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On a rigorous interpretation of the quantum Schrödinger–Langevin operator in bounded domains with applications

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ABSTRACT

In this paper we make it mathematically rigorous the formulation of the following quantum Schrödinger–Langevin nonlinear operator for the wavefunction

$$\mathcal{A}_{QSL} = i\hbar\partial_t + \frac{\hbar^2}{2m}\Delta_x - \lambda\big(S_\psi - \langle S_\psi\rangle\big) - \Theta_\hbar[n_\psi, J_\psi]$$

in bounded domains via its mild interpretation. The *a priori* ambiguity caused by the presence of the multi-valued potential λS_ψ, proportional to the argument of the complex-valued wavefunction

$$\psi = |\psi|\exp\left\{\frac{i}{\hbar}S_\psi\right\},$$

is circumvented by subtracting its positional expectation value,

$$\langle S_\psi\rangle(t) := \int_\Omega S_\psi(t,x)n_\psi(t,x)\,dx,$$

as motivated in the original derivation (Kostin, 1972 [45]). The problem to be solved in order to find S_ψ is mostly deduced from the modulus-argument decomposition of ψ and dealt with much like in Guerrero et al. (2010) [37]. Here $\hbar$ is the (reduced) Planck constant, m is the particle mass, λ is a friction coefficient, $n_\psi = |\psi|^2$ is the local probability density, $J_\psi = \frac{\hbar}{m}\operatorname{Im}(\overline{\psi}\nabla_x\psi)$ denotes the electric current density, and $\Theta_\hbar$ is a general operator (eventually nonlinear) that only depends upon the macroscopic observables n_ψ and J_ψ. In this framework, we show local well-posedness of the initial-boundary value problem associated with the Schrödinger–Langevin operator $\mathcal{A}_{QSL}$ in bounded domains. In particular, all of our results apply to the analysis of the well-known Kostin equation derived in Kostin (1972) [45] and of the Schrödinger–Langevin equation with Poisson coupling and enthalpy dependence (Jüngel et al., 2002 [41]).

1. Introduction, setting of the problem and main result

Dissipative theories are nowadays at the basis of current multidisciplinary research in quantum-mechanical motion at all levels of description (see [39] for a recent review). As a matter of fact, a great deal of nonlinear models of Schrödinger, Wigner, Heisenberg, and hydrodynamic type have proliferated in the literature, each of them incorporating different

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quantum corrected mechanisms accounting for dissipative and/or diffusive behaviors such as quantum decoherence and entanglement, anomalous diffusion, quantum Brownian motion or nonlinear frictional couplings, even the prevention of some other genuinely quantum problems such as Bloch oscillations or Anderson localization [24]. Among these models we can remark the quantum Fokker–Planck system [6,7,17,18,26,27,55], the viscous quantum hydrodynamic model [40,42,43], the quantum Smoluchowski equation [3,4,54], the logarithmic Schrödinger equation [10,20–22,25,36–38,47,50,51], the Doebner–Goldin family of nonlinear Schrödinger equations driven by diffusion currents [28,58], the modular Schrödinger equation [8,31], and many other nonlinear modifications of the Schrödinger Hamiltonian (Caldirola–Kanai model, quantum potential approach, etc.) [35,46,57].

The general physical background accommodating this sort of models is that of quantum open systems [23], consisting of a quantum particle ensemble (the system) interacting dissipatively with a local thermal environment (the reservoir), which in the simplest case can be thought of as an infinite set of harmonic oscillators in thermal equilibrium whose degrees of freedom are linearly coupled to those of the system of interest [29]. Then, after taking partial trace with respect to the reservoir degrees of freedom, a reduced description is obtained for the evolution of the particle ensemble under observation.

This is shown to be a suitable framework to analyze quantum Brownian motion. We are especially interested in the quantum version of the well-known stochastic Langevin equation (in the wavefunction picture), which was first derived by Kostin [45] as the following nonlinear (stochastic) Schrödinger model for the motion of a quantum particle immersed in a heat bath:

$$i\partial_t\psi = \left(-\frac{1}{2}\Delta_x + V\right)\psi + \lambda V_L\psi + V_R\psi, \tag{1}$$

$$V_L(\psi) = S_\psi - \langle S_\psi\rangle \tag{2}$$

(in a unit system in which $\hbar = m = 1$ hereafter, and where quantum states are also normalized to unity: $\|\psi\|^2_{L^2(\Omega)} = \int_\Omega n_\psi\,dx = 1$), where $\psi = \psi(t,x)$ is the complex-valued wavefunction, λ is the friction parameter (that sets the strength of the coupling between the system and the reservoir), $V = V(t,x)$ is an external potential, $V_R = V_R(t)$ is a random time-dependent potential that represents the fluctuational interaction of the quantum system with the surrounding environment, and where $V_L = V_L(\psi(t,x))$ is the nonlinear frictional term, which is proportional to $S_\psi - \langle S_\psi\rangle$. Here, S_ψ denotes any of the infinitely many argument functions associated with ψ satisfying

$$\psi(t,x) = \left|\psi(t,x)\right|e^{iS_\psi(t,x)}, \tag{3}$$

and $\langle S_\psi\rangle(t) := \int_\Omega S_\psi(t,x)n_\psi(t,x)\,dx$ describes the quantum expectation value of S_ψ, where $n_\psi = |\psi|^2$ stands for the local position density. Notice that Nelson's "stochastic quantization" [52] can be applied to obtain Eqs. (1)–(2) except for the action of V_R (see [56,61]), which provides a frame in which fluctuations might be neglected as well as gives rise to a deterministic Schrödinger–Langevin type equation which constitutes the main objective of our work.

On the other hand, various linear models related to quantum Brownian motion are also well known from previous literature. Some of them are based on non-stochastic equations (stemming from a master equation approach), as those dealt with in [16,26,27], while others are founded on quantum stochastic equations. Particularly, the generalized (linear) Schrödinger–Langevin picture introduced by van Kampen in [60],

$$i\partial_t\psi = \left\{H - i\lambda\big(U + i\mathcal{F}(t)\big)\right\}\psi, \tag{4}$$

where H denotes the system Hamiltonian, λ the coupling parameter, U is a linear Hermitian operator representing dissipation, and where $\mathcal{F}(t)$ is a zero-mean random operator accounting for fluctuating contributions, has been dealt with from a stochastic viewpoint in [13–15] (see also references therein), where the authors showed that in an adequate approach it is possible to go beyond the Markovian description under a completely positive evolution determined by a quantum dynamical semigroup. Among these models we can highlight for their mathematical relevance (in the sense that positivity and/or complete positivity is preserved along the evolution for the density matrix operator) those pertaining to the Kossakowski–Lindblad class [44,49,34,1,30,9,33].

It should be also observed that the analysis of Eqs. (1)–(2) cannot be compared to that of Eq. (4) because of the deep differences underlying these two Langevin approaches for the wavefunction. Indeed, van Kampen's scheme is stochastic in essence, whereas Kostin's equation is interpreted in a deterministic manner. Besides, the expectation values associated with the latter satisfy the classical frictional equations, a feature that is not guaranteed to be fulfilled by the former. Therefore, Eqs. (1)–(2) are shown to describe random fluctuations, dissipation, and probability preservation in an independent way. On the contrary, U and $\mathcal{F}(t)$ in Eq. (4) are typically constrained by the fluctuation–dissipation theorem (see for instance [13] and [60]). Anyhow, the nonlinear character exhibited by Eqs. (1)–(2) (which sets the most important separation with respect to other approaches) requires, at variance with Eq. (4), the specific use of genuine mathematical tools (nonlinear Sobolev estimates, fixed-point arguments based on the concept of mild solution, elliptic regularization, trace properties on the boundary, etc.) for the treatment of nonlinear potential operators, say S_ψ.

