \) and size \( k^2 - k + 4 \), having a vertex \( x \) such that \( G-x \) is Hamiltonian and bipartite. Then \( G \) is Hamiltonian.

**Proof.** We proceed by induction on \( k \), the case \( k = 3 \) being easily verified. Let the vertex classes of \( G-x \) be \( V_1 \) and \( V_2 \), with \( \Gamma(x) \cap V_1 = U \) and \( \Gamma(x) \cap V_2 = W \). We assume \(|U| \geq 2\), for the case \(|U| = |W| = 1\) follows from Lemma 6. The technique we use is similar to that used in Lemma 4. Let \( a \in V_1 \) and let \( b, c \in V_2 \) be its neighbours on the Hamilton cycle of \( G-x \). Form a graph \( G_a \) from \( G-a-b-c \) by joining a new vertex \( t \) to the neighbours of \( b \) and \( c \) in \( V_2 \). Also, join \( t \) to \( x \) if \( W \cap \{b, c\} \neq \emptyset \). Then \( e(G) - e(G_a) \leq d(a) + (k-1) + e(a) \), where \( e(a) = 1 \) if \( \{b, c\} \subset W \) and \( e(a) = 0 \) otherwise. Now \( G_a-x \) is Hamiltonian and \( G_a \) is not bipartite (because \( U \setminus \{a\} \neq \emptyset \)), so if \( e(G_a) \) is large enough the induction hypothesis tells us that \( G_a \) is Hamiltonian. The Hamilton cycle passes through \( t \), and can be augmented via \( bac \) to a Hamilton cycle in \( G \).

So we are done unless, for each \( a \in V_1 \), \( e(G_a) \leq (k-1)^2 - (k-1) + 3 \) holds, in which case \( 2k-1 \leq e(G) - e(G_a) \leq d(a) + (k-1) + e(a) \), or \( d(a) \geq k - e(a) \). Thus

\[
-k^2 - k + 4 = e(G) = |W| + \sum_{a \in V_1} d(a) \geq |W| + k^2 - \sum_{a \in V_1} e(a)
\]

\[
\geq |W| + k^2 - (|W| - 1),
\]

so \( k \leq 3 \). This completes the proof.
We are now in a position to handle the case that our minimal counterexample $G$ has a vertex $x$ with $G - x$ bipartite.

**Lemma 10.** Let $G$ be a non-bipartite graph of order $2k+1$ and size $k^2 - k + 10$, having a vertex $x$ such that $G - x$ is Hamiltonian and bipartite. Then $G$ is weakly pancyclic.

**Proof.** We know from Lemma 8 that $G$ contains all even cycles of length $2l$, $2 \leq l \leq k$, and from Lemma 7 that $G$ has a 5-cycle, so we need find only the odd cycles of length $2l + 1$, $3 \leq l \leq k$. If $|U| + |W| \geq 2k - 2$ then the existence of these odd cycles follows immediately from the existence of the even cycles, since $x$ is joined to all but 2 vertices of every $2l$-cycle. Thus we assume that $|U| + |W| \leq 2k - 3$, and proceed by induction on $k$ (there being nothing to prove for $k \leq 3$).

By Lemma 9, $G$ is Hamiltonian. Let $caxbd$ be the five consecutive vertices around $x$ on this cycle. Form a graph $G'$ from $G - a - b - x$ by adding a new vertex $x'$ joined to the neighbours of $a$ and $b$; also add the edge $cd$ if it is not already present. Then $G'$ is non-bipartite and $G' - x'$ is bipartite and Hamiltonian. Also, $e(G') \geq e(G) - d(x) = e(G) - (|U| - |W|) \geq (k - 1)^2 - (k - 1) + 11$, so by the induction hypothesis $G'$ contains all odd cycles of lengths $2l - 1$, $3 \leq l \leq k$. These cycles all contain the path $cx'd$ and none contain $cd$, so by replacing $cx'd$ with $caxbd$ we find in $G$ all odd cycles of lengths $2l + 1$, $3 \leq l \leq k$.

