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On Finite Linear Groups II

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1. INTRODUCTION

The primary purpose of this paper is to study the following situation.

(i) *G is a finite group. p is an odd prime and P is a Sylow p-group of G with $|P| = q$.*

(ii) *$G = G'$. $\mathbf{Z}(G)$ is cyclic and $G/\mathbf{Z}(G)$ is simple.*

(iii) *P is abelian. If $x \in G - \mathbf{N}_G(P)$ then $P \cap x^{-1}Px = \langle 1 \rangle$.*

(iv) *$\mathbf{C}_G(P) = P \times \mathbf{Z}(G)$ and $|\mathbf{N}_G(P) : \mathbf{C}_G(P)| = 2$.* $\qquad (*)$

The main result of this paper follows.

THEOREM 1. *Suppose that $(*)$ is satisfied with $q \geqslant 7$. Let χ be a faithful irreducible character of G with $\chi(1) \not\equiv 0 \pmod p$. Then one of the following holds.*

(I) $\mathbf{Z}(G) = \langle 1 \rangle$.

(II) $|\mathbf{Z}(G)| = 2$. *$2q \pm 1$ is a prime power and $G/\mathbf{Z}(G) \approx PSL_2(2q \pm 1)$.*

(III) *$|\mathbf{Z}(G)|$ is even and G contains exactly one conjugate class of non-central involutions.*

(IV) *For $q \geqslant 19$, $\chi(1) \geqslant q((q-7)/4) - 2$ if $q \equiv 3 \pmod 4$. $\chi(1) \geqslant q((q-9)/4) - 2$ if $q \equiv 1 \pmod 4$. Furthermore if $q = 7$ or 11 then $\chi(1) \geqslant 2q - 2$. If $q = 13$ or 17 then $\chi(1) > q - 2$.*

The following result which was announced previously [9, Theorem 8.3.4(iii)] is a consequence of Theorem 1.

THEOREM 2. *Let G be a finite group. Let $p > 5$ be a prime and let P be a Sylow p-group of G. Assume that G has a faithful complex irreducible represen-*

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tation of degree $p-2$. *Then either* $P \lhd G$ *or* $p-1=2^a$ *and* $G \approx SL_2(p-1) \times A$ *where* A *is an abelian group.*

In case $p=5$ Theorems 1 and 2 are both false since there exists a group G with a center Z of order 3 and $G/Z \approx \mathfrak{A}_6$ which has a faithful three-dimensional complex irreducible representation. As all three-dimensional complex groups are known it can be seen by inspection that up to an abelian factor the above mentioned group is the only counterexample for $p=5$.

Theorem 2 is proved by using Theorem 1 to reduce to the case where the center of G has order 1. In this case the result is an immediate consequence of [8, Theorem 2]. The main result of [6] is used to reduce to the case that P has order p. The argument here is quite straightforward and is similar to that used in [8].

Section 5 contains a proof of the following result which is handled by similar methods and is occasionally useful.

THEOREM 3. *Let* G *be a finite group with* $G = G'$. *Let* p *be an odd prime. Suppose* G *has an abelian Sylow* p*-group* P *which satisfies the following conditions*

(i) *If* $x \in G - \mathbf{N}_G(P)$ *then* $x^{-1}Px \cap \mathbf{N}_G(P) = \langle 1 \rangle$.

(ii) *There exists a subgroup* H *of* G *such that* $\mathbf{C}_G(P) = \mathbf{C}_G(u) = P \times H$ *for all* $u \in P$, $u \neq 1$.

(iii) $|\mathbf{N}_G(P) : \mathbf{C}_G(P)| = 3$.

Then if λ *is a nonprincipal linear character of* $P \times H/H$

$$(1_{P\times H} - \lambda)^G = 1_G + \theta - \zeta - \eta$$

where θ, ζ, η *are irreducible characters of* G *and* $\theta(1)\,\zeta(1)\,\eta(1)$ *is the square of a rational integer.*

As an immediate corollary of Theorem 3 one gets the following theorem.