Our equation of interest in this paper has the general structure $\mathcal{A}_{QSL}[\psi] = 0$ or, in other words,

$$i\partial_t\psi = \left(-\frac{1}{2}\Delta_x + V\right)\psi + \lambda V_L\psi + \Theta[n_\psi, J_\psi]\psi, \tag{5}$$

$$V_L(\psi) = S_\psi - \langle S_\psi\rangle, \tag{6}$$

where Θ stands for an arbitrary self-consistent interaction (Poissonian or thermodynamical, for example) depending upon the wavefunction through its associated observables, say the probability and current densities given by $n_\psi = |\psi|^2$ and $J_\psi = \mathrm{Im}(\overline{\psi}\nabla_x\psi)$, respectively.

The expectation value of S_ψ, that in principle does not contain any physical information, is typically removed from V_L by means of the following gauge transformation

$$\psi \mapsto \psi e^{i\nu(t)}, \quad \nu(t) = -\int_0^t e^{-\lambda(t-\tau)}\langle S_\psi\rangle(\tau)\,d\tau.$$

This is the way in which the simplest form of Kostin's equation, given by

$$i\partial_t\psi = \left(-\frac{1}{2}\Delta_x + V\right)\psi + \lambda S_\psi\psi, \tag{7}$$

comes up and turns out to dissipate energy from the system to the thermal bath [46].

The main difficulty in writing Eq. (7) is of course the ambiguity induced by the multi-valued nature of the function S_ψ, that is, associated with each wavefunction ψ there exist infinitely many functions S_ψ that fulfill the polar decomposition $\psi = |\psi|e^{iS_\psi}$. Thus, even taking into account the well-known fact that quantum mechanics is invariant by changes of global phase (i.e. no matter the chosen S_ψ be, the physics of the system remains unaltered), the questions that immediately arise are:

What is the mathematical sense of Eq. (7)?
How its analysis can be made rigorous?
What is the appropriate functional setting in order that all quantities are well defined?

In [41], the authors overcome this ambiguity by imposing the existence of an argument function S_0 associated with the initial datum ψ_0, and solving the following boundary value problem at any time:

$$\Delta_x S_\psi = \nabla_x \cdot \left\{\mathrm{Im}\left(\frac{\nabla_x\psi}{\psi}\right)\right\} \quad \text{in } \Omega, \tag{8}$$

$$S_\psi(t) = S_B \quad \text{in } \partial\Omega. \tag{9}$$

This system stems from the Madelung decomposition (3), when assumed that $S_B := S_0|_{\partial\Omega}$. Under this interpretation of the argument S_ψ, Eq. (7) is shown to have a unique solution 'separated from zero'. Nonetheless, this criterion still proves somehow unsatisfactory given that Eq. (7) turns S_0-dependent, in the sense that the equation is obviously changed with the initial datum. For example, take $S_0' = S_0 + 2k\pi$, with arbitrary $k \in \mathbb{Z}$, instead of S_0, so that $S_B' = S_0'|_{\partial\Omega} \neq S_B$. In a nutshell, a particular choice of the initial argument S_0 produces an alteration in the definition of the nonlinear term in Eq. (7).

On the other hand, along with the difficulties in the rigorous treatment of Eq. (7) already discussed, another inconvenience makes this model inappropriate from a physical viewpoint. It does consist in the fact that Eq. (7) is not invariant by constant (and global) changes of phase, i.e. if ψ is a solution of Eq. (7), there does not exist $0 \neq \nu \in \mathbb{R}$ such that $\phi = e^{i\nu}\psi$ is also a solution, given that $S_\phi = S_\psi + \nu \neq S_\psi$.

This paper aims to elucidate all of these issues. In particular, it is desirable to find a model that, on one hand, keeps the same observable behavior than (7), but on the other hand removes the inconsistencies concerning the nonlinear potential S_ψ. To this aim, we may take advantage of the following simple property: Given a wavefunction ψ such that both S and S' satisfy the Madelung relation (3), then $S - \langle S\rangle = S' - \langle S'\rangle$ and the mapping $\psi \mapsto V_L(\psi)$ is now univoquely determined. Furthermore, the formulation (5) (with V_L instead of just S_ψ) does enjoy the invariance by change of (constant and global) phase since $V_L(\psi) = V_L(\psi e^{i\nu})$ for all $\nu \in \mathbb{R}$.

Some authors have already analyzed different aspects of the (nonlinear) Schrödinger–Langevin equation, mainly from a formal point of view in the perspective of PDEs. We remark [12], in which the global existence of Gaussian solutions in the harmonic oscillator framework was established for the particular case of Kostin's equation, as well as the absence of L^2 solitary waves in the free-particle regime; [48], where the semiclassical limit of the Kostin–Poisson system is performed in the whole space; and [59], where the stability of stationary solutions was studied. Also, in [5] and [41] the Schrödinger–Langevin equation is dealt with as an auxiliary problem in analyzing the well-posedness of the associated quantum hydrodynamic system. In spite of that, there still remains an important lack of mathematical sense in the very core of the formulation of the problem, that is intended to be clarified throughout this paper.

Our main theorem here is the following.

Theorem 1.1. *Let $\delta > 0$, $\Omega \subset \mathbb{R}^d$ (with $1 \leqslant d \leqslant 3$), ψ_0 and ψ_B satisfying assumptions* (H1)–(H4)*. Then, there exist $T = T(\delta, \Omega, \psi_0, \psi_B, \Theta) > 0$ and a unique strong solution in $[0, T]$ to Eq.* (5) *subject to the initial-boundary conditions $\psi(t = 0) = \psi_0$ and $\psi|_{\partial\Omega} = \psi_B$.*

We structure the paper as follows: Section 2 is devoted to the presentation of the problem and the introduction of the appropriate function spaces in which it can be tackled successfully, as well as the main *a priori* estimates. In Section 3 we demonstrate our main result (cf. Theorem 1.1), which shows that the initial-boundary value problem associated with the Schrödinger–Langevin operator $\mathcal{A}_{QSL}$ is locally well-posed in bounded domains without the need of any prescription of the initial argument function. Finally, Section 4 is concerned with the application of this theory to the Schrödinger–Langevin equation with enthalpy and Poisson coupling.

2. Functional framework and *a priori* estimates

We open this section by introducing some notational conventions that will be useful in the sequel. Let Ω be a 'sufficiently smooth' (to be specified below) bounded domain, H a subset of $L^2(\Omega)$ and δ a positive constant, and denote by

$$H_\delta = \{\varphi \in H\colon |\varphi| > \delta \text{ a.e. } x \in \Omega\}$$

the space of functions belonging to H that are separated a minimum distance δ from zero. Given $T > 0$, we also denote

$$X^T = C([0, T], H^2(\Omega)) \cap C^1([0, T], L^2(\Omega))$$

and

$$X_\delta^T = \{\varphi \in X^T\colon |\varphi| > \delta \text{ a.e. } x \in \Omega,\ \forall 0 \leqslant t \leqslant T\},$$

where $H^k(\Omega)$ stands for the usual Sobolev space $W^{k,2}(\Omega)$.

