Universal bounds for eigenvalues of the biharmonic operator

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Abstract

In this paper, we study the eigenvalues of the clamped plate problem:

\[
\begin{align*}
\Delta^2 u &= \lambda u, \quad \text{in } D, \\
\frac{\partial^2 u}{\partial n} &= 0, \quad \text{on } \partial D,
\end{align*}
\]

where \( D \) is a bounded connected domain in an \( n \)-dimensional complete minimal submanifold of a unit \( m \)-sphere \( S^m(1) \) or of an \( m \)-dimensional Euclidean space \( \mathbb{R}^m \). Let \( 0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \) be the eigenvalues of the above problem. We obtain universal bounds on \( \lambda_{k+1} \) in terms of the first \( k \) eigenvalues independent of the domains. For example, when \( D \) is contained in an \( n \)-dimensional complete minimal submanifold of \( S^m(1) \), we show that

\[
\lambda_{k+1} \leq \frac{1}{k} \sum_{i=1}^{k} \lambda_i \leq \frac{1}{kn} \left\{ \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} \left[ (2n + 4)\lambda_i^{1/2} + n^2 \right] \right\}^{1/2} \cdot \left\{ \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} \left( 4\lambda_i^{1/2} + n^2 \right) \right\}^{1/2},
\]

from which one can obtain a more explicit upper bound on \( \lambda_{k+1} \) in terms of \( \lambda_1, \ldots, \lambda_k \) (see Corollary 1). When \( D \) is contained in a complete \( n \)-dimensional minimal submanifold of \( \mathbb{R}^m \), we prove the inequality

\[
\lambda_{k+1} \leq \frac{1}{k} \sum_{i=1}^{k} \lambda_i + \left( \frac{8(n+2)}{n^2} \right)^{1/2} \frac{1}{k} \sum_{i=1}^{k} (\lambda_i (\lambda_{k+1} - \lambda_i))^{1/2},
\]

which generalizes the main theorem in Cheng and Yang (2006) [10] that states that the same estimate holds when \( D \) is a connected and bounded domain in \( \mathbb{R}^n \).

1. Introduction

Let \( D \) be a connected bounded domain with smooth boundary in \( \mathbb{R}^n \), the \( n \)-dimensional Euclidean space. Denote by \( \Delta \) the Laplacian operator on \( \mathbb{R}^n \). Consider the eigenvalue problem of a fixed membrane or Dirichlet Laplacian on \( D \):

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\[
\begin{aligned}
&\Delta u = -\lambda u, \quad \text{in } D, \\
&u = 0, \quad \text{on } \partial D.
\end{aligned}
\]
This problem has a real and purely discrete spectrum \([\lambda_i]_{i=1}^{\infty}\) where
\[
0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots \to \infty.
\]
Here each eigenvalue is repeated according to its multiplicity. In 1955 and 1956, Payne, Pólya and Weinberger [24, 25] proved that
\[
\frac{\lambda_2}{\lambda_1} \leq 3 \quad \text{for } D \subset \mathbb{R}^2
\]
and conjectured that
\[
\frac{\lambda_2}{\lambda_1} \leq \frac{\lambda_2}{\lambda_1} \text{disk}
\]
with equality if and only if \(D\) is a disk. For \(n \geq 2\), the analogous statements are
\[
\frac{\lambda_2}{\lambda_1} \leq 1 + \frac{4}{n} \quad \text{for } D \subset \mathbb{R}^n,
\]
and the PPW conjecture
\[
\frac{\lambda_2}{\lambda_1} \leq \frac{\lambda_2}{\lambda_1} \text{n-ball},
\]
with equality if and only if \(D\) is an \(n\)-ball. This important PPW conjecture was solved by Ashbaugh and Benguria in their excellent papers [3–5].

In [25], Payne, Pólya and Weinberger also proved the bound
\[
\lambda_{k+1} - \lambda_k \leq \frac{2}{k} \sum_{i=1}^{k} \lambda_i, \quad k = 1, 2, \ldots,
\]
for \(D \subset \mathbb{R}^2\). This result easily extends to \(D \subset \mathbb{R}^n\) as
\[
\lambda_{k+1} - \lambda_k \leq \frac{4}{kn} \sum_{i=1}^{k} \lambda_i, \quad k = 1, 2, \ldots
\]
(1.2)
Two main advances in extending (1.2) were made by Hile and Protter in [18] and Yang [27], respectively. Namely, in 1980, Hile and Protter proved
\[
\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{4n}{kn}, \quad \text{for } k = 1, 2, \ldots
\]
(1.3)
In 1991, Yang proved the following much stronger inequality:
\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left( \lambda_{k+1} - \left( 1 + \frac{4}{n} \right) \lambda_i \right) \leq 0, \quad \text{for } k = 1, 2, \ldots
\]
(1.4)
By elementary calculations (cf. [2]), one can show that Yang’s inequality (1.4) is sharper than the inequality (1.3) of Hile and Protter and that (1.3) is sharper than the inequality (1.2) of Payne, Pólya and Weinberger.

The inequalities on the higher eigenvalues of the Laplacian on a connected bounded domain in \(\mathbb{R}^n\) obtained by Payne, Pólya and Weinberger, Hile and Protter, and Yang have also been extended to some Riemannian manifolds (cf. [9, 14–16, 21, 23, 28]).

In [25], Payne, Pólya and Weinberger also considered the eigenvalue problem for the Dirichlet biharmonic operator which describes the characteristic vibrations of a clamped plate. This problem is given by
\[
\Delta^2 u = \lambda u \quad \text{in } D \subset \mathbb{R}^n, \quad u|_{\partial D} = \frac{\partial u}{\partial v}|_{\partial D} = 0.
\]
(1.5)
Payne, Pólya and Weinberger proved in [25] that the eigenvalues \([\lambda_i]_{i=1}^{\infty}\) of the above problem satisfy
\[
\lambda_{k+1} - \lambda_k \leq \frac{8(n + 2)}{n^2} \sum_{i=1}^{k} \lambda_i.
\]
(1.6)
As a generalization of (1.6), Hile and Yeh obtained [19]
\[
\sum_{i=1}^{k} \frac{\lambda_i^{1/2}}{\lambda_{k+1} - \lambda_i} \geq \frac{n^2 k^{3/2}}{8(n+2)} \left( \sum_{i=1}^{k} \lambda_i \right)^{-1/2}. 
\]  
(1.7)

Hook [20], Chen and Qian [11] proved, independently, the following inequality
\[
\frac{n^2 k^2}{8(n+2)} \leq \left( \sum_{i=1}^{k} \frac{\lambda_i^{1/2}}{\lambda_{k+1} - \lambda_i} \right) \left( \sum_{i=1}^{k} \frac{\lambda_i^{1/2}}{\lambda_{k+1} - \lambda_i} \right).
\]  
(1.8)

Very recently, answering a problem proposed by Ashbaugh in [1] and [2], Cheng and Yang established inequalities for eigenvalues of the problem (1.5) which are similar to the Yang’s inequality (1.4). Namely, they proved [10]
\[
\lambda_{k+1} - \frac{1}{k} \sum_{i=1}^{k} \lambda_i \leq \left( \frac{8(n+2)}{n^2} \right)^{1/2} \frac{1}{k} \sum_{i=1}^{k} \left( \lambda_i (\lambda_{k+1} - \lambda_i) \right)^{1/2}.
\]  
(1.9)

It has been proven in [26] that the eigenvalues of the problem (1.5) in which \( D \) is a compact domain of an \( n \)-dimensional complete minimal submanifold in a Euclidean space satisfy
\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \left( \frac{8(n+2)}{n^2} \right)^{1/2} \left( \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^4 \lambda_i^{1/2} \right)^{1/2} \left( \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i^{1/2} \right)^{1/2}.
\]  
(1.10)

It should be pointed out that the proof of Theorem 3.2 in [26] contains a mistake which occurs in the equality “\(||\nabla (\Delta u)|^2 = -\int_\Omega \Delta u \Delta^2 u||^2\)” on page 345 since there is an extra term of integration on the boundary when one uses the divergence theorem to derive it. The authors don’t see a way of correcting this error nor do they know of an alternative proof although they believe that that theorem is true. We are also very grateful to the referee and Professor Cheng, Qingming for pointing out this.