THEOREM 4. *Suppose that* G *has a Sylow* p*-group* P *of order* $p > 3$. *Assume that* $G = G'$ *and* $|\mathbf{N}_G(P) : \mathbf{C}_G(P)| = 3$. *Let* $1_G, \theta, \eta, \chi_1, \ldots, \chi_{(p-1)/3}$ *be all the irreducible characters in the principal* p*-block of* G *where* $\chi_1, \ldots, \chi_{(p-1)/3}$ *are exceptional. Then* $\theta(1)\,\eta(1)\,\chi_1(1)$ *is the square of a rational integer.*

The situation described in Theorem 3 occurs infinitely often. For instance let $G_1(q) = PSL_3(q)$ and let $G_{-1}(q) = PSU_3(q)$. Then for $\epsilon = \pm 1$, $G_\epsilon(q)$ contains a subgroup $A = P \times H$ which satisfies the assumptions of Theorem 3 of order $(q^2+\epsilon q+1)/(q^2+\epsilon q+1, 3)$. The degrees of the given characters are q^3, $q^2+\epsilon q$, and $(q^2-1)(q-\epsilon)$.

The notation used is standard and is the same as that used in [8] except that the gothic letters have been replaced by ordinary capital letters.

2. Some Consequences of Condition (*)

Throughout this section it will be assumed that condition $(*)$ *is satisfied.*

Lemma 2.1. *P is cyclic.*

Proof. This follows from [11].

The results of [11] are not really necessary for this paper since the results of [2] yield sufficient information about the characters of G for what is needed here. However in view of Lemma 2.1 the theory of blocks with a cyclic defect group becomes available and can be used to simplify some of the arguments.

Let $Z = \mathbf{Z}(G)$. Let $|Z| = m$ and let $\mu_1 = 1_Z, \ldots, \mu_m$ be all the irreducible characters of Z. For $i = 1, \ldots, m$ let B_i be the p-block of G corresponding to μ_i.

Let $P = \langle y \rangle$.

Lemma 2.2. *Any two involutions in* G/Z *are conjugate.*

Proof. Since G/Z satisfies $(*)$ it may be assumed after a change of notation that $Z = \langle 1 \rangle$ and $G = G/Z$. Since $|\mathbf{N}_G(P) : P| = 2$ it follows that $\mathbf{N}_G(P)$ contains exactly one conjugate class of involutions. Suppose that x is an involution in G which is not conjugate to any element of $\mathbf{N}_G(P)$. Then y is not the product of two conjugates of x. Therefore

$$\sum \frac{\xi(x)^2\, \xi(y)}{\xi(1)} = \sum \frac{\xi(x)\, \xi(1)\, \xi(y)}{\xi(1)} = \sum \frac{\xi(1)^2\, \xi(y)}{\xi(1)} = 0, \qquad (2.3)$$

where ξ ranges over all the irreducible characters of G. Since $\xi(y) = 0$ unless ξ is in B_1 the same equations hold if ξ runs over the irreducible characters in B_1. Let

$$\underset{1}{\circ} \text{———} \underset{\eta}{\circ} \text{———} \underset{\zeta}{\circ}$$

be the tree corresponding to B_1. Then (2.3) implies that

$$1 + \frac{\zeta(1)^2}{\zeta(1)} - \frac{\eta(1)^2}{\eta(1)} = 1 + \frac{\zeta(1)\, \zeta(x)}{\zeta(1)} - \frac{\eta(1)\, \eta(x)}{\eta(1)}$$
$$= 1 + \frac{\zeta(x)^2}{\zeta(1)} - \frac{\eta(x)^2}{\eta(1)} = 0.$$

Hence the vectors $(1, \zeta(1), \eta(1))$, $(1, \zeta(x), \eta(x))$ lie in an isotropic subspace for the diagonal quadratic form $(1, \zeta(1)^{-1}, -\eta(1)^{-1})$. Since this is a non-degenerate form an isotropic subspace has dimension at most 1. Thus the two vectors are proportional and so $\zeta(x) = \zeta(1)$. Hence x is in the kernel of ζ contrary to the simplicity of G.