We can at last rule out the possibility of an $(n-1)$-cycle in our minimal counterexample.

**Lemma 11.** Let $G$ be a minimal counterexample to Theorem 1. Then $G$ has no $(n-1)$-cycle.

**Proof.** Let us suppose $G$ has an $(n-1)$-cycle $C$. Note that to derive a contradiction it is sufficient to show that $G$ has cycles of every length $l$, $4 \leq l \leq n - 2$. Let $x$ be the vertex not on $C$. If $x$ is of low degree then $G - x$ is not a counterexample. Now if $G - x$ is not bipartite it must be weakly pancyclic, and therefore so is $G$. If $G - x$ is bipartite then $n$ is odd and Lemma 10 tells us that $G$ is weakly pancyclic. So $x$ must be of high degree. But if $n = 2k + 2$ or $n = 2k + 1$ and $d(x) \geq k + 1$ then the edges from $x$ to $C$ immediately generate, together with $C$, cycles of every length. This leaves only the case $n = 2k + 1$ and $d(x) = k$. We first show there is a cycle $D$ of length $2k - 1 = n - 2$.

Suppose first that $x$ is joined to alternate vertices of $C$. Now $C$ must have a chord of even length, for otherwise $G$ would be bipartite. Label the vertices of $C u_1, u_2, ..., u_{2k}$ where $xu_i \in G$ if $i$ is odd, and let the even chord be $u_{2i-1}u_{2i}$ or $u_2u_{2k}$. We may assume the chord has length at least 4 since
otherwise we immediately have a \((2k-1)\)-cycle. Now if \(u_1u_2\ldots u_{2k-1} \in G\) then
\(u_1u_2\ldots u_{2k-1}xu_{2k-2}\ldots u_2\) is a \((2k-1)\)-cycle. If \(u_2u_3 \in G\) then
\(u_2u_3u_4\ldots u_{2k-1}xu_{2k-2}\ldots u_2u_3\) is a \((2k-1)\)-cycle.

Suppose on the contrary that the neighbours of \(x\) do not alternate around \(C\); in particular, we may assume that the cycle \(C\) is labelled \(x_1x_2\ldots x_{2k}\) where \(x_1, x_2 \in \Gamma(x)\). If two neighbours of \(x\) are distance three apart on \(C\) then we have our \((2k-1)\)-cycle. If not, then among every two vertices at distance three on \(C\), one is a neighbour of \(x\) and one is not, because \(d(x) = |C|/2\). Now if \(x_3 \notin \Gamma(x)\), then \(x_3x_2 \in \Gamma(x)\), in which case by relabelling we may assume \(x_3 \in \Gamma(x)\). It follows that \(\{x_4, x_5, x_6\} \subseteq \Gamma(x)\) so \(\{x_2, x_3, x_5\} \subseteq \Gamma(x)\) and so on around \(C\) (in particular, \(k\) is a multiple of 3). Now there are many chords of \(C\) whose existence would give rise to a \((2k-1)\)-cycle. For example, if \(x_2x_3\) is a chord and either \(\{x_{a+3}, x_{b+1}\} \subseteq \Gamma(x)\) or \(\{x_{a+2}, x_{b+2}\} \subseteq \Gamma(x)\) then one of \(xx_{a+3}x_{a+4}\) or \(xx_{a+2}x_{a+3}\ldots x_{b-1}\) is a \((2k-1)\)-cycle. It is now easily checked that if \(G\) has no such cycle then vertices of the form \(x_{a+4}, x_{a+5}\) and \(x_{a+6}\) each have only two neighbours in \(G\), namely, their neighbours on \(C\). Consequently, by summing the degrees of all vertices we obtain

\[2k^2 - 4k + 116 \leq 2\epsilon(G) \leq k + k + 2 + k(k + 2),\]

a contradiction since \(k \geq 57\).

We have now shown that \(G\) has a \((2k-1)\)-cycle \(D\); let \(a, b\) be the vertices of \(G - D\). If \(a\) is a vertex of low degree then, since \(D \subseteq G - a\), the graph \(G - a\) is not bipartite and so must be weakly pancyclic; thus \(G\) is also, a contradiction. Thus \(d(a) \geq k\). But if \(a\) has at least \(k\) neighbours on \(D\) then \(G\) is weakly pancyclic. So \(d(a) = d(b) = k\) and \(ab \in G\).