In this paper we study the eigenvalues of the Dirichlet biharmonic operator on compact domains of a complete minimal submanifold in a unit sphere or a Euclidean space. We give an estimate of the \((k+1)\)-st eigenvalue in terms of the first \(k\) eigenvalues which is similar to (1.9).

**Theorem 1.** Let \( M \) be an \( n \)-dimensional complete minimal submanifold in a unit sphere \( S^n(1) \) and denote by \( \Delta \) the Laplacian of \( M \). Let \( \lambda_i \) be the \( i \)-th eigenvalue of the following problem:
\[
\Delta^2 u = \lambda u \quad \text{in} \quad D, \quad u|_{\partial D} = \frac{\partial u}{\partial \nu}|_{\partial D} = 0,
\]  
(1.10)

where \( D \) is a bounded connected domain in \( M \). Then we have
\[
\lambda_{k+1} = \sum_{i=1}^{k} \lambda_i \leq \left( 1 + \frac{4(n+2)}{n^2} \right) \left( \sum_{i=1}^{k} \lambda_i \right)^{1/2} \left( \sum_{i=1}^{k} \lambda_i^{1/2} \right)^{1/2} \cdot \left( \sum_{i=1}^{k} \lambda_i \right)^{1/2} \left( 4 \lambda_i^{1/2} + n^2 \right)^{1/2}.
\]  
(1.11)

From Theorem 1, we can obtain a more explicit inequality which is weaker than (1.11):

**Corollary 1.** Under the same assumptions as in Theorem 1, we have
\[
\lambda_{k+1} \leq \left( 1 + \frac{4(n+2)}{n^2} \right) \left( \frac{n^2 + 2(n+4)k}{k} \sum_{i=1}^{k} \lambda_i^{1/2} \right) \left( \frac{n^2 + 2(n+4)k}{k} \sum_{i=1}^{k} \lambda_i^{1/2} \right)^{1/2} \left( \sum_{i=1}^{k} \lambda_i \right)^{1/2} \left( 4 \lambda_i^{1/2} + n^2 \right)^{1/2}.
\]  
(1.12)

Using the similar arguments as in the proof of Theorem 1, we also show

**Theorem 2.** Under the same assumptions as in Theorem 1, we have
\[
\sum_{i=1}^{k} \frac{\lambda_i^{1/2} + n^2}{4} (\lambda_{k+1} - \lambda_i) \geq \frac{n^2 k^2}{4}.
\]  
(1.13)
Next, we study the eigenvalues of the Dirichlet biharmonic operator on bounded domains of complete minimal submanifolds in a Euclidean space.

**Theorem 3.** Let $M$ be an $n$-dimensional complete minimal submanifold in $\mathbb{R}^m$ and let $D$ be a connected bounded domain in $M$. Denote by $\lambda_i$ the $i$-th eigenvalue of the problem:

$$
\begin{cases}
\Delta^2 u = \lambda u & \text{in } D, \\
u|_{\partial D} = \frac{\partial u}{\partial v}|_{\partial D} = 0.
\end{cases}
$$

Then it holds

$$
\lambda_{k+1} \leq \frac{1}{k} \sum_{i=1}^{k} \lambda_i + \left( \frac{8(n+2)}{n^2} \right)^{1/2} \left( \frac{1}{k} \sum_{i=1}^{k} \left( \lambda_{i} (\lambda_{k+1} - \lambda_i) \right) \right)^{1/2}
$$

and

$$
\frac{n^2 k^2}{8(n+2)} \leq \left( \sum_{i=1}^{k} \lambda_{i}^{1/2} \right) \left( \sum_{i=1}^{k} \lambda_{i} - \frac{1}{k} \sum_{j=1}^{k} \lambda_j \right).
$$

**Remark 1.** If we take $M = \mathbb{R}^n$ in Theorem 3, then the inequalities (1.15) and (1.14) become the Hook, Chen–Qian’s inequality (1.7) and the Cheng–Yang’s estimate (1.9), respectively.

**Remark 2.** The key element in Theorems 1 and 2 that we use is that the coordinate functions on minimal submanifolds in a Euclidean space (resp. a sphere) are harmonic functions (resp. eigenfunctions) (cf. [8]). This is the essence of the condition as used by earlier authors (cf. [2,12,21,23,27]).

It should be mentioned that the development of H.C. Yang inequality came to fruition thanks to the work of M.S. Ashbaugh [2] and that of Harrell and Stubbe [17]. It is not just H.C. Yang. In fact it was Harrell and Stubbe who first explained the key commutator facts behind the “trick” introduced by H.C. Yang in the traditional Payne–Pólya–Weinberger scheme and introduced the Yang inequality to the mathematical physics and geometry community. This “trick” was explained in further work of Ashbaugh (and later in the work of Ashbaugh and Hermi [6,7]) as an instance of the use of “optimal Cauchy–Schwarz” inequality. It was Ashbaugh who dubbed it the “Yang inequality”. The “optimal Cauchy–Schwarz” “trick” is what enabled Cheng and Yang [10] to extend the earlier work of H.C. Yang to the case of the clamped plate problem for bounded domains of Euclidean space (without the minimality condition). This is the “trick” that makes all extensions à la H.C. Yang. It appears in (2.19) and (2.67) in this paper. The arguments around this “trick” were later generalized by Levitin and Parnovski [22], following the commutator method via Rayleigh–Ritz. The simplified proof of Ashbaugh [2] is the final version which Yang adopted as an appendix to his paper with Cheng [9].

### 2. Proofs of the results

**Proof of Theorem 1.** Let $x_1, x_2, \ldots, x_{m+1}$ be the standard coordinate functions of the Euclidean space $\mathbb{R}^{n+1}$; then $M \subset S^m(1) = \{(x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1}; \sum_{\alpha=1}^{m+1} x_{\alpha}^2 = 1\}$. Since $M$ is minimal in $S^m(1)$, we have (cf. [8])

$$
\Delta x_\alpha = -nx_\alpha, \quad \alpha = 1, \ldots, m + 1.
$$

(2.1)

Let $u_i$ be the $i$-th orthonormal eigenfunction of the problem (1.10) corresponding to the eigenvalue $\lambda_i$, $i = 1, 2, \ldots$, that is, $u_i$ satisfies

$$
\Delta^2 u_i = \lambda_i u_i \quad \text{in } D, \quad u_i|_{\partial D} = \frac{\partial u_i}{\partial v}|_{\partial D} = 0, \quad \int_D u_i u_j = \delta_{ij}, \quad \forall i, j.
$$

(2.2)
For each $i = 1, \ldots, k$ and $\alpha = 1, \ldots, m + 1$, following Payne, Pólya and Weinberger we consider the functions $\phi_{\alpha i} : D \to \mathbb{R}$ given by

$$\phi_{\alpha i} = x_\alpha u_i - \sum_{j=1}^{k} a_{\alpha ij} u_j,$$

(2.3)

where

$$a_{\alpha ij} = \int_{D} x_\alpha u_i u_j.$$

We have

$$\int_{D} u_j \phi_{\alpha i} = 0, \quad \forall i, j = 1, \ldots, k, \alpha = 1, \ldots, m + 1.$$  

(2.4)

Moreover, since $\phi_{\alpha i} = \frac{\partial \phi_{\alpha i}}{\partial \nu} = 0$ on $\partial D$, we have the well-known inequality

$$\lambda_{k+1} \leq \frac{\int_{D} \phi_{\alpha i} \Delta^2 \phi_{\alpha i}}{\int_{D} \phi_{\alpha i}^2}. $$

(2.5)

From (2.1) we have

$$\Delta(x_\alpha u_i) = -nx_\alpha u_i + 2\langle \nabla x_\alpha, \nabla u_i \rangle + x_\alpha \Delta u_i$$