LEMMA 2.3. *One of the following holds*

(i) *Condition* (I), (II), *or* (III) *of Theorem* 1 *is satisfied.*

(ii) *There exists an involution w in $\mathbf{N}_G(P) - \mathbf{C}_G(P)$ which is not conjugate in G to any element of the form wz with z in Z, $z \neq 1$.*

Proof. Suppose that $\mathbf{N}_G(P) - \mathbf{C}_G(P)$ contains no involutions. Then there exists a unique involution z in $\mathbf{N}_G(P)$ and $z \in Z$. Then by Lemma 2.2 z is the unique involution in G. This implies that a Sylow 2-group of G is a generalized quaternion group. Thus a Sylow 2-group of G/Z is a dihedral group. Therefore the main result of [4] implies that G/Z is either isomorphic or $\mathfrak{A}_7$ or $PSL_2(r)$ for some prime power r. It is easily seen that $\mathfrak{A}_7$ does not satisfy $(*)$ for any prime p. Furthermore if $PSL_2(r)$ satisfies $(*)$ for p then $|P| = (r \pm 1)/2$. Since $q \geqslant 7$, $PSL_2(2q \pm 1)$ has a Schur multiplier of order 2. Thus either condition (I) or (II) of Theorem 1 is satisfied.

Suppose that w is an involution in $\mathbf{N}_G(P) - \mathbf{C}_G(P)$ and condition (ii) of the Lemma does not hold. Let z be an element in Z, $z \neq 1$, such that w is conjugate to wz in G. Then clearly z is an involution. Since every involution in $\mathbf{N}_G(P)$ is conjugate to w, wz or z in $\mathbf{N}_G(P)$ it follows from Lemma 2.2 that every involution in G is conjugate to w or z. Thus condition (III) of Theorem 1 is satisfied and the Lemma is proved.

Suppose that condition (ii) of Lemma 2.3 is satisfied. For $i = 1,..., m$ define

$$f_i = \sum_{\xi \text{ in } B_i} \frac{\xi(w)^2 \, \xi(y)}{\xi(1)} .$$

LEMMA 2.4. *Suppose that condition* (ii) *of Lemma* 2.3 *is satisfied. Then $f_i = f_1$ for $i = 1,..., m$.*

Proof. Induction on m. If $m = 1$ the result is clear.

Let H be any subgroup of Z. Let $\bar{G} = G/H$ and let $\bar{x}$ be the image of x in $\bar{G}$. The group $\bar{G}$ clearly satisfies condition $(*)$. If $H \neq \langle 1 \rangle$ and $\bar{x} = \overline{wz}$ for some element z in Z with $\bar{z} \neq 1$ then x has order at least 4. Thus w is not conjugate to x. Hence $\bar{G}$ also satisfies condition (ii) of Lemma 2.3.

Let n_H be the number of ordered pairs of involutions x_1, x_2 in $\bar{G}$ with $\bar{y} = x_1 x_2$ and x_1, x_2 conjugate to $\bar{w}$ in $\bar{G}$. Then $x_1, x_2 \in \mathbf{N}_{\bar{G}}(\bar{P})$. Hence condition (ii) of Lemma 2.3 implies that $n_H = p$. Furthermore

$|\mathbf{C}_{\bar{G}}(x_1)|\,|H| = |\mathbf{C}_G(w)|$. Thus

$$\frac{|G|}{|\mathbf{C}_G(w)|^2}\sum_{i=1}^{m} f_i = n_{\langle 1\rangle} = p = n_H = \frac{|G:H|}{|\mathbf{C}_G(x_1)|^2}\sum_{H \text{ in kernel of } \mu_i} f_i\,.$$