Since \(G\) is not weakly pancyclic there is some \(i, 3 \leq i \leq 2k - 4\), for which \(G\) has no \((i+2)\)-cycle. Let \(v_1, v_2, \ldots, v_{2k-1}\) be the vertices of \(D\). For each \(i, 1 \leq i \leq 2k - 1\), only one of \(v_i\) and \(v_{i+1}\) can be a neighbour of \(a\) (of course, we take suffices modulo \(2k - 1\)). Consider the orbits on the set of chords \(\{v_i, v_{i+1}; 1 \leq i \leq 2k - 1\}\) induced by the permutation \(v_i \mapsto v_{i+1}\) of \(\{v_i; 1 \leq i \leq 2k - 1\}\). Each orbit has the same size and the sum of the sizes is \(2k - 1\), so each orbit has odd size. Therefore each orbit contains a chord, no end of which is a neighbour of \(a\). But \(a\) is joined to \(k - 1\) of the \(2k - 1\) vertices of \(a\), so there is exactly one orbit. This means that \(l\) is coprime to \(2k - 1\). Let \(s\) be the inverse of \(l\) (modulo \(2k - 1\)). If we now relabel the vertex set of \(D\) as \(\{w_1, w_2, \ldots, w_{2k-1}\}\) where \(w_i = v_{is}\), then \(a\) is joined to alternate vertices \(w_i\) with a single "gap:" say \(\{w_3, w_5, \ldots, w_{2k-1}\}\) is the set of neighbours of \(a\) on \(D\).

By the same argument \(b\) is also joined to alternate \(w_i\) with a gap of size two. Now the gap for \(b\) must also be \(\{w_1, w_2\}\) (which implies that
$a$ and $b$ have the same neighbours): for otherwise, suppose $bw_1 \in G$. Let $t = (l - 1)s$ (modulo $2k - 1$), so $2 \leq t \leq 2k - 2$. If $t$ is even then $aw_{r+1} \in G$ and if $t$ is odd then $aw_{2k+1} \in G$. Either way, $a$ and $b$ have neighbours whose suffixes (when labelled in the form $w_n$) differ by $t$, so they have neighbours distance $l - 1$ apart on $D$. But then $G$ has an $l$-cycle.

Thus $a$ and $b$ have identical neighbours on $D$. But then $w_3$ and $w_{t+3}$ (if $t$ is even, $t \neq 2k - 2$) or $w_3$ and $w_{2k+2-t}$ (if $t$ is odd) are pairs of neighbours of $a$ and $b$ distance $l - 1$ apart on $D$, again giving an $l$-cycle. The special case $t = 2k - 2$, corresponding to $(l - 1)l^{-1} \equiv -1$ (modulo $2k - 1$) or $l = k$, requires further attention. In this case $a$ is joined to a sequence of vertices at intervals of length $2k \equiv 1$ (modulo $2k - 1$) around $D$; in other words $a$ is joined to a sequence of $k - 1$ consecutive vertices, say $v_1, v_2, ..., v_{k-1}$. Now it is easily checked that if there is any chord whose end is in $\{v_{k+1}, ..., v_{2k-2}\}$ then two neighbours of $a$ are distance $l$ apart, giving an $(l+2)$-cycle. Thus $G$ can have at most $(k+1) + k - 1$ edges. But this is less than $k^2 - k + 58$, so our lemma is proved.

5. HAMILTONIAN GRAPHS

In this final section we concentrate on Hamiltonian graphs. Our aim is to show that if a minimal counterexample to Theorem 1 is Hamiltonian then it has an $(n-1)$-cycle as well. We shall produce an $(n-1)$-cycle by first finding an $(n-2)$-cycle and then enlarging it.

Our main lemma is the next one. As mentioned in Section 2, it is here that we need $n \geq 115$ in order to avoid spending time on edge effects and small cases.

