On the Integrability of Double Cosine and Sine Series, I

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We consider double cosine and sine series whose coefficients form a null sequence of bounded variation. We prove that in this case the sum of a double cosine series is integrable in the sense of improper Riemann integral and the series in question is the Fourier series of its sum in the same sense. On the other hand, the sum of a double sine series is not necessarily integrable in the sense of improper Riemann integral, but the series is always a generalized Fourier sine series of its sum. These imply the important corollary that if the sum of a double cosine or sine series, with coefficients tending to zero and of bounded variation, is Lebesgue integrable, then the series is the Fourier series of its sum. Our results are the extensions of those by R. P. Boas and N. K. Bary from one-dimensional to two-dimensional trigonometric series.

1. INTRODUCTION

We study the double cosine series

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \lambda_j \lambda_k a_{jk} \cos jx \cos ky,$$

where $\lambda_0 = \frac{1}{2}$ and $\lambda_j = 1$ for $j \geq 1$, and the sine series

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} \sin jx \sin ky$$

on the positive quadrant $Q = [0, \pi] \times [0, \pi]$ of the two-dimensional torus. We assume that the real coefficients $a_{jk}$ are such that

$$a_{jk} \to 0 \quad \text{as} \quad j+k \to \infty$$

and

$$\sum_{j} \sum_{k} |A_{jk}| < \infty,$$
where \( j, k \) run over 0, 1, \( \ldots \), in the case of series (1.1), and \( j, k \) run over 1, 2, \( \ldots \), in the case of series (1.2). Here, as usual,
\[
A_{11}a_{jk} = a_{jk} - a_{j+1,k} - a_{j,k+1} + a_{j+1,k+1}.
\]

In the sequel, we also use the notations
\[
A_{10}a_{jk} = a_{jk} - a_{j+1,k} \quad \text{and} \quad A_{01}a_{jk} = a_{jk} - a_{j,k+1}.
\]

Conditions (1.3) and (1.4) express the fact that \( \{a_{jk}\} \) is a double null sequence of bounded variation.

In this paper, the pointwise convergence of series (1.1) and (1.2) is meant in Pringsheim's sense. (See, e.g., [6, Vol. 2, Chap. 17].) In other words, we form the rectangular partial sums
\[
s_{mn}(x, y) = \sum_{j=0}^{m} \sum_{k=0}^{n} \hat{a}_{jk} \cos jx \cos ky \quad (m, n \geq 0)
\]
of series (1.1), say; then let both \( m \) and \( n \) tend to \( \infty \) independently of one another, and assign the limit \( f(x, y) \) (if it exists) to the series (1.1) as its sum.

We have shown in [3] that under conditions (1.3) and (1.4), series (1.1) and (1.2) converge uniformly on the rectangle \([\delta, \pi] \times [\varepsilon, \pi]\) for all \( \delta, \varepsilon > 0 \) and, consequently, define their sums \( f(x, y) \) and \( g(x, y) \), respectively. These functions, periodic in each variable, are not Lebesgue integrable in general (cf. [4] for single series). However, both \( f \) and \( g \) are Lebesgue integrable if condition (1.3) is satisfied and
\[
\sum_{j} |A_{11}a_{jk}| \ln(j+2) \ln(k+2) < \infty. \tag{1.5}
\]

Furthermore, under conditions (1.3) and (1.5), series (1.1) and (1.2) are the Fourier series of their sums \( f \) and \( g \), respectively. (See again [3].)

2. MAIN RESULTS

Nevertheless, conditions (1.3) and (1.4) are themselves sufficient to ensure the integrability of \( f \) in the sense of improper Riemann integral and even more is true, as the following theorem shows.

**Theorem 1.** If conditions (1.3) and (1.4) are satisfied, then

(i) series (1.1) converges to a function \( f(x, y) \) for all \( 0 < x, y \leq \pi \);
(ii) \( f \) is integrable on \( Q \) in the sense of improper Riemann integral; and

(iii) series (1.1) is the Fourier series of \( f \) in the same sense:

\[
\begin{align*}
\Delta_{pq} & = \lim_{\delta, \epsilon \to 0} \frac{4}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \cos px \cos qy \, dx \, dy, \\
(p, q = 0, 1, \ldots).
\end{align*}
\]

This theorem is the extension of a theorem by Boas [2] from one-dimensional to two-dimensional cosine series.