If a pure quantum state ψ is assumed to be decomposable in Madelung's modulus-argument form, that is to say $\psi = |\psi| \exp\{iS_\psi\}$, then the identity

$$\nabla_x V_L = \operatorname{Im}\left(\frac{\nabla_x \psi}{\psi}\right) \tag{10}$$

formally holds for the gradient velocity field $u = \nabla_x V_L = \nabla_x S_\psi$, which allows for a clear connection between the fluid and the Schrödinger–Madelung descriptions of quantum mechanics. The solvability of Eq. (10) obviously requires the irrotationality of the field, that can be deduced from Schwartz's lemma if Ω is assumed to be simply connected. This hypothesis on the domain is of crucial importance for our purposes, since otherwise one is oblied to admit jump discontinuities of S_ψ. Under this assumption, the existence of a unique (up to an additive constant) solution V_L to Eq. (10) for any given $\psi \in H_\delta^2$ satisfying some additional regularity can be claimed [37]. Thus, a family of countably many functions $S_\psi^k \in H^2(\Omega)$, $k \in \mathbb{Z}$, exists such that the decomposition (3) is fulfilled along with the identity

$$S_\psi^k - S_\psi^l = 2\pi(k - l), \quad k, l \in \mathbb{Z}. \tag{11}$$

Furthermore, for any fixed $\mu \in \mathbb{R}$, the mapping $\psi \mapsto S_\psi$ is continuous from H_δ^2 to the subset $S^\mu = \{S \in H^2(\Omega)\colon \int_\Omega S\,dx = \mu\}$. Indeed, it is this property that makes it possible to find a continuous-in-time argument for any given $\psi \in X_\delta^T$ solving a general Schrödinger-like equation. As a matter of fact, the main result in [37] establishes that for any strong solution ψ in $[0, T]$ to the Schrödinger equation

$$i\partial_t \psi = -\frac{1}{2}\Delta_x \psi + \Theta[n, J]\psi, \tag{12}$$

with $\Theta : (H_{\delta^2}^2(\Omega)) \times (H^1(\Omega))^3 \to L^2(\Omega)$ being any continuous mapping, there exists a family of arguments $\{S_\psi^k\}_{k\in\mathbb{Z}} \subset X^T$ fulfilling Eqs. (3) and (11) a.e. Ω, for all $t \in [0, T]$. This is possible in accordance with the evolution law satisfied by S_ψ, stemming from Eq. (12), which can be described in terms of the only observables n_ψ and J_ψ. The required assumptions here are (see [37], where the hypotheses (H1)–(H3) were already justified as key ingredients of our analysis)

(H1) $\Omega \subset \mathbb{R}^d$ is a simply-connected, C^2 bounded domain.
(H2) $\psi_0 \in H^2(\Omega)$, $\psi_B \in H^{3/2}(\partial\Omega)$, and $\psi_0 = \psi_B$ in $\partial\Omega$.
(H3) There exists $\delta > 0$ such that

$$\text{ess-inf}\{|\psi_0(x)|\colon x \in \Omega\} > \delta, \qquad \text{ess-inf}\{|\psi_B(x)|\colon x \in \partial\Omega\} > \delta.$$

(H4) The operator $\Theta : H_\delta^2 \to H^2(\Omega)$ is locally Lipschitz continuous and there exists $\Theta_B \in H^{3/2}(\partial\Omega)$ such that $\Theta(\psi)|_{\partial\Omega} = \Theta_B$ when $\psi|_{\partial\Omega} = \psi_B$.

Note that the Rellich–Kondrachov compactness theorem (see for instance [11]), along with the regularity properties of ψ_0 and ψ_B, permit to consider the condition $\psi_0 = \psi_B$ stated in (H2) not only in the usual sense, $\psi_0 - \psi_B \in H_0^1(\Omega)$, but also as a pointwise identity in $\partial\Omega$.

The following two results were proved in [37]. We include them here for the sake of self-consistency, as they will play a relevant role in our subsequent study. In the sequel, the dependence of the norms upon the domain Ω will be also removed for clarity.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a simply connected, Lipschitz-continuous bounded domain and $0 \leqslant k \in \mathbb{Z}$. Then the following assertions hold.*

(i) *For all complex functions $\psi \in H^k(\Omega)$ such that*

$$\frac{\nabla\psi}{\psi} \in \left(H^{k-1}(\Omega)\right)^d, \qquad \frac{(\nabla\otimes\nabla)\psi}{\psi}, \frac{\nabla\psi}{\psi}\otimes\frac{\nabla\psi}{\psi} \in \left(H^{k-2}(\Omega)\right)^{d^2},$$

there exists a unique (up to an additive constant) function $S \in H^k(\Omega)$ that solves Eq. (10). Besides, given $\mu \in \mathbb{R}$ there exists a unique $S \in H^k(\Omega)$ solution to Eq. (10) such that $\int_\Omega S\,dx = \mu$, and also a unique $\beta_\mu \in [0, 2\pi)$ such that the family

$$S_l := S + \beta_\mu + 2\pi l, \quad l \in \mathbb{Z}, \tag{13}$$

satisfies (3)–(11).

(ii) *Under the hypotheses of statement (i), there exists $C > 0$ such that*

$$\|S_\psi - S_\phi\|_{H^k} \leqslant C\left\|\mathrm{Im}\left(\frac{\nabla\psi}{\psi}\right) - \mathrm{Im}\left(\frac{\nabla\phi}{\phi}\right)\right\|_{H^{k-1}}$$

for all S_ψ, S_ϕ solutions to Eq. (10) (associated with ψ and ϕ, respectively) satisfying $\int_\Omega S_\psi\,dx = \int_\Omega S_\phi\,dx$.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^d$ be a C^1 bounded domain, $\delta > 0$ and $\psi \in H^2_\delta$. Then, for $n = |\psi|^2$ and $J = \mathrm{Im}(\overline{\psi}\nabla\psi)$, the following identity holds*

$$\overline{\psi}\nabla\psi = \frac{1}{2}\nabla_x n + iJ. \tag{14}$$

As a consequence, $\sqrt{n} \in H^2_\delta$, $n \in H^2_{\delta^2}$, $J, \nabla n/n, J/n \in (H^1(\Omega))^d$, and the mappings

$$\psi \mapsto \sqrt{n}, n, J, \frac{\nabla n}{n}, \frac{J}{n}$$

are locally Lipschitz continuous from H^2_δ onto the corresponding functional space in each case.

We start by establishing some technical issues concerning the derivation of *a priori* estimates. An appropriate combination of Lemmata 2.1 and 2.2 leads to establish the existence of a well-defined and Lipschitz continuous operator that yields the correct frictional term in the Schrödinger–Langevin picture.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^d$ (with $1 \leqslant d \leqslant 3$) be a simply-connected, Lipschitz-continuous bounded domain. Let also δ and T be positive constants. Then, there exists a mapping $V_L : H^2_\delta \to H^2(\Omega)$ such that the following properties hold true:*

(i) *For all $\psi \in H^2_\delta$, $V_L(\psi)$ is a solution to Eq. (10) coupled with (6).*

(ii) *For all $M > 0$, there exists $C = C(\delta, \Omega, M) > 0$ such that*

$$\left\|V_L(\psi) - V_L(\phi)\right\|_{H^2} \leqslant C\|\psi - \phi\|_{H^2},$$
$$\left\|V_L(\psi) - V_L(\phi)\right\|_{L^2} \leqslant C\|\psi - \phi\|_{L^2},$$

for all $\psi, \phi \in H^2_\delta$ satisfying $\|\psi\|_{H^2}, \|\phi\|_{H^2} \leqslant M$.

Besides, if $S_\psi \in H^2(\Omega)$ is any argument of ψ, then $V_L(\psi) = S_\psi - \langle S_\psi\rangle$. In particular, it is fulfilled that $V_L(\psi) = V_L(e^{i\nu}\psi)$ for all $\nu \in \mathbb{R}$.

Proof. Lemma 2.2 claims that $\nabla_x\psi/\psi \in H^1(\Omega)$ for any $\psi \in H^2_\delta$, so that Lemma 2.1 can be applied with $\mu = 0$ and $k = 2$ to get a unique $S_\psi \in H^2(\Omega)$ such that

$$\nabla_x S_\psi = \mathrm{Im}\left(\frac{\nabla_x\psi}{\psi}\right) \quad \text{with} \int\limits_\Omega S_\psi(x)\,dx = 0.$$

We can now define

$$V_L(\psi)(x) := S_\psi(x) - \langle S_\psi \rangle \quad \text{a.e. } x \in \Omega,$$

in such a way that $V_L(\psi) \in H^2(\Omega)$ and satisfies (i).