**Lemma 12.** Let $H$ be a Hamiltonian graph of order $n \geq 115$ and size at least $\lfloor n^2/4 \rfloor - n + 59$. Then either $H$ has an $(n-1)$-cycle or it has an $(n-2)$-cycle with an edge $xy$ with $x$ and $y$ not on the cycle such that $d(x) + d(y) \geq n - 3$.

**Proof.** Let the cycle be labelled $v_1v_2\cdots v_n$. Given a set of vertices $A \subset V(H)$, let $A^+$ denote the set $\{v_i : v_i \in A\}$. Of course, indices are taken modulo $n$. Typically we shall apply this to the sets $A_i = \Gamma(v_i)$, $1 \leq i \leq n$. We shall suppose from now on that $H$ has no $(n-1)$-cycle. Therefore

$$A_i^+ \cap A_{i+1} = A_i^+ \cap A_{i+2} = \emptyset,$$

for every $i$, so in particular

$$d_i + d_{i+1} \leq n \quad \text{and} \quad d_i + d_{i+2} \leq n, \quad (1)$$
where $d_i = |A_i|$, $1 \leq i \leq n$. Given subsets $A, B, C \subseteq V(H)$ and $X \subseteq C \setminus (A \cup B)$, observe that

$$|A \cap B| \geq |A \cap C| + |B \cap C| - |C| + |X|. \quad (2)$$

Note also that if $A \cap C = B \cap C = \emptyset$ then

$$|A \cap B| \geq |A| + |B| - (n - |C|) = |A| + |B| + |C| - n. \quad (3)$$

Let us assume also that $H - \{v_2, v_3\}$ has no $(n - 2)$-cycle. Then $A^*_1 \cap A_4 = \{v_3\}$, so applying (3) with $A = A_3$, $B = A_4 \setminus \{v_3\}$ and $C = A^*_1$ gives

$$|A_3 \cap A_4| = |A \cap B| \geq d_1 + d_3 + d_4 - n - 1.$$ Note now that $|A^*_2 \cap A^*_4| = |A^*_2 \cap A_4|$, so applying (3) again with $A = A_3$, $B = A^*_2$ and $C = A^*_4$ gives

$$|A_3 \cap A^*_4| \geq d_2 + d_3 + d_4 - n.$$ Now $|A^*_3 \cap A^*_4| = |A_3 \cap A_4|$, so if we apply (2) to the previous two inequalities with $A = A_3$, $B = A^*_3$, $C = A^*_4$ and $X = \emptyset$ we obtain

$$|A_3 \cap A^*_4| \geq d_1 + d_3 + d_4 - 2n - 1.$$ Again, $|A^*_3 \cap A^*_4| = |A_3 \cap A^*_4|$, and applying (2) with $A = A_3$, $B = A^*_3$, $C = A_4^*$ and $X = \{v_3, v_5\}$ gives

$$|A_3 \cap A^*_4| \geq 2d_1 + 2d_2 + 3d_3 + 2d_4 - 4n.$$ Finally, applying (2) with $A = A_4$, $B = A_3^*$, $C = A_3$ and $X = \emptyset$ we obtain

$$|A_4 \cap A_3^*| \geq 3d_1 + 2d_2 + 3d_3 + 3d_4 - 5n - 1.$$ But $A_4 \cap A_3^* = \emptyset$: thus the assumption that $H - \{v_2, v_3\}$ has no $(n - 2)$-cycle leads to the inequality

$$3d_1 + 2d_2 + 3d_3 + 3d_4 \leq 5n + 1.$$ By symmetry,

$$3d_1 + 3d_2 + 2d_3 + 3d_4 \leq 5n + 1.$$
also, and hence

\[6d_i + 5d_2 + 5d_3 + 6d_4 \leq 10n + 2.\]

Let us call a pair of adjacent vertices \(\{v_i, v_{i+1}\}\) good if \(d_i + d_{i+1} \geq n - 3\), and call a single vertex \(v_j\) bad if \(d_j \leq (5n + 17)/12\). Note that if \(\{v_i, v_{i+1}\}\) is good and neither \(v_{i-1}\) nor \(v_{i+2}\) is bad then

\[6d_{i-1} + 5d_i + 5d_{i+1} + 6d_{i+2} > 10n + 2,\]

so \(H - \{v_i, v_{i+1}\}\) has an \((n - 2)\)-cycle and our lemma is proved. We shall now show that such a good pair exists.