Consequently

$$\sum_{i=1}^{m} f_i = |H| \sum_{H \text{ in kernel of } \mu_i} f_i\,. \tag{2.6}$$

By induction $f_i = f_1$ if μ_i is not a faithful character of Z. There are $\phi(m)$ faithful characters of Z and they are all algebraically conjugate. Since f_i is rational for all i this implies that if μ_m is a faithful character of Z then $f_j = f_m$ for any faithful character μ_j of Z. Hence (2.6) with $H = Z$ yields that

$$mf_1 = \sum_{i=1}^{m} f_i = \{m - \phi(m)\}f_1 + \phi(m)f_m\,.$$

Consequently $f_m = f_1$ and the result is proved.

3. Proof of Theorem 1

Throughout this section the following will be assumed.

Condition $(*)$ *and condition* (ii) *of Lemma* 2.3 *are both satisfied with* $q \geqslant 7$.

$\mathbf{Z}(G) \neq \langle 1\rangle$.

χ *is a faithful irreducible character of* G *with* $\chi(1) \not\equiv 0 \pmod{p}$.

The notation of Section 2 will be used without change and $W = \langle w\rangle$.

In view of Lemma 2.3 the proof of Theorem 1 will be complete once it is shown that the above mentioned hypotheses imply that condition (IV) of Theorem 1 holds. This will be done in a series of Lemmas in this section.

Lemma 3.1. *Let* ξ *be an irreducible character of* G *with* $\xi(1) \not\equiv 0 \pmod{p}$. *Then*

(i) $\xi(1) = kq \pm 1$ *or* $kq \pm 2$ *for some integer* k.

(ii) *If* $\xi(1) = kq + 1$ *then* $|\xi(w)| \leqslant k + 1$. *If* $\xi(1) = kq - 2$ *then* $|\xi(w)| \leqslant k$. *Furthermore*

$$\frac{|\xi(w)|}{\xi(1)} \leqslant \frac{1}{q} + \frac{1}{\xi(1)}\,.$$

(iii) *If* $\xi(1) = q + 1$ *then* $\xi(w) = \pm\{1 - (-1)^{(q+1)/2}\}$. *If* $\xi(1) = q - 2$ *then* $\xi(w) = (-1)^{(q+1)/2}$.

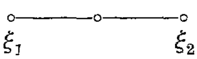
Proof. By [3] (i) is an immediate consequence of Lemma 2.1.

$$\xi_{PW} = \lambda + \sum_{i=1}^{k} \gamma_i \qquad \text{or} \qquad \gamma_k - \lambda + \sum_{i=1}^{k-1} \gamma_i$$

where each γ_i is a character of PW which vanishes on $P - \langle 1 \rangle$. λ is an irreducible character of PW and in the second case λ is a constituent of γ_k. This is easily seen to imply that $\gamma_i = \alpha_i^{PW}$ is the character of PW induced by the linear character α_i of W. Thus $\gamma_i(w) = \pm 1$ for all i and γ_i has a unique linear irreducible constituent. Furthermore every nonlinear irreducible character of PW vanishes on w. If $\xi(1) = kq + 1$ this implies that $|\xi(w)| = |\lambda(w) + \sum_{i=1}^{k} \gamma_i(w)| \leqslant k + 1$. If $\xi(1) = kq - 2$ then $\lambda(w) = 0$ and so $|\xi(w)| = |\sum_{i=1}^{k} \gamma_i(w)| \leqslant k$. The last statement in (ii) follows by inspection.

Let $M(w)$ be the linear transformation corresponding to w in the representation which affords ξ. All the characteristic roots of $M(w)$ are ± 1 and the determinant of $M(w) = 1$. Thus -1 occurs as a characteristic root with even multiplicity. Since the trace of $M(w)$ is $\xi(w)$ condition (iii) is a direct consequence of condition (ii).