To prove (ii), let $M > 0$ and denote

$$M_S = \sup\bigl\{ \|S_\psi\|_{H^2} \colon \psi \in H^2,\ \|\psi\|_{H^2} \leqslant M \bigr\} < \infty.$$

Also denote by $K(\Omega, M)$ the maximum among the constants stemming (i) from the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, (ii) from Lemma 2.1 (with $k = 0, 2$), and (iii) from Lemma 2.2 (applied to $\mathrm{Im}(\nabla_x \psi/\psi) = J_\psi/n_\psi$). Then, for all $\psi, \phi \in H^2_\delta$ with $\|\psi\|_{H^2}, \|\varphi\|_{H^2} \leqslant M$, we can estimate

$$\begin{aligned}
\bigl|\langle S_\psi \rangle - \langle S_\phi \rangle\bigr| &\leqslant \int_\Omega \bigl| |\psi(x)|^2 S_\psi(x) - |\phi(x)|^2 S_\phi(x) \bigr|\, dx \\
&\leqslant \int_\Omega \bigl\{ |\psi(x)|^2 \bigl|S_\psi(x) - S_\phi(x)\bigr| + \bigl| |\psi(x)|^2 - |\phi(x)|^2 \bigr| \bigl|S_\phi(x)\bigr| \bigr\}\, dx \\
&\leqslant \|\psi\|^2_{L^\infty} \|S_\psi - S_\phi\|_{L^2} + \bigl(\|\psi\|_{L^\infty} + \|\phi\|_{L^\infty}\bigr) \|\psi - \phi\|_{L^2} \|S_\phi\|_{L^\infty} \\
&\leqslant C\|\psi - \phi\|_{L^2},
\end{aligned}$$

for which we used Lemmata 2.1 and 2.2 Hence, we easily find

$$\bigl\|V_L(\psi) - V_L(\phi)\bigr\| \leqslant \|S_\psi - S_\phi\| + \bigl|\langle S_\psi \rangle - \langle S_\phi \rangle\bigr|, \tag{15}$$

$\|\cdot\|$ standing for $\|\cdot\|_{H^2}$ or $\|\cdot\|_{L^2}$ indistinctively. In addition, if $S_\psi \in H^2(\Omega)$ is an argument of ψ, it is enough to differentiate the Madelung identity $\psi = |\psi| e^{iS_\psi}$ and take imaginary parts to get $\nabla_x S_\psi = \mathrm{Im}(\nabla_x \psi/\psi)$. Thus, if considering $\sigma := S_\psi - \langle S_\psi \rangle - V_L(\psi)$ it is clear that $\nabla_x \sigma = 0$ and $\langle \sigma \rangle = 0$. Consequently we deduce $\sigma \equiv 0$, or equivalently $V_L(\psi) = S_\psi - \langle S_\psi \rangle$. In particular, for any $\nu \in \mathbb{R}$ we notice that $S_\psi + \nu$ is an argument of $\psi e^{i\nu}$, and that

$$V_L\bigl(\psi e^{i\nu}\bigr) = S_\psi + \nu - \langle S_\psi \rangle - \langle \nu \rangle = S_\psi - \langle S_\psi \rangle = V_L(\psi).$$

This ends the proof. $\square$

Once the problem (5)–(6) is known to be well defined, the proof of our main result (cf. Theorem 1.1 above) is reached by means of a standard fixed point argument at the level of mild solutions, although our treatment of V_L actually does allow for strong solutions as will be seen later. In connecting both mild and strong pictures of our problem, we shall use a well-known property of nonhomogeneous Schrödinger equations that is stated below for the sake of completeness (the interested reader can find the proof in [19]).

Lemma 2.4. *Let X be a Hilbert space, A a self-adjoint and negative definite operator defined in $\mathcal{D}$, and $T > 0$. Then, for any $f \in C([0, T], X)$ and $g \in X$ there exists a unique solution of the problem*

$$\begin{aligned}
&\psi \in C\bigl([0, T], X\bigr) \cap C^1\bigl([0, T], \mathcal{D}'\bigr), \\
&i\partial_t \psi + \overline{A}\psi + f = 0, \\
&\psi(0) = g,
\end{aligned}$$

where $\mathcal{D}'$ holds for the dual space of $\mathcal{D}$ and $\overline{A}$ denotes the extension of A to X. Besides, $\psi \in C([0, T], X)$ is a solution to this problem if and only if the following integral equation

$$\psi(t) = e^{iAt} g + i \int_0^t e^{iA(t-\tau)} f(\tau)\, d\tau$$

is satisfied for all $t \in [0, T]$, where e^{iAt} is the group of isometries associated with the infinitesimal generator iA. Also, if $g \in \mathcal{D}$ and $f \in W^{1,1}((0, T), X)$ or $f \in L^1((0, T), \mathcal{D})$, then $\psi \in C([0, T], \mathcal{D}) \cap C^1([0, T], X)$.

In our setting we shall consider $X = L^2(\Omega)$ and $\mathcal{D} = H^2(\Omega) \cap H^1_0(\Omega)$ the domain of the elliptic operator $A = \frac{i}{2}\Delta_x$, which generates the uniparametric group of operators $\{e^{(i/2)\Delta_x t}\}_{t \in \mathbb{R}}$ in virtue of Stone's theorem (see for instance [53]). It is noticeable the fact that $e^{(i/2)\Delta_x t}$ defines for all $t \in \mathbb{R}$ an isometry in $L^2(\Omega)$ and in $\mathcal{D}$, but not in $H^2(\Omega)$. Indeed, this oblies us to deal with the terms of the equation restricted to the boundary. The first drawback in this analysis arises from the fact that we have no *a priori* information available on the behavior of V_L at $\partial\Omega$, so that the H^2-norm preservation property of $e^{(i/2)\Delta_x t}$ cannot be exploited any more. It is therefore required an estimate in $H^2(\Omega)$ for the elements of the group. To this aim we will make essential use of the following result, of paramount importance in the analysis of elliptic PDEs. We include here the guidelines of the proof for the sake of consistency.

Lemma 2.5 *(Elliptic regularization). Let $\Omega \subset \mathbb{R}^d$ be a C^2 bounded domain and $f \in L^2(\Omega)$. Then, the following assertions hold true.*

(i) *There exists a unique weak solution $u \in H_0^1(\Omega)$ of $-\Delta u + u = f$ in Ω. Furthermore, $u \in H^2(\Omega)$ and there exists a positive constant $C = C(\Omega)$ such that $\|u\|_{H^2} \leqslant C\|f\|_{L^2}$.*
(ii) *There exists a unique strong solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ of $-\Delta u = f$ in Ω and a positive constant $C = C(\Omega)$ such that $\|u\|_{H^2} \leqslant C\|f\|_{L^2}$.*

Proof. The proof of (i) can be found in [11] (Theorems IX. 21 and IX. 25). To prove (ii), we first note the existence of a unique weak solution $u \in H_0^1(\Omega)$ of $-\Delta u = f$ in virtue of the Lax–Milgram theorem. It is then clear that u particularly solves $-\Delta u + u = f + u$ weakly, so that $\|u\|_{H^2} \leqslant C\|f + u\|_{L^2}$ according to part (i) of the lemma. Using now Poincaré's inequality we find that $\|u\|_{H^2} \leqslant C(\|f\|_{L^2} + \|\nabla u\|_{L^2})$. Finally, multiplying $-\Delta u = f$ by u, integrating by parts and applying the Cauchy–Schwarz inequality yields $\|\nabla u\|_{L^2}^2 \leqslant C\|f\|_{L^2}\|\nabla u\|_{L^2}$. This ends the proof. □

We are now in conditions to estimate the (nontrivial) action of the group of Schrödinger operators $\{e^{(i/2)\Delta_x t}\}_{t\in\mathbb{R}}$ in $H^2(\Omega)$.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a C^2 bounded domain. Then, there exists a positive constant C depending only upon Ω such that*

$$\left\|e^{(i/2)\Delta_x t} g\right\|_{H^2} \leqslant C\|g\|_{H^2},$$

for all $g \in H^2(\Omega)$.