Paint the bad vertices black and the others white. Now proceed clockwise round the cycle, starting at a white vertex, painting brown any black vertex which follows an unchanged black vertex: thus for example from a maximal block of five consecutive originally black vertices three would remain black, and from a maximal block of four black vertices two would remain black. Let \(k\) black vertices remain. Consider now the \(n - k\) non-black vertices and examine the \(n - k\) consecutive pairs therefrom. (Thus a pair may “jump over” a black vertex.) By (1), the degree sum of the vertices in a consecutive pair is at most \(n\). Summing over the \(n - k\) pairs and adding in the black degrees twice, we see that

\[n^2 - 4n + 235 \leq 4e(G) \leq (n - k) n + 2k(5n + 17)/12,\]

wherefore

\[(k - 24)(n - 17) + 1027 \leq 0.\]

If \(k \geq 24\) this is false, so \(k \leq 23\).

Let us call one of our \(n - k\) pairs a candidate pair if it satisfies two conditions: first, it is a pair of consecutive vertices on the original cycle (that is, it is not separated by a black vertex) and secondly, both vertices to either side on the original cycle are white. Each black vertex prevents at most four pairs from being candidates; namely if \(v_i\) is black, the pairs \(\{v_{i-2}, v_{i-1}\}, \{v_{i-1}, v_{i+1}\}, \{v_{i+1}, v_{i+2}\}\) and (because \(v_{i+1}\) may be brown) \(\{v_{i+2}, v_{i+3}\}\). Therefore at least \(n - 5k \geq n - 115 \geq 0\) pairs are candidates. If any candidate pair is also a good pair the lemma will be proved. So suppose to the contrary that each candidate pair has degree sum at most \(n - 4\). The other \(4k\) pairs will have degree sum at most \(n\), by (1). Adding these degree sums to twice the sum of the degrees of the black vertices, we obtain

\[n^2 - 4n + 235 \leq 4e(G) \leq (n - 5k)(n - 4) + 4kn + 2k(5n + 17)/12,\]
or
\[ k(n - 137) + 1435 \leq 0. \]

This is clearly false when \( n \geq 137 \), and otherwise, since \( k \leq 23 \), it implies \( 23n \leq 1716 \), which is false also. The lemma is now established.

The remaining possibilities for our minimal counterexample \( G \) can now be polished off with three relatively short lemmas.

**Lemma 13.** Let \( H \) be a graph of order \( n \) having an \((n - 2)\)-cycle \( C \), such that if \( \{a, b\} = V(H - C) \) then \( ab \in E(H) \) and \( d(a) + d(b) \geq n - 1 \). Then \( H \) has an \((n - 1)\)-cycle or it is bipartite.

**Proof.** Suppose \( H \) has no \((n - 1)\)-cycle. Let \( A = \Gamma(a) \cap C \) and \( B = \Gamma(b) \cap C \). In the notation of the proof of Lemma 12, we have \( A \cap A^+ = B \cap B^+ = A^+ \cap B = \emptyset \). In particular \(|A|, |B| \leq (n - 2)/2 \). But \(|A| + |B| \geq n - 3 \). If \( n \) is odd this implies \(|A| = |B| = (n - 3)/2 \). Thus \( A \) consists of an alternating sequence of vertices, there being a “gap” of two consecutive vertices not in \( A \). The same applies to \( B \). No vertex of \( A \) can lie in \( B \)'s gap because \( A^+ \cap B = B^+ \cap A = \emptyset \). So the gaps coincide, which again contradicts \( A^+ \cap B = B^+ \cap A = \emptyset \).