In the case of series (1.2), conditions (1.3) and (1.4) ensure the continuity of \( g(x, y) \sin x \sin y \), a fortiori, the continuity of \( g(x, y) \sin px \sin qy \) for all \( p, q = 1, 2, \ldots \).

**Theorem 2.** If conditions (1.3) and (1.4) are satisfied, then

(i) series (1.2) converges to a function \( g(x, y) \) for all \( x, y \);

(ii) \( g(x, y) \sin px \sin qy \) is continuous for all \( p, q = 1, 2, \ldots \); and

(iii) series (1.2) is the Fourier series of \( g \) in the sense that

\[
\begin{align*}
\Delta_{pq} & = \frac{4}{\pi^2} \int_{0}^{\pi} \int_{0}^{\pi} g(x, y) \sin px \sin qy \, dx \, dy, \\
(p, q = 1, 2, \ldots).
\end{align*}
\]

So, the integrals in (2.2) exist in the sense of the ordinary Riemann integral. Following the terminology in [6, Vol. I, p. 48], we can say that series (1.2) is a generalized Fourier sine series of its sum \( g \).

Theorem 2 is the extension of a result presented in [1, p. 656] from one-dimensional to two-dimensional sine series.

We draw the following corollary from Theorems 1 and 2.

**Corollary.** If conditions (1.3) and (1.4) are satisfied, and the sum \( f \) of series (1.1) is Lebesgue integrable, then (1.1) is the Fourier series of \( f \).

The same statement is true in the case of series (1.2).

The integrability of \( g \) in the sense of improper Riemann integral is a delicate question. In this paper, we deal with this problem only in the special case when

\[
A_{11} a_{jk} \geq 0 \quad (j, k = 1, 2, \ldots).
\]

It is obvious that conditions (1.3 and (2.3) imply that \( \{a_{jk}\} \) is a sequence of bounded variation and

\[
a_{jk} \geq 0, \quad A_{10} a_{jk} \geq 0, \quad A_{01} a_{jk} \geq 0.
\]
THEOREM 3. If conditions (1.3) and (2.3) are satisfied, then
\[
\lim_{\delta, \epsilon \downarrow 0} \int_{-\delta}^{\epsilon} \int_{-\epsilon}^{\epsilon} g(x, y) \, dx \, dy
\]
exists if and only if
\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_{jk}}{jk} < \infty.
\]

We note that, under (1.3) and (2.3), conditions (1.5) and (2.6) are equivalent. (See [3].)

Theorem 3 partially extends a result by Boas [2] from one-dimensional to two-dimensional sine series.

In closing, we study the integrability of \( g(x, y)/xy \) in the sense of improper Riemann integral provided series (1.2), which defines \( g \), is absolutely convergent. As in the one-dimensional case (cf. [2] again), this is closely related to the question of the integrability of \( f \) under conditions (1.3) and (1.4). Actually, Theorem 1 plays a key role in the proof of the next theorem.

THEOREM 4. If
\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |a_{jk}| < \infty,
\]
then the improper integral
\[
\lim_{\delta, \epsilon \downarrow 0} \int_{-\delta}^{\epsilon} \int_{-\epsilon}^{\epsilon} \frac{g(x, y)}{xy} \, dx \, dy
\]
exists.

We note that the problem of the integrability of \( f(x, y)/xy \) is more intricate and its solution would require knowledge of the exact conditions which ensure the integrability of \( g \) under conditions (1.3) and (1.4) (instead of under (2.3) as in Theorem 3).