Proof. After selection of a continuous representative of g in $\overline{\Omega}$, we may obtain a unique harmonic function $\widetilde{g_B} \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\widetilde{g_B} - g \in \mathcal{D}$ (see [32]). Then, Lemma 2.5(ii) can be applied with $f = \Delta_x g$ to find

$$\|\widetilde{g_B}\|_{H^2} \leqslant K\|g\|_{H^2}. \tag{16}$$

Define now

$$\psi(t) := \widetilde{g_B} + \phi(t), \quad \text{with } \phi(t) = e^{(i/2)\Delta_x t} g - \widetilde{g_B},\ \forall t \in \mathbb{R}. \tag{17}$$

It is clear that $e^{(i/2)\Delta_x t}$ is isometric on $\mathcal{D}$ for all $t \in \mathbb{R}$, so that according to (16) we can write

$$\|\psi\|_{H^2} \leqslant C\|g\|_{H^2}, \quad \forall t \in \mathbb{R}. \tag{18}$$

To conclude, it suffices to observe that $\psi = u$, where $u(t) := e^{(i/2)\Delta_x t} g$ for all $t \in \mathbb{R}$. To this aim we notice that u is the unique solution to the following problem

$$\begin{cases} u \in C\big([0,T], L^2(\Omega)\big) \cap C^1\big([0,T], \mathcal{D}'\big), \\ i\partial_t u + \dfrac{1}{2}\overline{\Delta_x} u = 0, \\ u(0) = g, \end{cases} \tag{19}$$

where $\overline{\Delta_x}$ denotes the extension of the Laplace operator to $L^2(\Omega)$. In the same spirit, making use of (17) we may also claim that ϕ is the unique solution to

$$\begin{cases} \phi \in C\big([0,T], \mathcal{D}\big) \cap C^1\big([0,T], L^2(\Omega)\big), \\ i\partial_t \phi + \dfrac{1}{2}\Delta_x \phi = 0, \\ \phi(0) = g - \widetilde{g_B}. \end{cases}$$

Hence, $\psi \in C([0,T], \mathcal{D}) \cap C^1([0,T], L^2(\Omega)) \subset C([0,T], L^2(\Omega)) \cap C^1([0,T], \mathcal{D}')$ solves

$$i\partial_t \psi + \frac{1}{2}\overline{\Delta_x}\psi = i\partial_t \psi(\widetilde{g_B} + \phi) + \frac{1}{2}\overline{\Delta_x}(\widetilde{g_B} + \phi) = \frac{1}{2}\overline{\Delta_x}\widetilde{g_B} = 0.$$

Finally, as $\psi(0) = g$ and the solution to problem (19) is unique, we are led to $\psi = u$. Now, the inequality (18) allows to conclude the proof. □

3. Well-posedness of the Schrödinger–Langevin system: Existence of a unique local-in-time strong solution

This section is mainly devoted to the proof of Theorem 1.1. First of all, in virtue of the Rellich–Kondrachov compactness theorem we can select a continuous representative of $\psi_B = \psi_0|_{\partial\Omega}$, so that there exists a unique function $\widetilde{\psi_B} \in C^2(\Omega) \cap C(\overline{\Omega})$ that is harmonic in Ω and equals ψ_B in $\partial\Omega$. Similarly, thanks to the assumption (H4) we obtain $\widetilde{\Theta_B} \in C^2(\Omega) \cap C(\overline{\Omega})$, the unique harmonic extension to Ω of Θ_B, and define

$$Z(t) = \widetilde{\psi_B} + e^{\frac{i}{2}\Delta_x t}(\psi_0 - \widetilde{\psi_B}) - i\int_0^t e^{\frac{i}{2}\Delta_x(t-\tau)}\widetilde{\Theta_B}\widetilde{\psi_B}\,d\tau, \quad \forall t \in \mathbb{R}.$$

As the mapping $t \mapsto \widetilde{\Theta_B}\widetilde{\psi_B}$ is constant, thus belonging to $W^{1,1}(\mathbb{R}, L^2(\Omega))$, Lemma 2.4 can be applied to find that Z is the unique solution to the following linear Schrödinger problem

$$\begin{cases} Z \in C\big(\mathbb{R}, H^2(\Omega)\big) \cap C^1\big(\mathbb{R}, L^2(\Omega)\big), \\ i\partial_t Z + \dfrac{1}{2}\Delta_x Z + \widetilde{\Theta_B}\widetilde{\psi_B} = 0, \\ Z(0) = \psi_0, \\ Z(t)|_{\partial\Omega} = \psi_B, \quad t \in \mathbb{R}. \end{cases} \tag{20}$$

Given $r, T > 0$, $Z(t)$ now allows to define the following set

$$Y_{r,T} = \big\{\psi \in C\big([0,T], H^2(\Omega)\big)\colon \|\psi - Z\|_{H^2} \leqslant r,\ \psi(t)|_{\partial\Omega} = \widetilde{\psi_B},\ \forall t \in [0,T]\big\},$$

which enjoys the following properties.

Lemma 3.6. *Under the assumptions of Theorem* 1.1 *and for every* $r, T > 0$, $Y_{r,T}$ *is a complete metric space* (*with respect to the norm* $\|\cdot\|_{L^\infty([0,T],H^2)}$). *Furthermore, there exist positive constants* $r^\star$, $T^\star$ *and* C (*depending upon* δ, Ω, ψ_0, ψ_B) *such that*

(i) $\|Z\|_{L^\infty([0,T],H^2)} \leqslant C$,
(ii) $Y_{r,T} \subset C([0,T], H^2_\delta)$,

for any arbitrary $0 < r < r^\star$ *and* $0 < T < T^\star$.

Proof. We first observe that $Y_{r,T} \subset B_r(Z)$, where

$$B_r(Z) = \big\{\psi \in C\big([0,T], H^2(\Omega)\big)\colon \|\psi - Z\|_{L^\infty([0,T],H^2)} \leqslant r\big\}$$

is a closed subset of a Banach space, thus complete. This reduces the problem to checking that $Y_{r,T}$ is closed in $B_r(Z)$. To this aim, consider the operator $G : C([0,T], H^2(\Omega)) \to C([0,T], H^2(\Omega))$ defined as $G(\psi)(t) := \psi(t) - \widetilde{\psi_B}$, which is continuous and allows to rewrite $Y_{r,T}$ as

$$Y_{r,T} = B_r(Z) \cap G^{-1}\big(C\big([0,T], \mathcal{D}\big)\big).$$

Since $\mathcal{D}$ is closed in $H^2(\Omega)$, $Y_{r,T}$ is straightforwardly noticed to be closed, thus complete with respect to the norm $\|\cdot\|_{L^\infty([0,T],H^2)}$.

Consider now

$$m_0 = \text{ess-inf}\big\{\big|\psi_0(x)\big|\colon x \in \Omega\big\},$$
$$m_B = \text{ess-inf}\big\{\big|\psi_B(x)\big|\colon x \in \partial\Omega\big\},$$
$$m = \min\{m_0, m_B\}.$$

Since $m > \delta$, there exists $\varepsilon = \varepsilon(\delta, \Omega, \psi_0, \psi_B) > 0$ such that $\varepsilon < m - \delta$. Also define $r^\star = \varepsilon/(2K_1)$, K_1 denoting the Sobolev constant stemming from the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, and use the continuity of Z with respect to time to guarantee the existence of $T^\star = T^\star(\delta, \Omega, \psi_0, \psi_B) > 0$ fulfilling

$$\big\|Z(t) - \psi_0\big\|_{H^2} \leqslant r^\star, \quad \forall 0 \leqslant t < T^\star. \tag{21}$$

Then we can define $C = r^\star + \|\psi_0\|_{H^2}$ such that (i) holds.