So \( n \) must be even. Colour the vertices of \( C \) alternately red and blue. We may assume \(|A| \geq |B| \), so \(|A| = (n - 2)/2 \) and \( a \) is joined exactly to every red vertex. Now \( B \cap A^+ = \emptyset \), so \( b \) is joined to all the blue vertices (except one). Either \( H \) is bipartite or there is a chord in \( C \) of length \( 2t < n/2 \). If \( t = 1 \) an \((n - 1)\)-cycle is easily found containing \( ab \). If \( t \geq 2 \) the cycle can be labelled \( v_1, \ldots, v_{n-2} \) so that the chord is \( v_2 v_{2t+2} \) and neither \( v_1 \) nor \( v_{2t} \) is the exceptional vertex joined neither to \( a \) nor \( b \). But then \( v_1 c d v_2 v_{2t-1} \cdots v_2 v_{2t+2} \cdots v_{n-2} \), where \( \{c, d\} = \{a, b\} \), is an \((n - 1)\)-cycle. This final contradiction proves the lemma.

**Lemma 14.** Suppose \( G \) is a minimal counterexample to Theorem 1, containing an edge \( ab \) with \( d(a) + d(b) \geq n - 3 \). Suppose moreover that the subgraph spanned by \( V(G) - \{a, b\} \) is both Hamiltonian and non-bipartite. Then \( G \) has an \((n - 1)\)-cycle.

**Proof.** Let us assume that \( G \) has no \((n - 1)\)-cycle. Let \( C \) be an \((n - 2)\)-cycle in \( G \) - \{a, b\}. Following the notation of the proof of Lemma 13, \( A \cap A^+ = B \cap B^+ = A^+ \cap B = B^+ \cap A = \emptyset \). Lemma 13 itself allows us to assume \( n - 5 \leq |A| + |B| \leq n - 4 \). If \( c \in A \cap B \) then none of the four vertices, two on each side of \( c \), is in \( A \cup B \). Thus
\[
 n - 2 \geq 3 |A \cap B| + 2 + |A \setminus B| + |B \setminus A| = |A \cap B| + 2 + |A| + |B|
\]
so \(|A \cap B| \leq 1\). If \(|A \cap B| = 1\) there is precisely one vertex in \(A \cap B\), a “gap” of two on either side, and the remainder are alternately in \(A\) or \(B\). If \(|A \cap B| = 0\) there are at most three vertices not in \(A \cup B\), and \(C\) comprises at most three intervals of alternating \(A\) and \(B\) vertices separated by gaps.

Let \(R\) consist of the gap vertices together with the endvertices of the intervals and the vertex in \(A \cap B\) if it exists. Then \(|R| \leq 9\). Now \(A \setminus R\) and \(B \setminus R\) must be independent sets, for otherwise we would obtain an \((n-1)\)-cycle precisely as was done at the end of the proof of Lemma 13. However, \(G[C]\) contains an \((n-3)\)-cycle \(D\) since \(e(G[C]) \geq e(G) - (n - 3) \geq ((n - 2)^2/4) - (n - 2) + 59\) and \(G[C]\) is not bipartite. If some edge of \(D\) joins \(A \setminus R\) to \(B \setminus R\) it can be replaced in \(D\) by a path of length three via \(ab\). But otherwise \((A \cup B) \setminus R\) contains no edge of \(D\), so every edge of \(D\) is incident with \(R\). Hence \(n - 3 = |D| \leq 2 |R| \leq 18\), contradicting the fact that \(n \geq 115\).

**Lemma 15.** Suppose \(G\) is a minimal counterexample to Theorem 1, containing an edge \(ab\) with \(d(a) + d(b) \geq n - 3\). Suppose moreover that the subgraph spanned by \(V(G) - \{a, b\}\) is both Hamiltonian and bipartite. Then \(G\) has an \((n-1)\)-cycle.