3. PROOFS

Proof of Theorem 1. Part (i) is proved in [3]. To prove (ii) and (iii), we extend the method used by Ul'janov [5] in the case of single cosine series.
A double summation by parts yields (cf. [3]) that, under conditions (1.3) and (1.4), the sum \( f \) of series (1.1) can be expressed in the form

\[
f(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} D_m(x) D_n(y) A_{11} a_{mn}
\]

and this convergence is uniform on each rectangle \([\delta, \pi] \times [\varepsilon, \pi]\), where \(0 < \delta, \varepsilon < \pi\). Here

\[
D_m(x) = \frac{1}{2} + \sum_{j=1}^{m} \cos jx = \frac{\sin(m + \frac{1}{2})x}{\sin \frac{1}{2}x} \quad (m \geq 0)
\]

is the well-known Dirichlet kernel. Thus,

\[
\int_{\delta}^{\pi} \int_{\varepsilon}^{\pi} f(x, y) \, dx \, dy = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_m^{(\delta)} b_n^{(\varepsilon)} A_{11} a_{mn}, \quad (3.1)
\]

where

\[
b_m^{(\delta)} = \int_{\delta}^{\pi} D_m(x) \, dx \quad \text{and} \quad b_n^{(\varepsilon)} = \int_{\varepsilon}^{\pi} D_n(y) \, dy. \quad (3.2)
\]

Clearly,

\[
b_m^{(\delta)} = \frac{\pi - \delta}{2} - \sum_{j=1}^{m} \frac{\sin j\delta}{j}.
\]

It is well known that the partial sums of the series \( \sum_{j=1}^{\infty} \frac{\sin jx}{j} \) are uniformly bounded. Hence the \( b_m^{(\delta)} \) are also uniformly bounded in \( m = 0, 1, \ldots \), and \( \delta > 0 \), and

\[
\lim_{\delta \to 0} b_m^{(\delta)} = \frac{\pi}{2}. \quad (3.3)
\]

Analogous observations are true for \( b_n^{(\varepsilon)} \). Consequently,

\[
\lim_{\delta, \varepsilon \to 0} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_m^{(\delta)} b_n^{(\varepsilon)} A_{11} a_{mn} = \frac{\pi^2}{4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{11} a_{mn} = \frac{\pi^2}{4} a_{00}.
\]

Combining this with (3.1), we see that \( f \) is integrable in the sense of improper Riemann integral, and in addition, (2.1) holds for \( p = q = 0 \).
Now let $p \geq 1$ be fixed. Then, in a similar manner,

$$
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y) \cos px \, dx \, dy = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mp}^{(\delta)} b_{n}^{(\varepsilon)} A_{11} a_{mn},
$$

(3.4)

where $b_{n}^{(\varepsilon)}$ is defined in (3.2) and

$$
c_{mp}^{(\delta)} = \int_{-\delta}^{\delta} D_{m}(x) \cos px \, dx.
$$

(3.5)

An elementary calculation shows that if $m < p$, then

$$
c_{mp}^{(\delta)} = -\frac{1}{2} \sum_{j=-p-m}^{p+m} \frac{\sin j\delta}{j}.
$$

Hence it follows that the $c_{mp}^{(\delta)}$ are uniformly bounded in $m = 0, 1, \ldots$, $p = 1, 2, \ldots$, and $\delta > 0$, and

$$
\lim_{\delta \downarrow 0} c_{mp}^{(\delta)} = 0 \quad (m < p).
$$

(3.6)

On the other hand, if $m \geq p$, then

$$
c_{mp}^{(\delta)} = \frac{\pi - \delta}{2} - \frac{1}{2} \sum_{j=-m-p+1}^{m+p} \frac{\sin j\delta}{j} - \sum_{j=1}^{m-p} \frac{\sin j\delta}{j},
$$

whence it follows that the $c_{mp}^{(\delta)}$ are uniformly bounded again, but this time

$$
\lim_{\delta \downarrow 0} c_{mp}^{(\delta)} = \frac{\pi}{2} \quad (m \geq p).
$$

(3.7)

Combining (3.4), (3.6), and (3.7) gives

$$
\lim_{\delta, \varepsilon \downarrow 0} \int_{-\delta}^{\delta} \int_{-\varepsilon}^{\varepsilon} f(x, y) \cos px \, dx \, dy = \frac{\pi^{2}}{4} \sum_{m=p}^{\infty} \sum_{n=0}^{\infty} A_{11} a_{mn} = \frac{\pi^{2}}{4} a_{p0}.
$$

This proves (2.1) for $p \geq 1$ and $q = 0$.