To show (ii), it suffices to verify that

$$\big\|\psi(t) - \psi_0\big\|_{H^2} \leqslant \big\|\psi(t) - Z(t)\big\|_{H^2} + \|Z(t) - \psi_0\|_{H^2} < 2r^\star = \frac{\varepsilon}{K_1}$$

for all $t \in [0, T]$. Hence, $|\psi_0(x) - \psi(t,x)| \leqslant |\psi_0(x) - \psi(t,x)| < \varepsilon$ a.e. $x \in \Omega$ according to $\|\psi - \psi_0\|_{L^\infty([0,T],H^2)} \leqslant \varepsilon/K_1$ and the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. As a consequence, we finally deduce

$$\left|\psi(t,x)\right| > \left|\psi_0(x)\right| - \varepsilon > m_0 - m + \delta \geqslant \delta.$$

This concludes the proof. □

We close this section with the

Proof of Theorem 1.1. Take $r^\star, T^\star, C > 0$ the constants shed by Lemma 3.6 and fix $0 < r < r^\star$. For $M := r + 2C > 0$, we have that $\|\psi\|_{L^\infty([0,T],H^2)} \leqslant M$ for all $\psi \in Y_{r,T}$. Lemma 2.3 and the assumption (H4) yield $K_{V_L}(M), K_\Theta(M) > 0$ the Lipschitz constants for V_L and Θ, respectively. Also, define

$$M_{V_L} = \sup\left\{\left\|V_L(\psi)\right\|_{H^2}\colon \|\psi\|_{H^2} \leqslant M\right\},$$
$$M_\Theta = \sup\left\{\left\|\Theta(\psi)\right\|_{H^2}\colon \|\psi\|_{H^2} \leqslant M\right\}.$$

Furthermore, we shall denote $K = K(\Omega)$ various positive constants. Let then

$$T_1 = \frac{\min\{r, 1/2\}}{KM\{\lambda K M_{V_L} + M_\Theta(1+K^3)\}},$$
$$T_2 = \frac{1}{2K\{\lambda(M_{V_L} + K_{V_L}M) + M_\Theta + K_\Theta M\}},$$

and choose $T = \min\{T^\star, T_1, T_2\}$. We first construct a solution in $[0, T]$ to

$$\begin{cases} i\partial_t\psi = \left(-\frac{1}{2}\Delta_x + V\right)\psi + \lambda V_L\psi + \Theta[n_\psi, J_\psi]\psi, \\ \psi(0) = \psi_0, \quad \psi(t)|_{\partial\Omega} = \psi_B, \\ \nabla_x V_L = \operatorname{Im}(\nabla_x\psi/\psi), \\ \langle V_L\rangle = 0, \end{cases}$$

as the limit of a sequence of elements of $Y_{r,T}$. To this purpose we define

$$\psi^0 := Z, \qquad \psi^{k+1} := Z + \Gamma_{V_L}\left(\psi^k\right) + \Gamma_\Theta\left(\psi^k\right), \quad 0 \leqslant k \in \mathbb{Z}, \tag{22}$$

where

$$\Gamma_{V_L}(\psi)(t) := -i\lambda\int_0^t e^{\frac{i}{2}\Delta_x(t-\tau)}V_L(\psi)(\tau)\psi(\tau)\,d\tau, \tag{23}$$

$$\Gamma_\Theta(\psi)(t) := -i\int_0^t e^{\frac{i}{2}\Delta_x(t-\tau)}\Theta(\psi)(\tau)\psi(\tau)\,d\tau, \tag{24}$$

for any $\psi \in Y_{r,T}$. Our objective consists in proving that the sequence $\{\psi^k\}$ defined in (22) converges in $Y_{r,T}$. We split the proof into two steps.

Step 1. *For all $0 \leqslant k \in \mathbb{Z}$, $\psi^k \in Y_{r,T}$ and there exists $L > 0$ (independent of k) such that*

$$\left\|V_L\left(\psi_k(t)\right)\psi_k(t) - V_L\left(\psi_k(s)\right)\psi_k(s)\right\|_{L^2} \leqslant L|t-s|, \quad \forall t, s \in [0, T]. \tag{25}$$

Define

$$L := K(M_{V_L} + K_{V_L}M)L_\psi, \qquad L_\psi := M\left(\frac{1}{2} + K\left(K^2M_\Theta + \lambda M_{V_L} + M_\Theta\right)\right),$$

and follow an inductive procedure. First of all we notice that $\psi^0 = Z \in Y_{r,T}$ according to (20). Besides, it can be also deduced that $\psi^0 \in X_\delta^T$ is a strong solution of

$$i\partial_t\psi^0 = -\frac{1}{2}\Delta_x\psi^0 + \widetilde{\Theta_B}\widetilde{\psi_B}.$$

Thus, by estimating the harmonic extensions as in (16), we find

$$\left\|\partial_t\psi^0\right\|_{L^2}\leqslant\frac{1}{2}\left\|\Delta_x\psi^0\right\|_{L^2}+K\|\widetilde{\Theta_B}\|_{H^2}\|\widetilde{\psi_B}\|_{H^2}\leqslant L_\psi$$

uniformly in time, which entails that $t\mapsto\psi^0(t)$ is Lipschitz continuous (with Lipschitz constant L_ψ). This along with Lemma 2.3 leads to

$$\left\|V_L\big(\psi^0(t)\big)\psi^0(t)-V_L\big(\psi^0(s)\big)\psi^0(s)\right\|_{L^2}\leqslant K(M_{V_L}+K_{V_L}M)L_\psi|t-s| \tag{26}$$

for all $t,s\in[0,T]$, which ends the first step of the proof for ψ^0. Let now $k\in\mathbb{N}$ and assume that $\psi^k\in Y_{r,T}$ and satisfies (25), so that the mapping $t\mapsto V_L(\psi^k(t))\psi^k(t)$ is Lipschitz continuous and thus belongs to $W^{1,1}([0,T],L^2(\Omega))$. In addition, (H4) informs that $\Theta(\psi^k)\psi^k-\widetilde{\Theta_B}\widetilde{\psi_B}\in C([0,T],\mathcal{D})$. Then, an application of Lemma 2.4 reveals that $\Gamma^k=\Gamma_{V_L}(\psi^k)+\Gamma_\Theta(\psi^k)$ is the unique solution to the following problem

$$\begin{cases}\Gamma^k\in C\big([0,T],\mathcal{D}\big)\cap C^1\big([0,T],L^2(\Omega)\big),\\ i\partial_t\Gamma^k+\frac{1}{2}\Delta_x\Gamma^k=\big(\Theta\big(\psi^k\big)\psi^k-\widetilde{\Theta_B}\widetilde{\psi_B}\big)+\lambda V_L\big(\psi^k\big)\psi^k,\\ \Gamma^k(0)=0,\end{cases}$$

thus $\psi^{k+1}(t)|_{\partial\Omega}=\psi^0(t)|_{\partial\Omega}=\psi_B$ or, equivalently, $\psi^{k+1}(t)-\psi_B\in\mathcal{D}$ for all $t\in[0,T]$. We then use Proposition 2.1 to deduce

$$\begin{aligned}\left\|\Gamma_{V_L}\big(\psi^k(t)\big)\right\|_{H^2}&\leqslant\lambda\int_0^t\left\|e^{(i/2)\Delta_x(t-\tau)}V_L\big(\psi^k(\tau)\big)\psi^k(\tau)\right\|_{H^2}d\tau\\&\leqslant\lambda K^2MM_{V_L}t\end{aligned} \tag{27}$$

for all $t\in[0,T]$. On the other hand, we also have

$$\begin{aligned}\left\|\Gamma_\Theta\big(\psi^k(t)\big)\right\|_{H^2}&\leqslant\int_0^t\left\|\Theta\big(\psi^k(\tau)\big)\psi^k(\tau)-\widetilde{\Theta_B}\widetilde{\psi_B}\right\|_{H^2}d\tau\\&\leqslant KMM_\Theta\big(1+K^3\big)t.\end{aligned} \tag{28}$$

Then, we can write

$$\left\|\psi^{k+1}-Z\right\|_{L^\infty([0,T],H^2)}\leqslant KM\big\{\lambda KM_{V_L}+M_\Theta\big(1+K^3\big)\big\}T\leqslant r,$$

which leads to $\psi^{k+1}\in Y_{r,T}$. To show that (25) is fulfilled, it is enough to realize that $\psi^{k+1}\in X_\delta^T$ solves the following Schrödinger equation

$$i\partial_t\psi^{k+1}+\frac{1}{2}\Delta_x\psi^{k+1}=\Theta\big(\psi^k\big)\psi^k+\lambda V_L\big(\psi^k\big)\psi^k,$$

so that the following estimate

$$\left\|\partial_t\psi^{k+1}(t)\right\|_{L^2}\leqslant\frac{M}{2}+KM(\lambda M_{V_L}+M_\Theta)\leqslant L_\psi,\quad\forall t\in[0,T]$$

holds. Finally, reasoning as in (26) we arrive at

$$\left\|V_L\big(\psi^k(t)\big)\psi^k(t)-V_L\big(\psi^k(s)\big)\psi^k(s)\right\|_{L^2}\leqslant K(M_{V_L}+K_{V_L}M)L_\psi|t-s| \tag{29}$$

for all $t,s\in[0,T]$, which ends the first step of the proof after application of the induction principle.