(2.6)

and

$$\Delta^2(x_\alpha u_i) = -n\Delta(x_\alpha u_i) - nx_\alpha \Delta u_i + 2\Delta(\nabla x_\alpha, \nabla u_i) + 2\langle \nabla x_\alpha, \nabla (\Delta u_i) \rangle + \lambda_i x_\alpha u_i,$$

(2.7)

where $\nabla$ is the gradient operator of $M$. It follows from (2.2), (2.3), (2.5) and (2.8) that

$$\int_{D} \phi_{\alpha i} \Delta^2 \phi_{\alpha i}$$

$$= \int_{D} \phi_{\alpha i} \left\{ \Delta^2 \left( x_\alpha u_i - \sum_{j=1}^{k} a_{\alpha ij} u_j \right) \right\}$$

$$= \int_{D} \phi_{\alpha i} \left( \Delta^2(x_\alpha u_i) - \sum_{j=1}^{k} a_{\alpha ij} \Delta u_j \right)$$

$$= \int_{D} \phi_{\alpha i} \Delta^2(x_\alpha u_i)$$

$$= \int_{D} \phi_{\alpha i} \left( -n\Delta(x_\alpha u_i) - nx_\alpha \Delta u_i + 2\Delta(\nabla x_\alpha, \nabla u_i) + 2\langle \nabla x_\alpha, \nabla (\Delta u_i) \rangle + \lambda_i x_\alpha u_i \right)$$

$$= \lambda_i \| \phi_{\alpha i} \|^2 + \int_{D} \left( x_\alpha u_i - \sum_{j=1}^{k} a_{\alpha ij} u_j \right) \left( -n\Delta(x_\alpha u_i) - nx_\alpha u_i + 2\Delta(\nabla x_\alpha, \nabla u_i) + 2\langle \nabla x_\alpha, \nabla (\Delta u_i) \rangle \right)$$

$$= \lambda_i \| \phi_{\alpha i} \|^2 + \int_{D} x_\alpha u_i \left( -n\Delta(x_\alpha u_i) - nx_\alpha \Delta u_i + 2\Delta(\nabla x_\alpha, \nabla u_i) + 2\langle \nabla x_\alpha, \nabla (\Delta u_i) \rangle \right) - \sum_{j=1}^{k} a_{\alpha ij} b_{\alpha ij},$$

(2.8)

where $\| g \|^2 = \int_{D} g^2$ and

$$b_{\alpha ij} = \int_{D} u_j \left( -n\Delta(x_\alpha u_i) - nx_\alpha \Delta u_i + 2\Delta(\nabla x_\alpha, \nabla u_i) + 2\langle \nabla x_\alpha, \nabla (\Delta u_i) \rangle \right).$$

(2.9)
Now we make some technical calculations. Using integration by parts, (2.1), (2.2) and (2.4), we have

\[
\int_D (\Delta u_j)(\nabla x_\alpha, \nabla u_i) - \int_D (\Delta u_i)(\nabla x_\alpha, \nabla u_j)
= \int_D (\nabla x_\alpha, (\Delta u_j)\nabla u_i) - \int_D (\nabla x_\alpha, (\Delta u_i)\nabla u_j)
= - \int_D x_\alpha \text{div}(\Delta u_j)\nabla u_i + \int_D x_\alpha \text{div}(\Delta u_i)\nabla u_j
\]

\[
= - \int_D [x_\alpha \nabla(\Delta u_j), \nabla u_i] + [x_\alpha \nabla(\Delta u_i), \nabla u_j]
= \int_D u_i \text{div}(x_\alpha \nabla(\Delta u_j)) - \int_D u_j \text{div}(x_\alpha \nabla(\Delta u_i))
\]

\[
= \int_D (\lambda_j x_\alpha u_i u_j + [u_i \nabla x_\alpha, \nabla(\Delta u_j)]) - \int_D (\lambda_i x_\alpha u_i u_j + [u_j \nabla x_\alpha, \nabla(\Delta u_i)])
= (\lambda_j - \lambda_i) a_{\alpha ij} - \int_D \Delta u_j \text{div}(u_i \nabla x_\alpha) + \int_D \Delta u_i \text{div}(u_j \nabla x_\alpha)
\]

\[
= (\lambda_j - \lambda_i) a_{\alpha ij} - \int_D \Delta u_j \left(\nabla u_i, \nabla x_\alpha - n x_\alpha u_i\right) + \int_D \Delta u_i \left(\nabla u_j, \nabla x_\alpha - n x_\alpha u_j\right),
\]

which implies that

\[
2 \int_D (\Delta u_j)(\nabla x_\alpha, \nabla u_i) - 2 \int_D (\Delta u_i)(\nabla x_\alpha, \nabla u_j) = (\lambda_j - \lambda_i) a_{\alpha ij} + \int_D n \left((\Delta u_j)x_\alpha u_i - (\Delta u_i)x_\alpha u_j\right),
\]

(2.10)

where \(\text{div}(Z)\) denotes the divergence of \(Z\). We also have

\[
2 \int_D u_j \Delta (\nabla x_\alpha, \nabla u_i) + 2 \int_D u_j [\nabla x_\alpha, \nabla(\Delta u_i)]
= 2 \int_D (\Delta u_j)(\nabla x_\alpha, \nabla u_i) - 2 \int_D (\Delta u_i) \text{div}(u_j \nabla x_\alpha)
\]

\[
= 2 \int_D (\Delta u_j)(\nabla x_\alpha, \nabla u_i) - 2 \int_D (\Delta u_i)(\nabla x_\alpha, \nabla u_j) + 2n \int_D x_\alpha u_j \Delta u_i
\]

(2.12)

and

\[
\int_D u_j (-n \Delta (x_\alpha u_i)) = -n \int_D x_\alpha u_i \Delta u_j.
\]

(2.13)

Substituting (2.11)–(2.13) into (2.9), we get

\[
b_{\alpha ij} = (\lambda_j - \lambda_i) a_{\alpha ij}.
\]

(2.14)

It follows from (2.5), (2.8) and (2.14) that

\[
(\lambda_{k+1} - \lambda_i) \|\phi_{\alpha i}\|^2 \leq p_{\alpha i} + \sum_{j=1}^{k} (\lambda_j - \lambda_i) a_{\alpha ij}^2,
\]

(2.15)

where

\[
p_{\alpha i} = \int_D x_\alpha u_i (-n \Delta (x_\alpha u_i) - n x_\alpha \Delta u_i + 2 \Delta (\nabla x_\alpha, \nabla u_i) + 2 \nabla x_\alpha, \nabla(\Delta u_i))
\]