A similar reasoning provides (2.1) for $p = 0$ and $q \geq 1$.

Finally, let $p, q \geq 1$ be fixed. Then

$$
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y) \cos px \cos qy \, dx \, dy = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mp}^{(\delta)} c_{nq}^{(\varepsilon)} A_{11} a_{mn},
$$

(3.8)
where \( c_{mn}^{(5)} \) is defined in (3.5) and \( c_{m0}^{(e)} \) is defined analogously. Hence, by (3.6) and (3.7),

\[
\lim_{\delta \to 0} \int_{-\delta}^{\pi-\delta} \int_{-\delta}^{\pi-\delta} f(x, y) \cos px \cos qy \, dx \, dy = \frac{\pi^2}{4} \sum_{m=p}^{\infty} \sum_{n=q}^{\infty} A_{11} a_{nm} = \frac{\pi^2}{4} \alpha_{pq},
\]

which proves (2.1) for \( p, q \geq 1 \).

**Proof of Theorem 2.** Part (i) is proved in [3]. To prove (ii) and (iii) we multiply series (1.2) by \( 4 \sin x \sin y \). As a result, we obtain that

\[
4g(x, y) \sin x \sin y = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{mn} (\cos(m-1)x - \cos(m+1)x) \cdot (\cos(n-1)y - \cos(n+1)y).
\]

Hence, a double summation by parts yields

\[
4g(x, y) \sin x \sin y = a_{11} + \sum_{m=1}^{\infty} (a_{m+1, 1} - a_{m-1, 1}) \cos mx + \sum_{n=1}^{\infty} (a_{1, n+1} - a_{1, n-1}) \cos ny + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{m+1, n+1} - a_{m-1, n+1} - a_{m+1, n-1} + a_{m-1, n-1}) \cdot \cos mx \cos ny
\]

with the agreement that \( a_{m0} = a_{0n} = a_{00} = 0 \) for all \( m, n \geq 1 \). By (1.4),

\[
\sum_{m=1}^{\infty} |a_{m+1, 1} - a_{m-1, 1}| + \sum_{n=1}^{\infty} |a_{1, n+1} - a_{1, n-1}| + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{m+1, n+1} - a_{m-1, n+1} - a_{m+1, n-1} + a_{m-1, n-1}| < \infty.
\]

Thus, series (3.8) converges uniformly. Consequently, \( g(x, y) \sin x \sin y \) is continuous. Multiplying by the continuous function \( (\sin px/\sin x) \) \( (\sin qy/\sin y) \), we conclude the continuity of \( g(x, y) \sin px \sin qy \) for all \( p, q \geq 1 \).
By (3.8), it is easy to see that

\[
\frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} g(x, y) \sin x \sin y \, dx \, dy
\]

\[
= \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} a_{11} \, dx \, dy = a_{11}.
\]  \hspace{1cm} (3.9)

\[
\frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} g(x, y) \sin x \sin y \cos x \, dx \, dy
\]

\[
= \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} a_{21} \cos^2 x \, dx = \frac{1}{2} a_{21},
\]

whence

\[
a_{21} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} g(x, y) \sin 2x \sin y \, dx \, dy,
\]  \hspace{1cm} (3.10)

and, in a similar way,

\[
\frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} g(x, y) \sin x \sin y \cos mx \, dx \, dy
\]

\[
= \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} (a_{m+1,1} - a_{m-1,1}) \cos^2 mx \, dx \, dy
\]

\[
= \frac{1}{2} (a_{m+1,1} - a_{m-1,1}).
\]

whence

\[
a_{m+1,1} - a_{m-1,1}
\]

\[
= \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} g(x, y) (\sin(m + 1) x - \sin(m - 1) x) \sin y \, dx \, dy
\]

\[
(m = 2, 3, \ldots).
\]  \hspace{1cm} (3.11)