Step 2. *The family $\{\psi^k\}$ defined in* (22)–(24) *is a Cauchy sequence in $Y_{r,T}$* (*with respect to the norm $\|\cdot\|_{L^\infty([0,T],H^2)}$*).

To this purpose, it suffices to show that

$$\left\|\psi^{k+1}-\psi^k\right\|_{L^\infty([0,T],H^2)}\leqslant\frac{1}{2^{k+1}},\quad\forall 0\leqslant k\in\mathbb{Z}. \tag{30}$$

We argue as in the previous step by following an inductive scheme. For $k=0$ we proceed as in (27)–(28) to get that $\|\psi^1-\psi^0\|_{L^\infty([0,T],H^2)}\leqslant\frac{1}{2}$. In addition, if $\psi,\phi\in Y_{r,T}$ we can use Proposition 2.1 to estimate

$$\left\|\Gamma_{V_L}\big(\psi(t)\big)-\Gamma_{V_L}\big(\phi(t)\big)\right\|_{H^2}\leqslant\lambda K^2(M_{V_L}+K_{V_L}M)\int_0^t\left\|\psi(\tau)-\phi(\tau)\right\|_{H^2}d\tau. \tag{31}$$

In the same spirit, (H4) entails

$$\left\| \Gamma_\Theta\big(\psi(t)\big) - \Gamma_\Theta\big(\phi(t)\big)\right\|_{H^2} \leqslant \lambda K(M_\Theta + K_\Theta M)\int_0^t \left\|\psi(\tau) - \phi(\tau)\right\|_{H^2} d\tau. \tag{32}$$

Therefore, for any $k \in \mathbb{N}$ we have

$$\left\|\psi^{k+1} - \psi^k\right\|_{L^\infty([0,T],H^2)} \leqslant K\left\{\lambda K(M_{V_L} + K_{V_L}M) + M_\Theta + K_\Theta M\right\}\left\|\psi^k - \psi^{k-1}\right\|_{L^\infty([0,T],H^2)},$$

which leads us straightforwardly to (31). This concludes this part of the proof.

Now it is clear that $\{\psi^k\}$ converges towards some $\psi \in Y_{r,T}$. Hence, in virtue of the estimates (31) and (32) we may go to the limit in (22) and deduce that

$$\psi = Z + \Gamma_{V_L}(\psi) + \Gamma_\Theta(\psi).$$

If in addition $\psi' \in Y_{r,T}$ enjoys the same property, then

$$\left\|\psi(t) - \psi'(t)\right\|_{H^2} \leqslant 2T_2 \int_0^t \left\|\psi(\tau) - \phi(\tau)\right\|_{H^2} d\tau, \quad \forall t \in [0,T].$$

Gronwall's lemma now implies $\psi = \psi'$, and thus the existence of a unique fixed point of the operator $\psi \mapsto Z + \Gamma_{V_L}(\psi) + \Gamma_\Theta(\psi)$. To conclude the proof we define $\Theta := \Theta(\psi)$ and $V_L := V_L(\psi)$ and observe that (H4) implies $\Theta \in C([0,T], H^2(\Omega))$ and that $V_L \in C([0,T], H^2(\Omega))$ solves Eq. (10) subject to (6), in virtue of Lemma 2.3. To conclude, it is clear that $\psi \in Y_{r,T} \subset C([0,T], H^2_\delta)$ satisfies the initial and boundary conditions, and that $V_L(\psi)\psi \in W^{1,1}([0,T], L^2(\Omega))$ after taking limits in (25). Moreover, from the boundary datum $\psi(t)|_{\partial\Omega} = \psi_B$ and the assumption (H4) it is deduced that $\Theta(\psi)\psi - \widetilde{\Theta_B}\widetilde{\psi_B} \in C([0,T], \mathcal{D})$. Finally, from Lemma 2.4 we obtain that $\psi \in X^T_\delta$ solves Eq. (5) pointwise. Now we are done with the proof. □

Remark 1. The shape of the generic potential Θ in Eq. (5) is ruled by the assumption (H4), stated like that in order to select some physical self-interaction potentials in a simple way. For instance, the Hartree electrostatic potential fits (H4) as will be analyzed throughout the next section. This led us to deal with Θ in a very different way that with V_L, mainly because the former is well known at $\partial\Omega$ while the latter not. On the contrary, V_L is proved to be Lipschitz continuous in $L^2(\Omega)$. It is a simple matter to verify that after replacement of (H4) by

(H4′) The operator $\Theta : H^2_\delta \to H^2(\Omega)$ is locally Lipschitz continuous with respect to $\|\cdot\|_{H^2}$ and $\|\cdot\|_{L^2}$,

Theorem 1.1 remains valid and its proof is even rather simplified, as Θ and V_L can be now treated in the same fashion.

Remark 2. Theorem 3.3 in [37] can be applied to our situation to find an argument $S_\psi \in X^T$ of the wavefunction ψ, which entails $V_L = S_\psi - \langle S_\psi\rangle \in X^T$.

Remark 3. The generalization of the results presented here to arbitrary dimension exhibits two main drawbacks. The first one concerns regularity, as Theorem 1.1 strongly depends on the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ (only valid if $1 \leqslant d \leqslant 3$). Indeed, in higher dimensions one should work in $H^s(\Omega)$ with $s > d/2$, which means an appropriate modification of Proposition 2.1 as well as of the assumptions on Θ. The second one stems from Lemma 2.1 (cf. [37]), which holds true thanks to the existence of scalar potentials associated with an irrotational field f (see [2]). In this case, the conditions $\partial_{x_j} f_k = \partial_{x_k} f_j$ for all $j \neq k$ would solve the problem.

Remark 4. The problem of extending our results to the whole space relies on an adequate extension to $\mathbb{R}^d$ of Lemmata 2.1 and 2.2 The main difficulty here consists in guaranteeing that the condition $\nabla_x\psi/\psi \in H^k(\mathbb{R}^d)$ still holds along the time evolution, at least locally. A detailed analysis of the action of the Schrödinger group on wavefunctions fulfilling this property might shed some light on this problem, that is postponed to future work.

4. An application: The Schrödinger–Poisson system with enthalpy dependence

In this section we deal with the following Schrödinger–Langevin–Poisson system for semiconductors

$$i\partial_t\psi = -\frac{1}{2}\Delta_x\psi + \lambda V_L\psi + \big(V + h(n_\psi)\big)\psi, \tag{33}$$

$$\lambda_D^2\Delta_x V = n_\psi - \mathcal{C}, \tag{34}$$

$$V|_{\partial\Omega} = V_B, \tag{35}$$

subject to the initial and boundary conditions $\psi(0)=\psi_0$, $\psi(t)|_{\partial\Omega}=\psi_B$. Here, $\lambda_D>0$ and $\mathcal{C}=\mathcal{C}(x)$ denote the scaled Debye length and the doping concentration profile of the device respectively, while the enthalpy $h(s)$ satisfies

$$sh'(s)=p'(s), \qquad h(1)=0, \tag{36}$$

where the pressure function $p(s)$ is typically assumed to depend (smoothly) only on the particle density, in such a manner that the profiles $p(n_\psi)=T_0 n_\psi^\beta$, with $\beta\geqslant 1$, are contemplated. Particularly, the choice $\beta=1$ accounts for the isothermal case at the hydrodynamical description while $\beta>1$ gives rise to the isentropic regime, T_0 standing for the temperature of the system.