Lemma 3.2. *Let*



be the tree corresponding to the block $B = B_i$. *Then after possibly interchanging* ξ_1 *and* ξ_2 *it follows that* $\xi_1(1) = 1 \pmod q$ *and*

$$f_i = \left\{ \frac{\xi_1(w)}{\xi_1(1)} - \frac{\xi_2(w)}{\xi_2(1)} \right\}^2 \left\{ \frac{1}{\xi_1(1)} + \frac{1}{\xi_2(1)} \right\}^{-1}.$$

Proof. It may be assumed that $\xi_1(1) \equiv \epsilon \pmod q$ for $\epsilon = \pm 1$. By definition

$$f_i = \frac{\xi_1(w)^2}{\xi_1(1)} \epsilon + \frac{\xi_2(w)^2}{\xi_2(1)} \epsilon - \frac{\{\xi_1(w) + \xi_2(w)\}^2}{\xi_1(1) + \xi_2(1)} \epsilon.$$

Direct computation shows that ϵf_i is equal to the expression in the statement of the lemma. Since $f_i = f_1 \geqslant 0$ this implies that $\epsilon = 1$.

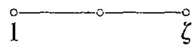
Lemma 3.3. *For* a, t *real let* $g(t) = (a + t)^2 t^{-1}$. *Then* $g(t)$ *is monotonic increasing for* $|t| > |a|$ *and* $g(t)$ *is monotonic decreasing for* $0 < |t| < |a|$.

Proof. Elementary calculus.

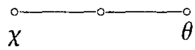
Throughout the rest of this section the following notation will be used.

$B = B_i$ is the block containing χ. $f = f_i$. Since χ is faithful and $Z \neq \langle 1 \rangle$ it follows that $B \neq B_1$. Thus every irreducible character in B is faithful on Z and hence faithful on G.

Let



be the tree corresponding to B_1 and let $\zeta(1) = c$. Let



be the tree corresponding to B and let $\chi(1) = d$, $\theta(1) = e$. Choose the notation so that $d \leqslant e$.

LEMMA 3.4. (i) $f_1 > 1 - 5/q$.

(ii) *If* $d = q - 2$ *then* $f < 9/2q$.

(iii) *If* $d > q - 2$ *then*

$$f \leqslant \frac{4}{q^2} d + \frac{4}{q} + \frac{1}{d}.$$

Proof (i). Suppose that $c = q - 2$. By Lemma 3.1(iii) and Lemma 3.2

$$f_1 \geqslant \left(1 - \frac{1}{q-2}\right)^2 \left(1 + \frac{1}{q-2}\right)^{-1} = \frac{(q-3)^2}{(q-2)(q-1)}$$

$$= 1 - \frac{(3q-7)}{q^2 - 3q + 2} > 1 - \frac{5}{q}.$$

Suppose that $c > q - 2$. By Lemma 3.2 $c > q$. Let $g_1(t) = (1/q - 2 + t)^2 t^{-1}$. Then $f_1 \geqslant g_1(1 + 1/c)$. Since $1 + 1/c < 2 - 1/q$. Lemma 3.3 implies that

$$f_1 \geqslant g_1\left(1 + \frac{1}{q}\right) = \left(1 - \frac{2}{q}\right)^2 \left(1 + \frac{1}{q}\right)^{-1} = \frac{(q-2)^2}{q(q+1)}$$

$$= 1 - \frac{(5q-4)}{q^2 + q} > 1 - \frac{5}{q}.$$

(ii) If $e = q - 2$ then Lemma 3.1(iii) and Lemma 3.2 imply that $f = 0$. If $e = q + 1$ then by Lemma 3.1(iii)

$$f \leqslant \left(\frac{1}{q-2} + \frac{2}{q+1}\right)^2 \left(\frac{1}{q-2} + \frac{1}{q+1}\right)^{-1}$$

$$= \frac{9(q-1)^2}{(2q-1)(q-2)(q+1)} < \frac{9}{2q}$$

for $q \geqslant 7$.