On the basis of (3.9)–(3.11), an induction argument yields

\[
a_{p1} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} g(x, y) \sin px \sin y \, dx \, dy \quad (p = 2, 3, \ldots). \]  \hspace{1cm} (3.12)

In a similar fashion, we deduce that

\[
a_{1q} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} g(x, y) \sin x \sin qy \, dx \, dy \quad (q = 2, 3, \ldots).
\]  \hspace{1cm} (3.13)

Finally, by (3.8) again,
\[ a_{m+1,n+1} - a_{m-1,n+1} - a_{m+1,n-1} + a_{m-1,n-1} \]
\[ = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi 4g(x, y) \sin x \sin y \cos mx \cos ny \, dx \, dy \]
\[ = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi g(x, y)(\sin(m+1)x - \sin(m-1)x)(\sin(n+1)y - \sin(n-1)y) \, dx \, dy \quad (m, n = 1, 2, \ldots). \]

In particular, for \( n = 1 \) and/or \( m = 1 \) (recall that \( a_{0k} = a_{j0} = 0 \) by agreement),
\[ a_{22} = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi g(x, y) \sin 2x \sin 2y \, dx \, dy, \]
\[ a_{m+1,2} - a_{m-1,2} \]
\[ = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi g(x, y)(\sin(m+1)x - \sin(m-1)x) \sin 2y \, dx \, dy \quad (m = 2, 3, \ldots), \]
\[ a_{2,n+1} - a_{2,n-1} \]
\[ = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi g(x, y) \sin 2x(\sin(n+1)y - \sin(n-1)y) \, dx \, dy \quad (n = 2, 3, \ldots). \]

Applying a double induction argument, while relying on the "initial values" in (3.9), (3.12), and (3.13), we find that
\[ a_{pq} = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi g(x, y) \sin px \sin qy \, dx \, dy \quad (p, q = 2, 3, \ldots). \tag{3.14} \]

Relations (3.9) and (3.12)-(3.14) give the proof of (2.2).

**Proof of Theorem 3:** Sufficiency. Assume that conditions (1.3), (2.3), and (2.6) are satisfied. Since series (1.2) converges uniformly, except in a cross neighborhood of 0 (see [3] again), we have, for all \( 0 < \delta, \varepsilon < \pi \),
\[ \int_\delta^\pi \int_\varepsilon^\pi g(x, y) \, dx \, dy \]
\[ = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_{jk}}{jk} \left( \cos j\delta - \cos j\pi \right) \left( \cos k\varepsilon - \cos k\pi \right). \tag{3.15} \]

Due to (2.3) and the fact that \( a_{jk} \geq 0 \) (see (2.4)) we have
\[ \lim_{N \to \infty} \left( \sum_{j=1}^{N} \sum_{k=N}^{\infty} + \sum_{j=N}^{\infty} \sum_{k=1}^{N} + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \right) \frac{a_{jk}}{jk} = 0. \]
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Using this, it is routine to show that

\[
\lim_{\delta, \varepsilon \to 0} \left\{ \int_0^\pi \int_0^\pi g(x, y) \, dx \, dy - \sum_{j=1}^{[1/\delta]} \sum_{k=1}^{[1/\varepsilon]} \frac{a_{jk}}{jk} (1 - \cos j\pi)(1 - \cos k\pi) \right\} = 0. \quad (3.16)
\]

Here \([ \cdot ]\) means the greatest integral part. Now (2.5) follows from (2.6) and (3.16).

Necessity. This time we assume that conditions (1.3), (2.3), and (2.5) are satisfied. Due to (3.15) and the fact again that \(a_{jk} \geq 0\), we conclude that

\[
\sum_{j=1}^\infty \sum_{k=1}^\infty \frac{a_{jk}}{jk} (1 - \cos j\pi)(1 - \cos k\pi) = 4 \sum_{j=1}^\infty \sum_{k=1}^\infty \frac{a_{ij-1,2k-1}}{(2j-1)(2k-1)}.
\]

Taking into account that the \(a_{jk}\) decreases monotonically in both \(j\) and \(k\) (see (2.4)), (2.6) follows immediately.