(2.16)
Setting
\[ c_{\alpha ij} = \int_D u_j \left( (\nabla x_\alpha, \nabla u_i) - \frac{n x_\alpha u_i}{2} \right), \]  
we have
\[ c_{\alpha ij} + c_{\alpha ji} = \int_D \left( (\nabla x_\alpha, \nabla (u_i u_j)) - n x_\alpha u_i u_j \right) = -\int_D u_i u_j \Delta x_\alpha - \int_D (n x_\alpha u_i u_j) = 0 \]  
and
\[ \int D (\nabla x_\alpha, \nabla u_i) \frac{n x_\alpha u_i}{2} = \int D (\nabla x_\alpha, \nabla u_i) - \frac{n x_\alpha u_i}{2} \int D c_{\alpha ij} c_{\alpha ij} \]  
For any \( \theta > 0 \), it follows from (2.4) and Schwarz inequality that
\[ \int D (\nabla x_\alpha, \nabla u_i) \frac{n x_\alpha u_i}{2} = \int D (\nabla x_\alpha, \nabla u_i) - \frac{n x_\alpha u_i}{2} \int D c_{\alpha ij} c_{\alpha ij} \]  
and combining (2.19) and (2.20), one gets
\[ q_{\alpha i} = \int D (n x_\alpha^2 u_i^2 - 2 x_\alpha u_i (\nabla x_\alpha, \nabla u_i)) \]  
Multiplying (2.22) by \((\lambda_{k+1} - \lambda_i)\) and using (2.15), we obtain
\[ (\lambda_{k+1} - \lambda_i) \left( q_{\alpha i} + 2 \sum_{j=1}^k a_{\alpha ij} c_{\alpha ij} \right) \leq (\lambda_{k+1} - \lambda_i) \left( \| \phi_{\alpha i} \|^2 + \frac{1}{\theta} \left( \| (\nabla x_\alpha, \nabla u_i) - \frac{n x_\alpha u_i}{2} \|^2 - \sum_{j=1}^k c_{\alpha ij}^2 \right) \right) \]  
Putting \( \theta = (\lambda_{k+1} - \lambda_i)^{1/2} \gamma, \gamma > 0 \), in the above inequality, we get
\[ (\lambda_{k+1} - \lambda_i) q_{\alpha i} + 2 \sum_{j=1}^k (\lambda_{k+1} - \lambda_i) a_{\alpha ij} c_{\alpha ij} \leq \gamma (\lambda_{k+1} - \lambda_i)^{1/2} \left( p_{\alpha i} + \sum_{j=1}^k (\lambda_i - \lambda_j) a_{\alpha ij}^2 \right) + \frac{(\lambda_{k+1} - \lambda_i)^{1/2}}{\gamma} \left( \| (\nabla x_\alpha, \nabla u_i) - \frac{n x_\alpha u_i}{2} \|^2 - \sum_{j=1}^k c_{\alpha ij}^2 \right). \]
Summing over \( i \) for (2.24) we obtain

\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) q_{ai} + 2 \sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_j) a_{aij} c_{aij} \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} \left( p_{ai} + \sum_{j=1}^{k} (\lambda_i - \lambda_j) a_{aij} \right) + \frac{1}{\gamma} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} \left( (\nabla x_i, \nabla u_i) - \frac{nx_a u_i}{2} \right)^2 \leq \sum_{j=1}^{k} c_{aij}^2
\]

\[
= \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} \left( p_{ai} + \frac{1}{\gamma} \left( (\nabla x_i, \nabla u_i) - \frac{nx_a u_i}{2} \right)^2 \right) + \frac{1}{\gamma} \sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} (\lambda_i - \lambda_j) a_{aij}^2 \sum_{j=1}^{k} c_{aij}^2.
\]

Let us compute

\[
\sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} (\lambda_i - \lambda_j) a_{aij}^2 = \sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} (\lambda_{k+1} - \lambda_j)^{1/2} (\lambda_i - \lambda_j) a_{aij}^2 \]

\[
= -\frac{1}{\gamma} \sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} (\lambda_{k+1} - \lambda_j)^{1/2} (\lambda_i - \lambda_j)^2 a_{aij}^2
\]

(2.26)

and

\[
-\frac{1}{\gamma} \sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} c_{aij}^2 = -\frac{1}{\gamma} \sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} (\lambda_{k+1} - \lambda_j)^{1/2} c_{aij}^2.
\]

(2.27)

Since \( a_{aij} = a_{aji}, c_{aij} = -c_{aji} \), we have

\[
2 \sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_i) a_{aij} c_{aij} = -\sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_i - \lambda_j) a_{aij} c_{aij}
\]

\[
\geq -\frac{1}{\gamma} \sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} (\lambda_{k+1} - \lambda_j)^{1/2} (\lambda_i - \lambda_j)^2 a_{aij}^2
\]

\[
- \frac{2}{\gamma} \sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} (\lambda_{k+1} - \lambda_j)^{1/2} c_{aij}^2.
\]

(2.28)

Introducing (2.26)–(2.28) into (2.25), we get

\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) q_{ai} \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} \left( p_{ai} + \frac{1}{\gamma} \left( (\nabla x_i, \nabla u_i) - \frac{nx_a u_i}{2} \right)^2 \right)
\]

(2.29)

Summing over \( \alpha \) we have

\[
\sum_{i=1}^{m+1} (\lambda_{k+1} - \lambda_i) \sum_{\alpha=1}^{m+1} q_{\alpha i} \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} \sum_{\alpha=1}^{m+1} \sum_{\alpha=1}^{m+1} \left( p_{\alpha i} + \frac{1}{\gamma} \left( (\nabla x_i, \nabla u_i) - \frac{nx_a u_i}{2} \right)^2 \right)
\]

(2.30)

From the definition of \( p_{\alpha i} \) and \( \sum_{\alpha=1}^{m+1} x_{\alpha 2} = 1 \), it follows that

\[
\sum_{\alpha=1}^{m+1} p_{\alpha i} = n \int_{D} \sum_{\alpha=1}^{m+1} \left| \nabla (x_a u_i) \right|^2 - n \int_{D} \left( \sum_{\alpha=1}^{m+1} x_{\alpha 2} \right) u_i \Delta u_i
\]

\[
+ 2 \int_{D} \sum_{\alpha=1}^{m+1} \Delta (x_a u_i) (\nabla x_a, \nabla u_i) + \int_{D} \left( \sum_{\alpha=1}^{m+1} x_{\alpha 2} \right) u_i \nabla (\Delta u_i)
\]
Thus we have

\[
\sum_{\alpha=1}^{m+1} |\nabla x_{\alpha}|^2 = \frac{1}{2} \Delta \left( \sum_{\alpha=1}^{m+1} x_{\alpha}^2 \right) - \left( \sum_{\alpha=1}^{m+1} x_{\alpha} \Delta x_{\alpha} \right) = n. \tag{2.32}
\]

Therefore,

\[
\sum_{\alpha=1}^{m+1} \langle \nabla x_{\alpha}, \nabla u_i \rangle^2 = \sum_{\alpha=1}^{m+1} \left( \langle \nabla u_i \rangle x_{\alpha} \right)^2 = |\nabla u_i|^2. \tag{2.33}
\]

Thus we have

\[
\sum_{\alpha=1}^{m+1} p_{\alpha i} = n^2 + (2n + 4) |\nabla u_i|^2. \tag{2.34}
\]

Also one has

\[
\sum_{\alpha=1}^{m+1} q_{\alpha i} = \int_D \left( n u_i^2 \sum_{\alpha=1}^{m+1} x_{\alpha}^2 - \frac{1}{2} \left( \nabla \left( \sum_{\alpha=1}^{m+1} x_{\alpha}^2 \right) \cdot \nabla u_i^2 \right) \right) = \int_D n u_i^2 = n \tag{2.35}
\]

and

\[
\sum_{\alpha=1}^{m+1} \left\| \langle \nabla x_{\alpha}, \nabla u_i \rangle - \frac{n x_{\alpha} u_i}{2} \right\|^2 = \sum_{\alpha=1}^{m+1} \int_D \left( \langle \nabla x_{\alpha}, \nabla u_i \rangle - \frac{n x_{\alpha} u_i}{2} \right)^2.
\]

\[
\quad = \sum_{\alpha=1}^{m+1} \int_D \left( \langle \nabla x_{\alpha}, \nabla u_i \rangle^2 - n x_{\alpha} u_i \langle \nabla x_{\alpha}, \nabla u_i \rangle + \frac{n x_{\alpha}^2 u_i^2}{4} \right)
\]

\[
\quad = \int_D \left( |\nabla u_i|^2 + \frac{n^2 u_i^2}{4} \right)
\]

\[
\quad = |\nabla u_i|^2 + \frac{n^2}{4}, \tag{2.36}
\]

\[
|\nabla u_i|^2 = \int_D u_i (-\Delta u_i) \leq \left( |\nabla u_i|^2 \Delta u_i \right)^{1/2} = \lambda_i^{1/2}. \tag{2.37}
\]