Suppose that $e > q+1$. Then $e \geqslant 2q-2$ by Lemma 3.2 and so $1/q < 1/d + 1/e < 2/q$. Let $g(t) = (1/q + t)^2 t^{-1}$. Then $f \leqslant g(1/d + 1/e)$. Hence by Lemma 3.3

$$f \leqslant g\left(\frac{2}{q}\right) = \left(\frac{3}{q}\right)^2 \frac{q}{2} = \frac{9}{2q}.$$

(iii) Let $g(t) = (2/q + t)^2 t^{-1}$. Then $f \leqslant g(1/d + 1/e)$. Since $1/d < 1/d + 1/e < 2/q$ it follows from Lemma 3.3 that

$$f \leqslant g\left(\frac{1}{d}\right) = \left(\frac{2}{q} + \frac{1}{d}\right)^2 d = \frac{4}{q^2} d + \frac{4}{q} - \frac{1}{d}.$$

LEMMA 3.5. *Condition* (IV) *of the theorem holds for* $q \geqslant 13$.

Proof. Suppose that $q \geqslant 13$. By Lemma 2.4 and 3.4 $1 - 5/q < f_1 = f$. If $d = q-2$ then Lemma 3.4 implies that $1 - 5/q < 9/2q$ and so $2q < 19$ which is not the case. Suppose that $d > q-2$. Thus it may be assumed that $q \geqslant 19$. Then Lemma 3.4 implies that

$$1 - \frac{5}{q} < \frac{4}{q^2} d + \frac{4}{q} + \frac{1}{d}.$$

Therefore $0 < 4d^2 - (q^2 - 9q)d + q^2$. Hence one of the following possibilities must occur.

$$8d \leqslant q^2 - 9q - q(q^2 - 18q + 65)^{1/2} < q\{(q-9) - (q-10)\} = q,$$

$$8d \geqslant q^2 - 9q + q(q^2 - 18q + 65)^{1/2} > q\{(q-9) + (q-10)\} = q(2q-19).$$

Since $d \geqslant q-2$ the first possibility cannot occur. Therefore $d > q(2q-19)/8$. If $q \equiv 3 \pmod 4$ this implies that $d > q((q-7)/4) - (3/8)q$. Since $d \equiv 1$ or $-2 \pmod q$ this implies that $d \geqslant q((q-7)/4) - 2$. If $q = 1 \pmod 4$ then $d > q((q-9)/4) - 5q/8$. Hence $d \geqslant q((q-9)/4) - 2$ in this case. The last statement in Condition (IV) is an immediate consequence as $q \geqslant 13$ and the lemma is proved.

To complete the proof of condition (IV) of Theorem 1 it is now sufficient to consider the cases where $q = 7$ or $q = 11$ and $d = q+1$ or $q-2$.

Suppose that $d = q+1$. By Lemma 3.1(iii) $\chi(w) = 0$ for $q = 7$ or 11. If $e = q+1$ then $f = 0$ by Lemma 3.1(iii). Then by Lemmas 2.4 and 3.4 $0 = f_1 > 1 - 5/q$ which is not the case. If $e > q+1$ then $e \geqslant 2q-2$. Lemmas 3.1(ii) and 3.2 imply that if $g(t) = ((1/q(q+1)) + t)^2 t^{-1}$ then

$$f \leqslant g\left(\frac{1}{e} + \frac{1}{q+1}\right) \leqslant g\left(\frac{1}{2q-2} + \frac{1}{q+1}\right) = \frac{(3q-2)^2 (q+1)}{2q^2(q-1)(3q-1)}.$$

If $q = 7$ or 11 this implies by direct computation that $f < 1 - 5/q$ contrary to Lemmas 2.5 and 3.4.