Proof of Theorem 4. We introduce a new sequence \(\{b_{mn}\}\):

\[ b_{mn} = \sum_{j=m}^\infty \sum_{k=n}^\infty a_{jk} \quad (m, n = 0, 1, \ldots). \]

By (2.7),

\[ b_{mn} \to 0 \quad \text{as} \quad m + n \to \infty, \quad (3.17) \]

and since \(A_{11}b_{mn} = a_{mn}\) we have

\[ \sum_{m=0}^\infty \sum_{n=0}^\infty |A_{11}b_{mn}| < \infty. \quad (3.18) \]

A double summation by parts yields

\[
g(x, y) = b_{11} \sin x \sin y \\
+ \sum_{m=1}^\infty b_{m1}(\sin mx - \sin(m - 1)x) \sin y \\
+ \sum_{n=1}^\infty b_{1n}\sin x(\sin ny - \sin(n - 1)y) \\
+ \sum_{m=2}^\infty \sum_{n=2}^\infty b_{mn}(\sin mx - \sin(m - 1)x) \\
\cdot (\sin ny - \sin(n - 1)y).
\]
Hence it follows easily that

\[ g(x, y) = \sin x \sin y \left( b_{11} + \sum_{m=2}^{\infty} b_{m1} \cos mx \right) \]

\[ + \sum_{n=2}^{\infty} b_{1n} \cos ny + \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} b_{mn} \cos mx \cos ny \]

\[ + 2 \sin^2 \frac{x}{2} \sin y \left( \sum_{m=2}^{\infty} b_{m1} \sin mx \right) \]

\[ + \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} b_{mn} \sin mx \sin ny \]

\[ + 2 \sin x \sin^2 \frac{y}{2} \left( \sum_{n=2}^{\infty} b_{1n} \sin ny \right) \]

\[ + \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} b_{mn} \cos mx \sin ny \]

\[ + 4 \sin^2 \frac{x}{2} \sin^2 \frac{y}{2} \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} b_{mn} \sin mx \sin ny \]

\[ = \frac{g_1(x, y) + g_2(x, y) + g_3(x, y) + g_4(x, y)}{4} \quad (3.19) \]

say.

Since the function \((\sin x \sin y)/xy\) has the limit 1 as \(x, y \to 0\),

\[ \lim_{\delta, \epsilon \downarrow 0} \int_{\delta}^{\pi} \int_{\epsilon}^{\pi} \frac{g_1(x, y)}{xy} \, dx \, dy \quad (3.20) \]

exists if and only if

\[ \lim_{\delta, \epsilon \downarrow 0} \int_{\delta}^{\pi} \int_{\epsilon}^{\pi} \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \cos mx \cos ny \right\} \, dx \, dy \]

exists. Due to (3.17) and (3.18), Theorem 1 is applicable and furnishes the existence of the latter improper integral. Consequently, (3.20) is true.

By a single summation by parts,

\[ g_2(x, y) = 2 \sin^2 \frac{x}{2} \sin y \]

\[ \cdot \sum_{m=2}^{\infty} \left( A_{10} b_{m1} + \sum_{n=2}^{\infty} (A_{10} b_{mn}) \cos ny \right) \sum_{j=2}^{m} \sin jx, \]
whence
\[
\frac{\varrho_2(x, y)}{xy} = \frac{\sin(x/2) \sin y}{xy} \cdot \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \left( A_{10} b_{m1} + A_{10} b_{mn} \right) \cos ny
\]
\[
\cdot \left( \cos \frac{3}{2} x - \cos \left( m + \frac{1}{2} \right) x \right).
\]

Again, it is enough to prove that the limit of
\[
I_2(\delta, \varepsilon) = \int_{\delta}^{\pi} \int_{\varepsilon}^{\pi} \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \left( A_{10} b_{m1} + A_{10} b_{mn} \right) \cos ny
\]
\[
\cdot \left( \cos \frac{3}{2} x - \cos \left( m + \frac{1}{2} \right) x \right) \, dx \, dy
\]
exists as \( \delta, \varepsilon \downarrow 0 \). Another single summation by parts gives
\[
\sum_{n=2}^{\infty} (A_{10} b_{mn}) \cos ny = \sum_{n=2}^{\infty} (A_{10} b_{mn}) \sum_{k=2}^{n} \cos ky.
\]