Lemma 4.7. *Let $\Omega\subset\mathbb{R}^d$ be a C^2 bounded domain, $\mathcal{C}\in L^2(\Omega)$ and $V_B\in H^{3/2}(\partial\Omega)$. Then, there exists $V: H^2(\Omega)\to H^2(\Omega)$ locally Lipschitz continuous and such that, for all $\psi\in H^2(\Omega)$, $V(\psi)$ is the unique solution to the elliptic problem* (34)–(35)*.*

Proof. As in the previous section, we first select a continuous representative of V_B defined in $\partial\Omega$ and take $\widetilde{V_B}\in C^2(\Omega)\cap C(\overline{\Omega})$ the harmonic extension of V_B to Ω. Define now $V(\psi):=\widetilde{V_B}+u_\psi$, where $u_\psi\in\mathcal{D}$ is the unique solution to $\Delta_x u_\psi=-\frac{1}{\lambda_D^2}(n_\psi-\mathcal{C})\in L^2(\Omega)$ as given by Lemma 2.5(ii). Then, it is a simple matter to conclude that $V(\psi)$ solves the elliptic problem (34)–(35). Furthermore, again according to Lemma 2.5(ii) with $f=f_\psi-f_\phi$ along with Lemma 2.1, it is straightforwardly concluded that

$$\big\|V(\psi)-V(\phi)\big\|_{H^2}\leqslant\frac{C}{\lambda_D^2}\|\psi-\phi\|_{H^2}$$

for all $\psi,\phi\in H^2(\Omega)$ satisfying $\|\psi\|_{H^2},\|\phi\|_{H^2}\leqslant M$. This ends the proof. □

In proving Theorem 4.2 below, we shall make use of the following

Lemma 4.8. *Let $\Omega\subset\mathbb{R}^d$ be a C^2 bounded domain, $\delta>0$ and $p\in C^{2,1}([\delta,\infty[)$. Then, $U: H^2_\delta\to H^2(\Omega)$ defined by $U(\psi)(x):=h(n_\psi(x))$ a.e. Ω, with $h:[\delta,\infty[\to\mathbb{R}$ as in* (36)*, is a locally Lipschitz continuous operator.*

Proof. Define

$$h(s):=\int_1^s\frac{p'(r)}{r}\,dr,\quad\forall s>\delta.$$

Then, according to the regularity of p and the fundamental theorem of calculus it is clear that $h\in C^{2,1}([\delta,\infty[)$ and satisfies (36). Then, we have

$$\big\|U(\psi)-U(\phi)\big\|_{L^2}\leqslant K_h\|n_\psi-n_\phi\|_{L^2}\leqslant C\|\psi-\phi\|_{H^2}.$$

Also, using the chain rule in Sobolev spaces we find $\nabla_x U(\psi)=h'(n_\psi)\nabla_x n_\psi\in L^2(\Omega)$, so that $U\in H^1(\Omega)$. In addition,

$$\begin{aligned}\big\|\nabla_x U(\psi)-\nabla_x U(\phi)\big\|_{L^2}&\leqslant\big\|h'(n_\psi)\big\|_{L^\infty}\|\nabla_x n_\psi-\nabla_x n_\phi\|_{L^2}+\big\|h'(n_\psi)-h'(n_\phi)\big\|_{L^\infty}\|\nabla_x n_\phi\|_{L^2}\\&\leqslant(M_h+M_nK_h)\|n_\psi-n_\phi\|_{H^1}\leqslant C\|\psi-\phi\|_{H^2}.\end{aligned}$$

Finally, for the second order derivatives we get

$$\nabla_x\otimes\nabla_x U(\psi)=h''(n_\psi)\nabla_x n_\psi\otimes\nabla_x n_\psi+h'(n_\psi)\nabla\otimes\nabla_x n_\psi\in L^2(\Omega)$$

and the following estimate holds:

$$\begin{aligned}\big\|\Delta_x U(\psi)-\Delta_x U(\phi)\big\|_{L^2}&\leqslant\big\|h''(n_\psi)\big\|_{L^\infty}\big(\|\nabla_x n_\psi\|_{L^4}+\|\nabla_x n_\phi\|_{L^4}\big)\|\nabla_x n_\psi-\nabla_x n_\phi\|_{L^4}\\&\quad+\big\|h''(n_\psi)-h''(n_\phi)\big\|_{L^\infty}\|\nabla_x n_\phi\|_{L^4}^2\\&\quad+\big\|h'(n_\psi)\big\|_{L^\infty}\|\Delta_x n_\psi-\Delta_x n_\phi\|_{L^2}\\&\quad+\big\|h'(n_\psi)-h'(n_\phi)\big\|_{L^\infty}\|\Delta_x n_\phi\|_{L^2}\\&\leqslant C\|\psi-\phi\|_{H^2},\end{aligned}$$

where $C>0$ denotes various positive constants. Now we are done with the proof. □

We can finally prove our main result in this section.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^d$ (with $1 \leqslant d \leqslant 3$), $\lambda, \lambda_D > 0$, $\mathcal{C} \in L^2(\Omega)$, $p \in C^{2,1}([\delta, \infty[)$, $V_B \in H^{3/2}(\partial\Omega)$, and assume that the hypotheses* (H1)–(H3) *are fulfilled. Then, there exist $T > 0$ and a unique local strong solution (ψ, V_L, V) of* (33)–(35)*, subject to the initial-boundary conditions*

$$\psi(0) = \psi_0, \qquad \psi(t)|_{\partial\Omega} = \psi_B,$$

and to Eq. (10) *with $\langle V_L \rangle = 0$ for the determination of the Langevin potential, that satisfies*

$$\psi \in X_\delta^T, \qquad V_L, V \in C\big([0, T], H^2(\Omega)\big).$$

Proof. The hypotheses of the theorem clearly imply those of Lemmata 4.7 and 4.8. Given $\psi \in H^2(\Omega)$, if we define $\Theta(\psi) = V(\psi) + U(\psi)$ and

$$\Theta_B = V_B + h(n_B), \qquad n_B = |\psi_B|^2,$$

it is a simple matter to check that (H4) is straightforwardly fulfilled. Indeed, Θ inherits the local Lipschitz continuity from V and U, and

$$\Theta(\psi)|_{\partial\Omega} = V(\psi)|_{\partial\Omega} + U(\psi)|_{\partial\Omega} = V_B + h(n_\psi|_{\partial\Omega}) = \Theta_B$$

for any $\psi \in H^2(\Omega)$ satisfying $\psi|_{\partial\Omega} = \psi_B$, for which we used the regularity of h and n_ψ. Applying now Theorem 1.1 yields $T > 0$ and a unique triplet (ψ, Θ, V_L) such that

(a) $V_L \in C([0, T], H^2(\Omega))$ solves Eq. (10) subject to the constraint $\langle V_L \rangle = 0$, and
(b) $\psi \in X_\delta^T$ solves Eq. (5) subject to the initial-boundary data $\psi(0) = \psi_0$, $\psi(t)|_{\partial\Omega} = \psi_B$, with $\Theta(\psi) = V(\psi) + U(\psi) \in C([0, T], H^2(\Omega))$.

Finally, Lemmata 4.7 and 4.8 imply that $V := V(\psi) \in C([0, T], H^2(\Omega))$ solves (34)–(35), hence (ψ, V, V_L) is the unique solution to our problem. This concludes the proof. □

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