Introducing (2.34)-(2.37) into (2.30), we have

\[
n \sum_{i=1}^{k} \left( \lambda_{k+1} - \lambda_i \right) \leq \sum_{i=1}^{k} \left( \lambda_{k+1} - \lambda_i \right)^{1/2} \left\{ \gamma \left( n^2 + (2n + 4) |\nabla u_i|^2 \right) + \frac{1}{\gamma} \left( |\nabla u_i|^2 + \frac{n^2}{4} \right) \right\}
\]

\[
\quad \leq \sum_{i=1}^{k} \left( \lambda_{k+1} - \lambda_i \right)^{1/2} \left\{ \gamma \left( n^2 + (2n + 4) \lambda_i^{1/2} + n^2 \right) + \frac{1}{\gamma} \left( \frac{n^2}{4} + \lambda_i^{1/2} \right) \right\},
\]

that is

\[
\lambda_{k+1} - \frac{1}{k} \sum_{i=1}^{k} \lambda_i \leq \frac{1}{kn} \left\{ \gamma \sum_{i=1}^{k} \left( \lambda_{k+1} - \lambda_i \right)^{1/2} \left( (2n + 4) \lambda_i^{1/2} + n^2 \right) + \frac{1}{\gamma} \sum_{i=1}^{k} \left( \lambda_{k+1} - \lambda_i \right)^{1/2} \left( \frac{n^2}{4} + \lambda_i^{1/2} \right) \right\}. \tag{2.38}
\]
Taking
\[
\gamma = \left\{ \frac{\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^{1/2}(\lambda_i^{1/2} + n^2)}{\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^{1/2}(2n + 4)\lambda_i^{1/2} + n^2} \right\}^{1/2}
\]
in (2.38), one gets
\[
\lambda_{k+1} = \frac{1}{k} \sum_{i=1}^{k} \lambda_i \leq \frac{1}{kn} \left( \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^{1/2}(2n + 4)\lambda_i^{1/2} + n^2 \right)^{1/2} \cdot \left( \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^{1/2}(4\lambda_i^{1/2} + n^2) \right)^{1/2}.
\]
This completes the proof of Theorem 1. □

Before proving Corollary 1, let us recall the following version of the “Chebyshev inequality” (see, for example, p. 43 of [13]).

**Lemma 1** (Weighted Chebyshev inequality). Let \( \{a_i\}_{i=1}^{k} \) and \( \{b_i\}_{i=1}^{k} \) be two similarly ordered real sequences, and let \( \{\omega_i\}_{i=1}^{k} \) be a sequence of nonnegative weights. Then the following inequality holds:
\[
\left( \sum_{i=1}^{k} \omega_i a_i \right) \left( \sum_{i=1}^{k} \omega_i b_i \right) \leq \left( \sum_{i=1}^{k} \omega_i \right) \left( \sum_{i=1}^{k} \omega_i a_i b_i \right).
\]
(2.39)

**Proof of Corollary 1.** We take \( \omega_i = (\lambda_{k+1} - \lambda_i), a_i = (\lambda_{k+1} - \lambda_i)^{-1/2}, \) and \( b_i = (\lambda_{k+1} - \lambda_i)^{-1/2}\lambda_i^{1/2} \) in Lemma 1. By elementary calculations, \( a_i \) and \( b_i \) are similarly ordered (see [7] and [13] for further insight). Thus we have from (2.39) that
\[
\left\{ \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^{1/2} \right\} \left\{ \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^{1/2}\lambda_i^{1/2} \right\} \leq \left\{ \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i) \right\} \left\{ \sum_{i=1}^{k}\lambda_i^{1/2} \right\}.
\]
(2.40)
Set
\[
\Lambda_k = \frac{1}{k} \sum_{i=1}^{k} \lambda_i, \quad Q_k = \sum_{i=1}^{k} \lambda_i^{2};
\]
then we have from (1.11), (2.40) and Schwarz inequality that
\[
n^2k(\lambda_{k+1} - \Lambda_k)^2 \leq \frac{1}{k} \left\{ \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^{1/2}(2n + 4)\lambda_i^{1/2} + n^2 \right\} \cdot \left\{ \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^{1/2}(4\lambda_i^{1/2} + n^2) \right\}
\]
\[
= \frac{8(n + 2)}{k} \left\{ \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^{1/2}\lambda_i^{1/2} \right\}^2 + \frac{n^4}{k} \left\{ \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^{1/2} \right\}^2
\]
\[
+ 2\frac{(n + 4)n^2}{k} \left\{ \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i) \} \right\} \left\{ \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^{1/2}\lambda_i^{1/2} \right\}
\]
\[
\leq 8(n + 2) \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)\lambda_i + n^4 \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i) + \frac{2(n + 4)n^2}{k} \left\{ \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i) \right\} \left\{ \sum_{i=1}^{k}\lambda_i^{1/2} \right\}
\]
\[
= \left( n^4 + \frac{2(n + 4)n^2}{k} \sum_{i=1}^{k}\lambda_i^{1/2} + 8(n + 2)k\Lambda_k \right)(\lambda_{k+1} - \Lambda_k) + 8(n + 2)\left( k\Lambda_k^2 - Q_k \right).
\]
(2.41)
Dividing (2.41) by \( n^2k \) and simplifying, we get
\[
\left\{ \lambda_{k+1} - \Lambda_k - \frac{1}{2n^2k} \left( n^4 + \frac{2(n + 4)n^2}{k} \sum_{i=1}^{k}\lambda_i^{1/2} + 8(n + 2)k\Lambda_k \right) \right\}^2
\]
\[
\leq \frac{1}{4n^4k^2} \left( n^4 + \frac{2(n + 4)n^2}{k} \sum_{i=1}^{k}\lambda_i^{1/2} + 8(n + 2)k\Lambda_k \right)^2 - \frac{8(n + 2)}{n^2k} \left( Q_k - k\Lambda_k^2 \right).
\]
(2.42)
Consequently, we have
\[
\lambda_{k+1} \leq \left( 1 + \frac{4(n+2)}{n^2} \right) \lambda_k + \frac{1}{2k} \left( n^2 + \frac{2(n+4)}{k} \sum_{i=1}^{k} \lambda_i^{1/2} \right)
\]
\[\quad + \left\{ \frac{1}{4n^4k^2} \left( n^4 + \frac{2(n+4)n^2}{k} \sum_{i=1}^{k} \lambda_i^{1/2} + 8(n+2)k \lambda_k \right)^2 - \frac{8(n+2)}{n^2k} \left( Q_k - k \lambda_k^2 \right) \right\}^{1/2}, \tag{2.43}
\]
which shows that (1.12) holds. \(\square\)