Suppose that $d = q - 2$. Then Lemmas 2.4 and 3.4 yield that $1 - 5/q < 9/2q$ and so $2q < 19$. Thus $q \neq 11$ and so $q = 7$. Hence $d = 5$. If χ is a monomial character then $P \lhd G$ contrary to assumption. If χ is not monomial then G is one of the groups listed in [1]. It can be seen by inspection that 7 does not divide $|G|$. Hence this case cannot occur. The proof of Theorem 1 is complete.

4. Proof of Theorem 2

Lemma 4.1. *Let G be a minimal counterexample to Theorem 2. Then G satisfies condition $(*)$ with $|P| = p$.*

Proof. Since G is a counterexample to Theorem 2 $P \ntriangleleft G$. The main result of [6] yields the existence of a subgroup P_0 of P with $|P : P_0| = p$ and $P_0 \lhd G$. Since G has a faithful representation of degree $p - 2 < p$ it follows that P is abelian. Hence $P \subseteq \mathbf{C}_G(P_0) \lhd G$. Let G_0 be the subgroup of G generated by all elements of order p. Thus $G_0 \lhd G$ and $G_0 \subseteq \mathbf{C}_G(P_0)$. Hence Burnside's transfer theorem implies that $G_0 = G_1 \times P_0$ for some characteristic subgroup G_1 of G_0. Therefore $G_1 \lhd G$. Let $P_1 = P \cap G_1$. Thus $|P_1| = p$ and $P_1 \ntriangleleft G_1$.

Suppose that $G_1 \neq G$. The minimality of G implies that $p = 2^a + 1$ and $G_1 = G_2 \times A$ where $G_2 \approx SL_2(p-1)$ and A is abelian. The smallest degree of a faithful irreducible character of $SL_2(p-1)$ is $p - 2$ and no outer automorphism of $SL_2(p-1)$ stabilizes a character of degree $p - 2$. Thus χ_{G_2} is irreducible and $G = G_2\mathbf{C}_G(G_2) = G_2 \times \mathbf{C}_G(G_2)$ with $\mathbf{C}_G(G_2)$ abelian contrary to the fact that G is a counterexample to Theorem 1.

Therefore $G_1 = G$. Hence $P_1 = P$ and $G = G_0$ is generated by all elements of order p. Thus $G = G'$. Since G has a faithful irreducible character $\mathbf{Z}(G)$ is cyclic.

Let H be a normal p'-subgroup of G. Then PH is a p-solvable group which has a faithful complex representation of degree $p - 2$. Thus by a result of Ito $P \lhd PH$. See e.g. [5, (24.6)]. Hence $P \subseteq \mathbf{C}_G(H) \lhd G$. Since G is generated by its p-elements this implies that $H \subseteq \mathbf{Z}(G)$. Thus $G/\mathbf{Z}(G)$ is simple and condition (ii) of $(*)$ is satisfied.

Condition (iii) of $(*)$ is satisfied as $|P| = p$. By [7, Theorem 1] $\mathbf{C}_G(P) = P \times \mathbf{Z}(G)$. Thus by [3] $|\mathbf{N}_G(P) : \mathbf{C}_G(P)| = 2$. Hence also condition (iv) of $(*)$ is satisfied.

Theorem 2 can now be proved.

Let G be a minimal counterexample to Theorem 2. By Lemma 4.1 $(*)$ is satisfied with $|P| = p$. Thus Theorem 1 can be applied. Since $\chi(1) = p - 2$

is odd it follows that $|\mathbf{Z}(G)|$ is odd. Therefore (II), (III), or (IV) of Theorem 1 cannot hold. Consequently by Theorem 1 $|\mathbf{Z}(G)| = 1$. Theorem 2 now follows directly from [8, Theorem 2].

