JOURNAL OF COMPLEXITY **14**, 333–377 (1998)
ARTICLE NO. CM980480

On the Asymptotic Average Number of Efficient Vertices in Multiple Objective Linear Programming

K.-H. Küfer

Department of Mathematics, University of Kaiserslautern, Erwin-Schrödinger-Str., Kaiserslautern 67663, Germany
E-mail: kuefer@mathematik.uni-kl.de

Received March 13, 1997

Let $a_1, \ldots, a_m, c_1, \ldots, c_k$ be independent random points in $\mathbb{R}^n$ that are identically distributed spherically symmetrical in $\mathbb{R}^n$ and let $X := \{x \in \mathbb{R}^n | a_i^T x \leqslant 1, i = 1, \ldots, m\}$ be the associated random polyhedron for $m \geqslant n \geqslant 2$. We consider multiple objec-

with respect to $c_1, \ldots, c_k$ for fixed n, k and $m \to \infty$. This expected number of efficient vertices is the most significant indicator for the average-case complexity of the multiple objective linear programming problem.

1. INTRODUCTION AND RESULTS

We consider multiple objective linear programming problems (MOLPs) with k linear *objective functionals*

$$\max_{x \in X} c_1^T x, \max_{x \in X} c_2^T x, \ldots, \max_{x \in X} c_k^T x \tag{1}$$

with $c_j, x \in \mathbb{R}^n$ subject to m linear *constraints*

$$X := \{x \in \mathbb{R}^n | a_i^T x \leqslant b_i, i = 1, \ldots, m\}, \tag{2}$$

where we assume $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for $1 \leqslant k \leqslant n \leqslant m$. Finally, we assume $\{c_1, \ldots, c_k\}$ linearly independent. The polyhedral set X defined by (2) is the *set of feasible solutions* for the MOLP (1)–(2). Multiple objective linear

0885-064X/98 $25.00

programming problems are generalizations of linear programming problems with $k=1$.

In contrast to the common linear programming problem, MOLPs with $k \geqslant 2$ deal with different objective functionals that have to be maximized simultaneously. Thus, it would be natural to define optimal solutions of (1)–(2) in the following way: a point $\tilde{x}$ is called optimal for the MOLP (1)–(2) if for all $i=1, ..., k$ and all $x \in X$ holds $c_i^T \tilde{x} \geqslant c_i^T x$. Unfortunately, in most cases such solutions do not exist as the objective functionals (1) might be contradictive. So, one has to weaken the assumptions on optimality in some sense.

The most familiar characterization of "optimal solutions" of MOLPs is due to Pareto: a point $\tilde{x} \in X$ is called *efficient* or *Pareto-optimal* for the MOLP (1)–(2) if and only if there is no $x \in X$ such that $c_i^T x \geqslant c_i^T \tilde{x}$ for all $i=1, ..., k$ and $c_j^T x > c_j^T \tilde{x}$ for some $j \in \{1, ..., k\}$. Roughly speaking: a solution $\tilde{x} \in X$ is efficient if we cannot improve some objective $x \to c_j^T x$ while we guarantee at least the actual values of the others.

In general, the efficient solutions of MOLPs will not be uniquely determined. The set of all efficient solutions for the objectives $x \to c_j^T x$, $j=1, ..., k$, in X is called *efficient frontier* of X with respect to $\{c_1, ..., c_k\}$. The efficient frontier of X with respect to $\{c_1, ..., c_k\}$ is a connected closed subset of X's boundary.

Figure 1 illustrates the efficient frontier of a polyhedron in $\mathbb{R}^3$ with respect to 2 linearly independent objective functionals on the left and with respect to 3 linearly independent objective functionals on the right hand side.

It will be useful for further considerations to have an alternative characterization of efficiency, which we will introduce for *non-degenerate* MOLPs. A multiple objective linear programming problem satisfies the *non-degeneracy condition* (NC) if any arrangement of n hyperplanes chosen from $\{x | a_i^T x = b_i\}$, $i=1, ..., m$, $\{x | c_j^T x = 0\}$, $j=1, ..., k$, has a unique intersection point and if any $n+1$ such hyperplanes have no points in common.