**Proof of Theorem 2.** In the proof of Theorem 1, we multiplied (2.22) by the factor \((\lambda_{k+1} - \lambda_i)\) so that the unwanted terms can be eliminated smoothly. For the proof of Theorem 2, we will not multiply (2.22) by this factor. Setting
\[
\theta = (\lambda_{k+1} - \lambda_i) \delta, \quad \delta = \left( kn^2 + (2n+4) \sum_{i=1}^{k} \lambda_i^{1/2} \right)^{-1/2} \left( \sum_{i=1}^{k} \lambda_i^{1/2} + \frac{n^2}{4} \right)^{1/2}
\]
and summing over \(i\) for (2.22), we have from (2.15) that
\[
\sum_{i=1}^{k} q_{ai} + 2 \sum_{j=1}^{k} \sum_{i=1}^{k} a_{aij} c_{aij}
\]
\[\leq \delta \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \| \phi_{ai} \|^2 + \frac{1}{(\lambda_{k+1} - \lambda_i) \delta} \left\| \langle \nabla u_\alpha, \nabla u_i \rangle - \frac{n x_\alpha u_{i}}{2} \right\|^2 - \sum_{j=1}^{k} c_{aij}^2
\]
\[\leq \delta \sum_{i=1}^{k} \left( p_{ai} + \sum_{j=1}^{k} (\lambda_i - \lambda_j) a_{aij} \right) + \sum_{i=1}^{k} \frac{1}{(\lambda_{k+1} - \lambda_i) \delta} \left\| \langle \nabla u_\alpha, \nabla u_i \rangle - \frac{n x_\alpha u_{i}}{2} \right\|^2 - \sum_{j=1}^{k} c_{aij}^2. \tag{2.45}
\]
From the antisymmetry of \(a_{aij} c_{aij}\) and \((\lambda_i - \lambda_j) a_{aij}^2\) with respect to \(i\) and \(j\), we have
\[
\sum_{i,j=1}^{k} q_{aij} c_{aij} = 0, \quad \sum_{i,j=1}^{k} (\lambda_i - \lambda_j) a_{aij}^2 = 0.
\]
Thus
\[
\sum_{i=1}^{k} q_{ai} \leq \delta \sum_{i=1}^{k} p_{ai} + \frac{1}{(\lambda_{k+1} - \lambda_i) \delta} \left\| \langle \nabla u_\alpha, \nabla u_i \rangle - \frac{n x_\alpha u_{i}}{2} \right\|^2. \tag{2.46}
\]
Summing over \(\alpha\) from 1 to \(m+1\) for (2.46), we get
\[
\sum_{i=1}^{k} \sum_{\alpha=1}^{m+1} q_{ai} \leq \delta \sum_{i=1}^{k} \sum_{\alpha=1}^{m+1} p_{ai} + \sum_{i=1}^{k} \frac{1}{(\lambda_{k+1} - \lambda_i) \delta} \sum_{\alpha=1}^{m+1} \left\| \langle \nabla u_\alpha, \nabla u_i \rangle - \frac{n x_\alpha u_{i}}{2} \right\|^2. \tag{2.47}
\]
Substituting (2.34)-(2.36) into (2.47) and using (2.37) and (2.44), we find
\[
k \leq \delta \sum_{i=1}^{k} \left( n^2 + (2n+4) \| \nabla u_i \|^2 \right) + \frac{1}{(\lambda_{k+1} - \lambda_i) \delta} \left( \| \nabla u_i \|^2 + \frac{n^2}{4} \right)
\]
\[\leq \delta \left( kn^2 + (2n+4) \sum_{i=1}^{k} \lambda_i^{1/2} \right) + \sum_{i=1}^{k} \frac{\lambda_i^{1/2} + \frac{n^2}{4}}{(\lambda_{k+1} - \lambda_i) \delta}
\]
\[= 2 \left( kn^2 + (2n+4) \sum_{i=1}^{k} \lambda_i^{1/2} \right) \left( \sum_{i=1}^{k} \lambda_i^{1/2} + \frac{n^2}{4} \right)^{1/2}. \tag{2.48}
\]
that is,
\[
\sum_{i=1}^{k} \frac{\lambda_i^{1/2} + \frac{n^2}{4}}{(\lambda_{k+1} - \lambda_i)} \geq \frac{n^2 k^2}{4}. \tag{2.49}
\]
Hence, Theorem 2 is true. \(\square\)
Proof of Theorem 3. Denote by $\Delta$ and $\nabla$ the Laplacian and the gradient operator of $M$, respectively. Let $u_i$ be the $i$-th orthonormal eigenfunction corresponding to the eigenvalue $\lambda_i$, $i = 1, \ldots, k$, that is, $u_i$ satisfies
\begin{equation}
\Delta^2 u_i = \lambda_i u_i, \quad \text{in } D, \quad u_i|_{\partial D} = \frac{\partial u_i}{\partial \nu}|_{\partial D} = 0, \quad \int_D u_i u_j = \delta_{ij}, \quad \forall i, j.
\end{equation}

Let $x_1, x_2, \ldots, x_m$ be the standard Euclidean coordinates of $\mathbb{R}^m$ and define $\phi_{\alpha i} : D \to \mathbb{R}$ by
\begin{equation}
\phi_{\alpha i} = x_\alpha u_i - \sum_{j=1}^{k} r_{\alpha ij} u_j, \quad i = 1, \ldots, k \text{ and } \alpha = 1, \ldots, m.
\end{equation}

where
\begin{equation}
r_{\alpha ij} = \int_D x_\alpha u_i u_j
\end{equation}
so that
\begin{equation}
\int_D u_j \phi_{\alpha i} = 0, \quad \forall i, j = 1, \ldots, k, \alpha = 1, \ldots, m.
\end{equation}

Hence
\begin{equation}
\lambda_{k+1} \leq \frac{\int_D \phi_{\alpha i} \Delta^2 \phi_{\alpha i}}{\int_D \phi_{\alpha i}^2}.
\end{equation}

Since $M^n$ is a minimal submanifold in $\mathbb{R}^m$, it is well known that (cf. [8])
\begin{equation}
\Delta x_\alpha = 0, \quad \alpha = 1, \ldots, m,
\end{equation}
and so
\begin{equation}
\Delta (x_\alpha u_i) = 2 \langle \nabla x_\alpha, \nabla u_i \rangle + x_\alpha \Delta u_i.
\end{equation}

It follows that
\begin{align*}
\int_D \phi_{\alpha i} \Delta^2 \phi_{\alpha i} &= \int_D \phi_{\alpha i} \left( \Delta^2 (x_\alpha u_i) - \sum_{j=1}^{k} r_{\alpha ij} \lambda_j u_j \right) \\
&= \int_D \phi_{\alpha i} \Delta^2 (x_\alpha u_i) \\
&= \int_D \phi_{\alpha i} \Delta \left( 2 \langle \nabla x_\alpha, \nabla u_i \rangle + x_\alpha \Delta u_i \right) \\
&= \int_D \phi_{\alpha i} \left( 2 \Delta (\nabla x_\alpha, \nabla u_i) + 2 \langle \nabla x_\alpha, \nabla (\Delta u_i) \rangle + \lambda_i x_\alpha u_i \right) \\
&= \lambda_i \| \phi_{\alpha i} \|^2 + 2 \int_D \left( x_\alpha u_i - \sum_{j=1}^{k} r_{\alpha ij} u_j \right) \left( \Delta (\nabla x_\alpha, \nabla u_i) + \langle \nabla x_\alpha, \nabla (\Delta u_i) \rangle \right) \\
&= \lambda_i \| \phi_{\alpha i} \|^2 + 2 \int_D x_\alpha u_i \left( \Delta (\nabla x_\alpha, \nabla u_i) + \langle \nabla x_\alpha, \nabla (\Delta u_i) \rangle \right) - 2 \sum_{j=1}^{k} r_{\alpha ij} s_{\alpha ij},
\end{align*}
where
\begin{equation}
s_{\alpha ij} = \int_D u_j (\Delta (\nabla x_\alpha, \nabla u_i) + \langle \nabla x_\alpha, \nabla (\Delta u_i) \rangle),
\end{equation}
and as before
\begin{equation}
\| \phi_{\alpha i} \|^2 = \int_D \phi_{\alpha i}^2.
\end{equation}
From Lagrange–Green theorem and (2.55), we have

\[
\int_D x_\alpha u_1 \Delta (\nabla x_\alpha, \nabla u_i) = \int_D \Delta (x_\alpha u_1) (\nabla x_\alpha, \nabla u_i)
\]

\[= 2 \int_D (\nabla x_\alpha, \nabla u_i)^2 + \frac{1}{2} \int_D \Delta u_i (\nabla x_\alpha^2, \nabla u_i), \quad (2.58)\]

\[
\int_D x_\alpha u_1 (\nabla x_\alpha, \nabla (\Delta u_i)) = \frac{1}{2} \int_D [u_i (\nabla x_\alpha^2, \nabla (\Delta u_i))]
\]

\[= -\frac{1}{2} \int_D \Delta u_i \left( (\nabla u_i, \nabla x_\alpha^2) + 2u_i|\nabla x_\alpha|^2 \right). \quad (2.59)\]

Thus

\[
\int_D x_\alpha u_1 (\nabla (\nabla x_\alpha, \nabla u_i) + (\nabla x_\alpha, \nabla (\Delta u_i))) = 2 \int_D (\nabla x_\alpha, \nabla u_i)^2 - \int_D u_i |\Delta u_i|^2. \quad (2.60)\]