5. Proof of Theorem 3

Suppose that G satisfies the hypotheses of Theorem 3. Let $N = \mathbf{N}_G(P)$. Let $\lambda = \lambda_1, \lambda_2, \ldots$ be all the nonprincipal irreducible characters of $(P \times H)/H$. Let $\tilde{\lambda}_i$ be the character of N induced by λ_i. Then each $\tilde{\lambda}_i$ is an irreducible character of N and after relabeling it may be assumed that $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n$ are all the distinct irreducible characters of N in the principal p-block which do not contain P in their kernel where $n = (|P| - 1)/3$.

There exists a sign $\epsilon = \pm 1$ and irreducible characters $\chi_1, \ldots, \chi_n$ of G with $(\tilde{\lambda}_i - \tilde{\lambda}_j)^G = \epsilon(\chi_i - \chi_j)$. See for instance [2]. Furthermore

$$\|(1_{P\times H} - \lambda)^G\|^2 = 4 \qquad \text{and} \qquad (1_{P\times H} - \lambda)^G = 1_G + \epsilon\chi_1 + \delta_1\theta_1 + \delta_2\theta_2,$$

where $\theta_1 \neq \theta_2$ are irreducible characters distinct from 1 and each χ_i, and $\delta_j = \pm 1$ for $j = 1, 2$.

Let y be an element of $P - \langle 1 \rangle$. Since $|N : \mathbf{C}_G(P)| = 3$ it follows that no p-singular element of G is the product of two involutions. Thus if B denotes the principal p-block of G then

$$\sum_{\xi \text{ in } B} \frac{\xi(x_1)\, \xi(x_2)\, \xi(y)}{\xi(1)} = 0 \tag{5.1}$$

for $x_1^2 = x_2^2 = 1$.

By the results of [2] it follows that $\{1_G, \theta_1, \theta_2, \chi_1, \ldots, \chi_n\}$ is precisely the set of all irreducible characters in the principal p-block. Furthermore $\chi_i(y) = -\epsilon\tilde{\lambda}_i(y)$ and $\theta_i(y) = \delta_i$ for $i = 1, 2$. Thus (5.1) becomes

$$1 + \epsilon\frac{\chi_1(x_1)\,\chi_1(x_2)}{\chi_1(x)} + \frac{\delta_1\theta_1(x_1)\,\theta_1(x_2)}{\theta_1(1)} + \frac{\delta_2\theta_2(x_1)\,\theta_2(x_2)}{\theta_2(1)} = 0 \tag{5.2}$$

for $x_1^2 = x_2^2 = 1$.

Let V be a four-dimensional rational vector space. Define the symmetric bilinear from B on V by

$$B(u, v) = u_1v_1 + \frac{\epsilon}{\chi_1(1)}u_2v_2 + \frac{\delta_1}{\theta_1(1)}u_3v_3 + \frac{\delta_2}{\theta_2(1)}u_4v_4 .$$

It suffices to show that V has a two-dimensional isotropic subspace because in that case exactly one of $\epsilon, \delta_1, \delta_2$ is $+1$ and V is the orthogonal sum of two hyperbolic planes. See for instance [10, Chapter XIV]. Thus the discriminant of B is a square. Since the discriminant of B is $\{\chi_1(1)\,\theta_1(1)\,\theta_2(1)\}^{-1}$ Theorem 2 is proved.

By (5.2) $u = (1, \chi_1(1), \theta_1(1), \theta_2(1))$ is an isotropic vector in V. Since $G = G'$ there exists an involution x in G which is not in $\mathbf{O}_{p'}(G)$ and so is not in the kernel of χ_1, θ_1, and θ_2. Let $v = (1, \chi_1(x), \theta_1(x), \theta_2(x))$. Thus u and v are linearly independent vectors. By (4.2)

$$B(u, u) = B(u, v) = B(v, v) = 0.$$

Hence the two-dimensional subspace of V which is spanned by u and v is isotropic as required.

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