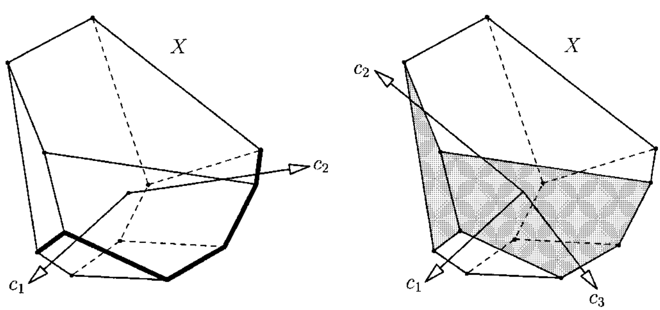

FIG. 1. The efficient frontier of X w.r.t. $\{c_1, c_2\}$ resp. $\{c_1, c_2, c_3\}$.

In our presentation, we will profit from a parametric description of efficiency. In case of non-degeneracy a point $\tilde{x} \in X$ is efficient with respect to $\{c_1, ..., c_k\}$ if and only if $\tilde{x}$ maximizes the parametric objective functional $x \to c_\alpha^T x$ with $c_\alpha := \sum_{i=1}^k \alpha_i c_i$ for some positive $\alpha = (\alpha_1, ..., \alpha_k) \neq 0$. This can be shown by an application of Farkas' lemma. For proofs and a more detailed presentation of the background theory of multiple objective linear programming we refer to the monographs of Zeleny (1974) and Steuer (1985).

The complete solution of a MOLP determines the total efficient frontier. The first and main step of most algorithms for that purpose is to calculate all efficient vertices and infinite edges. Based on this information it is possible to determine all efficient faces and hence all efficient points as convex-combinations of face-generators. The most effective algorithms for vertex-finding like gift-wrapping, shelling, etc., calculate the efficient vertices successively.

The analysis of the computational complexity of such an algorithm requires a detailed analysis of the number of efficient vertices and infinite edges and the cost for finding the next efficient vertex and for organizing the search, i.e., data-accounting and administration. In this contribution we will concentrate only on the analysis of the number of efficient vertices, which we consider the most important indicator of the computational complexity.

In worst-case situations for any triple (k, n, m) all vertices might be efficient. This means that MOLPs might be *intractable* in a worst-case situation as there might be exponentially many efficient vertices in terms of n and m even if k is fixed.

The more important question for practitioners is: What number of vertices in the efficient frontier is expected? Empirical evidence and many statistical experiments suggest that at least for fixed k the average number of efficient vertices should be polynomially bounded in terms of n and m.

In order to investigate the average number of efficient vertices theoretically, one has to analyze the expected number of efficient vertices for randomly generated problem instances in a stochastic model that simulates real world's random as good as possible. The first attempt here is due to Haimovich (in press), who analyzed the expected number of efficient vertices of X, cf. (2), with respect to the MOLP (1). As a stochastic model for the generation of random problem instances he used the *Flipping Model*, which is characterized by a symmetry assumption and a non-degeneracy assumption.

Let a_i, b_i, and c_i be distributed jointly in such a way that flipping the sign (i.e., the orientation) of any inequality $a_i^T x \leqslant b_i$ or any objective c_j is a measure preserving transformation. This symmetry condition is called *sign invariance condition* (SIC) in the literature and was used in works of

Adler and Berenguer (1981), Adler, Karp, and Shamir (1987), Adler and Meggido (1985), May and Smith (1982), and others for simulating random polytopes and random linear programs.

In addition, let a_i, b_i, and c_j be distributed in such a way that the non-degeneracy condition (NC) fulfilled almost surely.

Let v denote the number of efficient vertices of the MOLP (1) with respect to the constraints (2). Haimovich proved that for distributions satisfying (SIC) and MOLPs of the form (1)–(2) holds

$$E(v|X \neq \emptyset) = \frac{\binom{m}{n} \sum_{i=0}^{k-1} \binom{n+k-1}{i}}{2^{k-1} \sum_{i=0}^{n} \binom{m}{i} - \sum_{i=0}^{k-1} \binom{m+k-1}{i}}. \tag{3}$$

In order to understand the asymptotic behaviour of $E(v)$ for fixed k and big m, n it is useful to simplify (3). It holds

$$E(v|X \neq \emptyset) \leqslant (1+o(1)) \cdot \min\left(\frac{(n/2)^{k-1}}{(k-1)!}, \frac{(m-n)^{k-1}}{(k-1)!}\right) \tag{4}$$

if $\min(n, m-n)/k \to \infty$ and even more particularly

$$E(v|X \neq \emptyset) \leqslant (1+o(1)) \cdot \frac{\sum_{i=0}^{k-1} \binom{n+k-1}{i}}{2^{k-1}} \tag{5}$$

if k, n are fixed and $m \to \infty$.