It is easy to see that

\[
s_{\alpha ij} = \int_D \Delta u_j (\nabla x_\alpha, \nabla u_i) + \int_D [u_j \nabla x_\alpha, \nabla (\Delta u_i)]
\]

\[= \int_D (\Delta u_j) (\nabla x_\alpha, \nabla u_i) - \int_D (\Delta u_i) \text{div}(u_j \nabla x_\alpha)
\]

\[= \int_D (\Delta u_j) (\nabla x_\alpha, \nabla u_i) - \int_D (\Delta u_i) (\nabla u_j, \nabla x_\alpha). \quad (2.61)\]

and so we have

\[
s_{\alpha ij} = \int_D (\nabla x_\alpha, \Delta u_j \nabla u_i) - \int_D (\nabla x_\alpha, \Delta u_i \nabla u_j)
\]

\[= -\int_D x_\alpha \text{div}(\Delta u_j \nabla u_i) + \int_D x_\alpha \text{div}(\Delta u_i \nabla u_j)
\]

\[= -\int_D x_\alpha (\nabla (\Delta u_j), \nabla u_i) + \int_D x_\alpha (\nabla (\Delta u_i), \nabla u_j)
\]

\[= \int_D u_i \text{div}(x_\alpha \nabla (\Delta u_j)) - \int_D u_j \text{div}(x_\alpha \nabla (\Delta u_i))
\]

\[= (\lambda_j - \lambda_i) r_{\alpha ij} + \int_D u_i (\nabla x_\alpha, \nabla (\Delta u_j)) - \int_D u_j (\nabla x_\alpha, \nabla (\Delta u_i))
\]

\[= (\lambda_j - \lambda_i) r_{\alpha ij} - \int_D \Delta u_j \text{div}(u_i \nabla x_\alpha) + \int_D \Delta u_i \text{div}(u_j \nabla x_\alpha)
\]

\[= (\lambda_j - \lambda_i) r_{\alpha ij} - \int_D (\Delta u_j (\nabla x_\alpha, \nabla u_i) - \Delta u_i (\nabla x_\alpha, \nabla u_j))
\]

\[= (\lambda_j - \lambda_i) r_{\alpha ij} - s_{\alpha ij}, \quad (2.62)\]

which gives

\[2s_{\alpha ij} = (\lambda_j - \lambda_i) r_{\alpha ij}. \quad (2.63)\]
Substituting (2.56), (2.60) and (2.63) into (2.53), we infer

\[(\lambda_{k+1} - \lambda_i) \|\phi_{ai}\|^2 \leq \int_D \left( 4(\nabla x_\alpha \cdot \nabla u_i)^2 - 2u_i \Delta u_i |\nabla x_\alpha|^2 \right) + \sum_{j=1}^k (\lambda_i - \lambda_j) r_{aij}^2. \quad (2.64)\]

Set

\[t_{aij} = \int_D u_j (\nabla x_\alpha \cdot \nabla u_i);\]

then

\[t_{aij} = - \int_D u_i \text{Div}(u_j \nabla x_\alpha) = - \int_D u_i (\nabla u_j \cdot \nabla x_\alpha) = - t_{aiji} \quad (2.65)\]

and

\[\int_D \phi_{ai} \left( -2(\nabla x_\alpha \cdot \nabla u_i) \right) = -2 \int_D \left( x_\alpha u_i - \sum_{j=1}^k r_{aij} u_j \right) (\nabla x_\alpha \cdot \nabla u_i) \]

\[= -\frac{1}{2} \int_D u_i^2 \Delta x_\alpha^2 + 2 \sum_{j=1}^k r_{aij} t_{aij}. \quad (2.66)\]

For any constant \(\beta > 0\), we have from (2.66) and Schwarz inequality that

\[\frac{1}{2} \int_D u_i^2 \Delta x_\alpha^2 + 2 \sum_{j=1}^k r_{aij} t_{aij} = \int_D \phi_{ai} \left( -2(\nabla x_\alpha \cdot \nabla u_i) + 2 \sum_{j=1}^k t_{aij} u_j \right) \]

\[\leq \int_D \left\{ \beta \phi_{ai}^2 + \frac{1}{\beta} \left( - (\nabla x_\alpha \cdot \nabla u_i) + \sum_{j=1}^k t_{aij} u_j \right)^2 \right\} \]

\[= \beta \|\phi_{ai}\|^2 + \frac{1}{\beta} \left( \|\nabla x_\alpha \cdot \nabla u_i\|^2 - \sum_{j=1}^k t_{aij}^2 \right). \quad (2.67)\]

Multiplying (2.67) by \((\lambda_{k+1} - \lambda_i)\) and using (2.64), one gets

\[(\lambda_{k+1} - \lambda_i) \left( \frac{1}{2} \int_D u_i^2 \Delta x_\alpha^2 + 2 \sum_{j=1}^k r_{aij} t_{aij} \right) \]

\[\leq (\lambda_{k+1} - \lambda_i) \left\{ \beta \|\phi_{ai}\|^2 + \frac{1}{\beta} \left( \|\nabla x_\alpha \cdot \nabla u_i\|^2 - \sum_{j=1}^k t_{aij}^2 \right) \right\} \]

\[\leq \beta \int_D \left( 4(\nabla x_\alpha \cdot \nabla u_i)^2 - 2u_i \Delta u_i |\nabla x_\alpha|^2 \right) + \sum_{j=1}^k (\lambda_i - \lambda_j) r_{aij}^2 \]

\[+ \frac{\lambda_{k+1} - \lambda_i}{\beta_1} \left( \|\nabla x_\alpha \cdot \nabla u_i\|^2 - \sum_{j=1}^k t_{aij}^2 \right). \quad (2.68)\]

Putting \(\beta = (\lambda_{k+1} - \lambda_i)^{1/2} \beta_1, \beta_1 = (2n + 4)^{-1/2}\) in the above inequality, we get

\[(\lambda_{k+1} - \lambda_i) \left( \frac{1}{2} \int_D u_i^2 \Delta x_\alpha^2 + 2 \sum_{j=1}^k (\lambda_{k+1} - \lambda_i) r_{aij} t_{aij} \right) \]

\[\leq \beta_1 (\lambda_{k+1} - \lambda_i)^{1/2} \left( \int_D \left( 4(\nabla x_\alpha \cdot \nabla u_i)^2 - 2u_i \Delta u_i |\nabla x_\alpha|^2 \right) + \sum_{j=1}^k (\lambda_i - \lambda_j) r_{aij}^2 \right) \]

\[+ \frac{(\lambda_{k+1} - \lambda_i)^{1/2}}{\beta_1} \left( \|\nabla x_\alpha \cdot \nabla u_i\|^2 - \sum_{j=1}^k t_{aij}^2 \right). \quad (2.69)\]
We infer from (2.69) by summing over $i$ that

$$
\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i) \left( \frac{1}{2} \int_{D} u_i^2 \Delta x_\alpha^2 \right) + 2 \sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_j) r_{aij} t_{aij}
$$

$$\leq \beta_1 \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} \left( \int_{D} (4\langle \nabla x_\alpha, x_i \rangle^2 - 2u_i \Delta u_i \langle x_i \rangle^2) + \sum_{j=1}^{k} (\lambda_i - \lambda_j) r_{aij}^2 \right)
$$

$$+ \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} \beta_1 \left( \| \langle \nabla x_\alpha, x_i \rangle \|^2 - \sum_{j=1}^{k} t_{aij}^2 \right)
$$

$$= B_\alpha + \beta_1 \sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} (\lambda_i - \lambda_j) r_{aij}^2 - \frac{1}{\beta_1} \sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} t_{aij}^2.
$$

(2.70)

where

$$B_\alpha = \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} \left( \beta_1 \int_{D} (4\langle \nabla x_\alpha, x_i \rangle^2 - 2u_i \Delta u_i \langle x_i \rangle^2) + \frac{1}{\beta_1} \| \langle \nabla x_\alpha, x_i \rangle \|^2 \right).
$$

(2.71)