Thus,
\[
I_2(\delta, \varepsilon) = \sum_{m=2}^{\infty} \int_{\delta}^{\pi} \left( \cos \frac{3}{2} x - \cos \left( m + \frac{1}{2} \right) x \right) \, dx
\]
\[
\cdot \int_{\varepsilon}^{\pi} \left( A_{10} b_{m1} + \sum_{n=2}^{\infty} (A_{11} b_{mn}) \sum_{k=2}^{n} \cos ky \right) \, dy
\]
\[
= \sum_{m=2}^{\infty} \left[ \frac{2}{3} \sin \frac{3}{2} x - \frac{2}{2m+1} \sin \left( m + \frac{1}{2} \right) x \right]_{\delta}^{\pi}
\]
\[
\cdot \left[ (A_{10} b_{m1}) y + \sum_{n=2}^{\infty} (A_{11} b_{mn}) \sum_{k=2}^{n} \frac{\sin ky}{k} \right]_{\varepsilon}^{\pi}.
\]

The limit \( \lim_{\delta, \varepsilon \downarrow 0} I_2(\delta, \varepsilon) \) exists, due to (3.18) the facts that
\[
A_{10} b_{m1} = \sum_{n=1}^{\infty} A_{11} b_{mn}
\]
implies
\[
\sum_{m=2}^{\infty} |A_{10} b_{m1}| \leq \sum_{m=2}^{\infty} \sum_{n=1}^{\infty} |A_{11} b_{mn}| < \infty.
\]
and that the partial sums of the series \( \sum_{k=2}^{\infty} (\sin ky)/k \) are uniformly bounded. Combining the above reasoning gives that

\[
\lim_{\delta, \epsilon \searrow 0} \int_{-\delta}^{\delta} \int_{-\epsilon}^{\epsilon} \frac{g_2(x, y)}{xy} \, dx \, dy
\]  

exists.

In a similar manner, we can conclude that

\[
\lim_{\delta, \epsilon \searrow 0} \int_{-\delta}^{\delta} \int_{-\epsilon}^{\epsilon} \frac{g_3(x, y)}{xy} \, dx \, dy
\]  

exists.

Finally, a double summation by parts yields

\[
\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} b_{mn} \sin mx \sin ny
\]

\[
= \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} (A_{11} b_{mn}) \sum_{j=2}^{m} \sin jx \sum_{k=2}^{n} \sin ky,
\]

whence

\[
\frac{g_4(x, y)}{xy} = \frac{\sin(x/2) \sin(y/2)}{xy} \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} (A_{11} b_{mn}) \cdot \left( \cos \frac{3}{2} x - \cos \left( m + \frac{1}{2} \right) x \right) \left( \cos \frac{3}{2} y - \cos \left( n + \frac{1}{2} \right) y \right).
\]

Thus,

\[
\lim_{\delta, \epsilon \searrow 0} \int_{-\delta}^{\delta} \int_{-\epsilon}^{\epsilon} \frac{g_4(x, y)}{xy} \, dx \, dy
\]  

exists if and only if

\[
\lim_{\delta, \epsilon \searrow 0} \int_{-\delta}^{\delta} \int_{-\epsilon}^{\epsilon} \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} (A_{11} b_{mn}) \left( \cos \frac{3}{2} x - \cos \left( m + \frac{1}{2} \right) x \right) \cdot \left( \cos \frac{3}{2} y - \cos \left( n + \frac{1}{2} \right) y \right) \, dx \, dy
\]

exists. But the latter exists even in the sense of (absolute) Lebesgue integral, due to (3.18).

(3.19)–(3.23) together provide the proof of (2.8).
REFERENCES

5. P. L. UL'JANOV, Application of $A$-integration to a class of trigonometric series, Mat. Sb. (N.S.) 35, No. 77 (1954), 469–490. [In Russian]