The very short and elegant proof of (3) is purely combinatorial and is given in [10] in generalization of (3) for expectations of the number of efficient j-faces of X for arbitrary $j \in \{0, ..., n\}$. The chance for such a short combinatorial proof is due to the finite number of symmetries of (SIC) and to the non-parametric stochastic setting.

In particular, (4) shows that the expected number of efficient vertices of X with respect to $\{c_1, ..., c_k\}$ is polynomial in n and m if k is fixed. Hence, (4) confirms at first sight completely the empirical suggestion that we mentioned above.

But there is one astonishing observation: if n and k are fixed and m is growing to infinity, the expected number of efficient vertices remains bounded. This contradicts a bit our feeling, as we know that an additional non-redundant constraint will generate new vertices, which might be efficient as well. So, at least a significant number of additional constraints should cause a growth of the expected number of efficient vertices.

What might be the reason why the bound of the expectation does not grow when we add a number of constraints? The only possible explanation could be that for large m and fixed n there are only a few non-redundant constraints that contribute to the number of (efficient) vertices. Thus, a large number of harmless problems with few vertices seems to compensate a small number of hard problems in expectation. It is not hard to prove and to quantify this, which was done by Borgwardt (1987): the expected number of non-redundant constraints tends to $2n$ for fixed n and $m \to \infty$. Qualitatively, this means for large m and fixed n that the average rate of redundancy is nearly 100 percent. Thus, it is no surprise that the expectation of efficient vertices does not significantly exceed $(n/2)^{k-1}/(k-1)!$ for large m.

So, as it is on the other hand a known fact that in many large practical problems few constraints are redundant, one should look for another stochastic model that covers distributions with a small redundancy rate for large problems and fixed dimension.

A completely different way of generating random polyhedra is the point-based and parameter dependent *Rotation Symmetry Model*, which is well known from stochastic geometry, cf. Rényi and Sulanke (1963), and which was introduced by Borgwardt in optimization theory within his probabilistic analysis of the simplex algorithm, cf. Borgwardt (1980, 1982a, 1982b, 1987, 1996).

We concentrate on a subclass of MOLPs given by

$$\max_{x \in X} c_1^T x, \max_{x \in X} c_2^T x, \ldots, \max_{x \in X} c_k^T x \tag{6}$$

with $c_j, x \in \mathbb{R}^n$ subject to the constraints

$$X := \{x \in \mathbb{R}^n | a_i^T x \leqslant 1, i = 1, \ldots, m\} \tag{7}$$

with $a_i \in \mathbb{R}^n$ and $1 \leqslant k \leqslant n \leqslant m$. This is a subclass of the MOLPs defined by (1)–(2), as we restrict ourselves essentially to positive b_i's at the right hand side of the constraints. Without loss of generality we take ones at the right hand side of the constraints dividing by b_i.

In particular, this restriction means that 0 is always an interior point of X. It is the special flavour of this class of MOLPs that there is a single central point—here the origin—which always belongs to X's interior.

In the Rotation Symmetry Model, the particular role of the central point is emphasized by taking spherically symmetric distributions for the generation of the vectors a_i defining the constraints, cf. (7).

Let $a \in \mathbb{R}^n$ be given in polar representation $a = \|a\|_2 \omega$ with an appropriate vector ω of unit length. a is spherically symmetrically distributed, if ω is uniformly distributed on the unit sphere $\mathcal{S}^{n-1}$ in $\mathbb{R}^n$ and if the radius $\|a\|_2$ of a is distributed with an arbitrary *radial distribution function* F, i.e., $F(t) = \Pr(\|a\|_2 \leqslant t)$ for $t \in \mathbb{R}_0^+$. Without loss of generality we may assume that F is continuous from the right. If the vectors $a_i, i = 1, ..., m$, and $c_j, j = 1, ..., k$, are independent and identically distributed spherically symmetric in $\mathbb{R}^n$, they satisfy the *rotation invariance condition* (RIC). In addition, we propose $F(0) = 0$, which warrants that the non-degeneracy condition (NC) is fulfilled with probability one. The radial distribution function F can be considered as the parameter of this stochastic model.

So, the Rotation Symmetry Model—like the Flipping Model—is given by a symmetry assumption—spherical symmetry of the distribution of the a_i and c_j—and a non-degeneracy assumption—no mass of the distribution in the origin.