Since $r_{aij} = r_{aji}$, $t_{aij} = -t_{aji}$, one can use the same arguments as in (2.26)-(2.28) to show that

$$2 \sum_{i,j=1}^{k} (\lambda_{k+1} - \lambda_i) r_{aij} t_{aij} \geq \beta_1 \sum_{i,j=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} (\lambda_i - \lambda_j) r_{aij}^2 - \frac{1}{\beta_1} \sum_{i,j=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} t_{aij}^2.
$$

(2.72)

Substituting (2.72) into (2.70), we have

$$\sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_i)^{1/2} (\lambda_i - \lambda_j) r_{aij}^2 \leq B_\alpha.
$$

(2.73)

Using the definition of $B_\alpha$ and summing over $\alpha$ from (2.73), we have

$$\sum_{i=1}^{k} \sum_{j=1}^{m} (\lambda_{k+1} - \lambda_i) \left( \frac{1}{2} \int_{D} u_i^2 \Delta x_\alpha^2 \right)
$$

$$\leq \sum_{i=1}^{k} \sum_{j=1}^{m} (\lambda_{k+1} - \lambda_i)^{1/2} \sum_{\alpha=1}^{m} \left( 4\beta_1 + \frac{1}{\beta_1} \right) \| \langle \nabla x_\alpha, x_i \rangle \|^2 - 2\beta_1 \int_{D} u_i \Delta u_i \| x_i \|^2 \right).
$$

(2.74)

Since $M$ is an $n$-dimensional minimal submanifold in $\mathbb{R}^m$, we have (cf. [12, p. 208])

$$\sum_{\alpha=1}^{m} \Delta x_\alpha^2 = 2 \sum_{\alpha=1}^{m} |x_\alpha|^2 = 2n.
$$

(2.75)

Also, it is easy to see that

$$\sum_{\alpha=1}^{m} \| \langle \nabla x_\alpha, x_i \rangle \|^2 = \| \nabla x_i \|^2
$$

(2.76)

and

$$\| \nabla x_i \|^2 = \int_{D} u_i (-\Delta u_i) \leq (\| u_i \|^2 \Delta u_i)^{1/2} \lambda_i^{1/2}.
$$

(2.77)

Substituting (2.75)-(2.77) into (2.74) and using $\beta_1 = (2n + 4)^{-1/2}$, we get

$$k \lambda_{k+1} - \sum_{i=1}^{k} \lambda_i \leq \left( \frac{8(n + 2)}{\pi^2} \right)^{1/2} \sum_{i=1}^{k} (\lambda_i (\lambda_{k+1} - \lambda_i))^{1/2}.
$$

This proves (1.14).
In order to prove (1.15), let us put
$$\beta = \frac{(\lambda_{k+1} - \lambda_i) \mu}{\sum_{l=1}^{k} \lambda_{l}^{1/2}}, \quad \mu = \frac{nk}{4(n+2)}$$
and sum over $i$ for (2.67); then
$$\frac{1}{2} \sum_{i=1}^{k} \int_{D} u_i^2 \Delta X_{\alpha}^2 + 2 \sum_{i=1}^{k} \sum_{j=1}^{k} r_{\alpha ij} t_{\alpha ij}$$
$$\leq \frac{\mu}{\sum_{l=1}^{k} \lambda_{l}^{1/2}} \sum_{i=1}^{k} \left( \frac{\lambda_{k+1} - \lambda_i}{\lambda_{l}^{1/2}} \| \phi_{\alpha ij} \|^2 + \sum_{l=1}^{k} \lambda_{l}^{1/2} \sum_{i=1}^{k} \frac{1}{\lambda_{k+1} - \lambda_i} \left( \| \nabla X_{\alpha}, \nabla u_i \|^2 - \sum_{j=1}^{k} r_{\alpha ij}^2 \right) \right)$$
$$\leq \frac{\mu}{\sum_{l=1}^{k} \lambda_{l}^{1/2}} \sum_{i=1}^{k} \left( \int_{D} (4(\nabla X_{\alpha}, \nabla u_i)^2 - 2u_i \Delta u_i |\nabla X_{\alpha}|^2) + \sum_{j=1}^{k} (\lambda_i - \lambda_j) r_{\alpha ij}^2 \right)$$
$$+ \sum_{i=1}^{k} \frac{1}{\lambda_{k+1} - \lambda_i} \left( \| \nabla X_{\alpha}, \nabla u_i \|^2 - \sum_{j=1}^{k} r_{\alpha ij}^2 \right).$$
(2.78)
Since $r_{\alpha ij} = r_{\alpha ji}, t_{\alpha ij} = -t_{\alpha ji}$, we have
$$2 \sum_{i=1}^{k} \sum_{j=1}^{k} r_{\alpha ij} t_{\alpha ij} = 0, \quad \sum_{i=1}^{k} \sum_{j=1}^{k} (\lambda_i - \lambda_j) r_{\alpha ij}^2 = 0.$$
(2.79)
It then follows from (2.78) that
$$\frac{1}{2} \sum_{i=1}^{k} \int_{D} u_i^2 \Delta X_{\alpha}^2 \leq \frac{\mu}{\sum_{l=1}^{k} \lambda_{l}^{1/2}} \sum_{i=1}^{k} \left( \int_{D} (4(\nabla X_{\alpha}, \nabla u_i)^2 - 2u_i \Delta u_i |\nabla X_{\alpha}|^2) \right) + \sum_{i=1}^{k} \frac{1}{\lambda_{k+1} - \lambda_i} \sum_{i=1}^{k} r_{\alpha ij}^2.$$
(2.80)
After summing over $\alpha$ for (2.80) and using (2.75)-(2.77) we get
$$nk \leq \frac{\mu}{\sum_{l=1}^{k} \lambda_{l}^{1/2}} \sum_{i=1}^{k} (4 + 2n) \lambda_{l}^{1/2} + \sum_{i=1}^{k} \frac{\lambda_{l}^{1/2}}{\lambda_{k+1} - \lambda_i} \sum_{i=1}^{k} \frac{\lambda_{l}^{1/2}}{\lambda_{k+1} - \lambda_i} \sum_{i=1}^{k} r_{\alpha ij}^2.$$
(2.81)
Thus (1.15) is true. The proof of Theorem 3 is complete. \(\square\)

**Proof of Corollary 2.** Set $A_k = \frac{1}{k} \sum_{i=1}^{k} \lambda_i, W_k = \sum_{i=1}^{k} \lambda_i^2$; then we have from (2.14) and Schwarz inequality that
$$(\lambda_{k+1} - A_k)^2 \leq \frac{8(n+2)}{nk^2} \left( \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_{l}^{1/2} \right)^2$$
$$\leq \frac{8(n+2)}{nk^2} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i$$
$$= \frac{8(n+2)}{nk^2} (k\lambda_{k+1} A_k - W_k)$$
$$= \frac{8(n+2)}{nk^2} (\lambda_{k+1} - A_k) - \frac{8(n+2)}{nk^2} (W_k - kA_k^2),$$
(2.82)
that is,
$$(\lambda_{k+1} - A_k - \frac{4(n+2)}{nk^2} A_k)^2 \leq \frac{16(n+2)^2}{nk^2} A_k^2 - \frac{8(n+2)}{nk^2} (W_k - kA_k^2)$$
$$= \frac{16(n+2)^2}{nk^2} A_k^2 - \frac{8(n+2)}{nk^2} \sum_{j=1}^{k} (\lambda_j - A_k)^2.$$
(2.83)
Consequently, we have

\[\lambda_{k+1} \leq \left(1 + \frac{4(n+2)}{n^2} \right) \left[ \frac{1}{k} \sum_{i=1}^{k} \lambda_i + \left( \frac{4(n+2)}{n^2} \right)^2 \sum_{i=1}^{k} \lambda_i - \frac{8(n+2)}{n^2} \sum_{i=1}^{k} \left( \lambda_i - \frac{1}{k} \sum_{j=1}^{k} \lambda_j \right)^2 \right]^{1/2}.\]

This completes the proof of Corollary 2. □

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