**Proof.** Let \(C\) be an \((n-2)\)-cycle in \(G - \{a, b\}\). Since \(G[C]\) is bipartite, \(|C|\) is even, and so is \(n\). Suppose \(G\) has no \((n-1)\)-cycle. Let \(A\) and \(B\) be defined as in the proofs of the previous two lemmas. Colour the vertices of \(G[C]\) red and blue. Now \(|A| + |B| \geq n - 5\), and assuming \(|A| \geq |B|\) we have \(|A| \geq n/2 - 2\). If \(A\) comprises only, say, red vertices, then it contains all except at most one. Since \(A^+ \cap B = \emptyset\), the set \(B\) consists entirely of blue vertices, contradicting the fact that \(G\) itself is not bipartite.

Otherwise \(|A| = n/2 - 2\) and, since \(A^+ \cap A = \emptyset\), we see that \(A\) consists of precisely two “intervals” of alternate vertices, one red interval and one blue interval; the intervals are separated by “gaps” of two adjacent vertices. Now \(A^+ \cap B = \emptyset\). So \(B\) cannot contain a vertex in a gap, and moreover if both intervals of \(A\) have more than one vertex then \(|B| \leq |A| - 2\) so \(|A| + |B| \leq n - 6\), a contradiction.

The only case which remains is that in which \(A = \{v_1, v_4, v_6, \ldots, v_{n-4}\}\) and \(B = \{v_2, v_5, v_7, \ldots, v_{n-5}\}\), for some labelling \(v_1v_2\ldots v_{n-2}\) of \(C\). Let \(d' \in B - \{v_1\}\). Notice that \(G - \{d', b\}\) is Hamiltonian (the place of \(d'\) in \(C\) being taken by \(a\)) and non-bipartite (since \(av_1v_2v_4\) is a 5-cycle). Therefore we are done by Lemma 14 if \(d(a') + d(b') \geq n - 3\). So we may assume that \(d(a') + d(b') \leq n - 4\), which is to say \(d(a') \leq n/2 - 2\). Likewise, if \(b' \in A - \{v_1, v_4, v_{n-4}\}\) then \(G - \{a, b'\}\) is Hamiltonian and bipartite, so by our earlier argument \(G\) has an \((n-1)\)-cycle unless \(d(a) + d(b') \leq n - 3\), which is to say \(d(b') \leq n/2 - 2\).
Now each vertex of $C$ has at most $n/2 - 1$ neighbours in $G[C]$ since $G[C]$ is bipartite. Moreover each vertex in $\{v_5, v_6, \ldots, v_{n-5}\}$ has at most $n/2 - 3$ neighbours in $G[C]$. Therefore

$$e(G) \leq d(a) + d(b) - 1 + e(G[C])$$
$$\leq n - 4 + (n - 2)^2/4 - (n - 9) = \lceil n^2/4 \rceil - n + 6,$$

contradicting $G$ being a counterexample to Theorem 1.

Proof of Theorem 1. Suppose the theorem is false, and let $G$ be a counterexample of minimal order. By Lemma 5, $G$ has an $(n-1)$-cycle or an $n$-cycle. Now if $G$ has an $n$-cycle, Lemmas 12, 14 and 15 imply that $G$ has an $(n-1)$-cycle. Therefore in every case $G$ must have an $(n-1)$-cycle. But this directly contradicts Lemma 11. Therefore our proof is complete.

6. CONCLUDING REMARKS

It is probable that in order to reduce $\lceil n^2/4 \rceil - n + 59$ to $\lceil n^2/4 \rceil - n + 5$, which we expect to be the correct value, our approach would need to be augmented by a case analysis whose length would be out of proportion to its value. However it is only Lemma 12 which would need serious attention; those occasional other places where we made use of the assumption $n \geq 115$ can be modified without too much difficulty to cope with small $n$.

A stronger condition than (weak) pancyclicity is that a graph have a cycle of each size whose vertex sets are nested: that is, a cycle of length $l+1$ can be found among $l+1$ vertices $l$ of which span an $l$-cycle, and so on. This property will not hold if the graph can be made bipartite by the removal of a single vertex; however, what is the smallest $d$ such that non-bipartite graphs satisfying the conditions of our theorem contain a set of cycles for which the vertex sets of successive cycles differ in at most $d$ vertices?

ACKNOWLEDGMENT

We thank the referee for a careful reading and thoughtful comments.

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