Note. In the light of the discussion following Haimovich's result (4) —that there exist particular distributions in the Rotation Symmetry Model, namely those that are concentrated on a unique sphere—whose redundancy rate is zero for all combinations of n and m if the points $a_i \in \mathbb{R}^n$, $i = 1, ..., m$ are pairwise different. This is due to the fact that a constraint $a_i^T x \leqslant 1$ is redundant if and only if a_i can be represented as a convex combination of the points a_j, $j = 1, ..., m$, $j \neq i$, and 0.

In this work, we will focus on distributions in the unit ball that have an *algebraically decreasing* tail $\bar{F} := 1 - F$, i.e., there exist positive reals γ and N such that for $t \to 0+$ holds

$$\bar{F}(1 - t) = N t^{\gamma} \cdot (1 + o(1)). \tag{8}$$

We will prove for the number v of efficient vertices:

THEOREM 1. *For distributions in the unit ball satisfying the* (*RIC*) *that have, in addition, an algebraically decreasing tail with* $\gamma > 0$ *holds*:

$$\begin{aligned} &\underline{K}_\gamma(n, k)\, m^{(k-1)/(n-1+2\gamma)}(1 + o(1)) \\ &\qquad \leqslant E(v) \leqslant \bar{K}_\gamma(n, k)\, m^{(k-1)/(n-1+2\gamma)}(1 + o(1)) \end{aligned} \tag{9}$$

for $m \to \infty$ *and fixed* n, k, *where* $\underline{K}_\gamma(n, k) = (k/\binom{n}{k-1})\, \bar{K}_\gamma(n, k)$ *and*

$$\bar{K}_\gamma(n, k) = \left(\frac{\sqrt{\pi}}{2}\right)^{k-1} \cdot \frac{1}{k!\,\Gamma((k+1)/2)} \cdot n^{2(k-1)} \cdot (1 + o(1)), \tag{10}$$

for $n \to \infty$ *and fixed* k.

Thus, we observe that the expectation of $E(v)$ grows sublinearly in m reflecting the above stated feeling that a growing number of constraints must influence the growth of $E(v)$. The uniform distribution in the unit ball satisfies the assumptions of Theorem 1 with $\gamma = 1$ and $N = n$. A corresponding result for the uniform distribution on the unit sphere can be gained as a limiting case of Theorem 1 if we consider, e.g., the sequence F_γ with $\bar{F}_\gamma(1-t) = t^\gamma$ for $\gamma \to 0$.

We remark that the result of Theorem 1 is in some sense a generalization of Borgwardt's asymptotic results in the analysis of the average complexity of the simplex method in Borgwardt (1980, 1982a). The relation to Borgwardt's work will be precisely stated in Section 2.

The rest of the work is organized as follows: Section 2 shows that we can embed the question for the average number of efficient vertices of a MOLP into stochastic geometry of polytopes. Section 3 gives a detailed proof of Theorem 1.

2. FROM OPTIMIZATION TO STOCHASTIC GEOMETRY

The heart of our analysis of the average number of efficient vertices of multiple objective linear programming problems (MOLPs) of type (6)–(7) is the close interaction between combinatorial structures from optimization theory—in particular the number of efficient vertices of the polyhedron X of feasible solutions with respect to k objectives $x \to c_1^T x, ..., x \to c_k^T x$—and geometrical figures of the set of feasible solutions.

It will be the role of this section to build a bridge between complexity theory of multiple objective programming and stochastic geometry.

In the first paragraph of this section we will show that the number of efficient vertices of X w.r.t. $\{c_1, ..., c_k\}$ equals the number of facets of X's polar polytope Y that are intersected by $C := \operatorname{cone}(\{c_1, ..., c_k\})$. For fixed Y the expected number of Y-facets that are intersected by a random C can be estimated in terms of the expected numbers of Y-facets that are intersected by random linear subspaces of dimensions $\ell = 1, ..., k$ as will be shown in the second subsection. We will see in the third subsection that the expected number of Y-facets intersected by a random linear subspace can be estimated by the expectation of a weighted sum of spherical angles generated by certain faces of Y and henceforth we will have arrived in the world of stochastic geometry.

2.1. *Polarization—A Link to Polytope Functionals*

Multiple objective linear programming problems (MOLPs) of type (6)–(7) can be described by two matrices $\mathscr{A}$ and $\mathscr{C}$, the first of which is

$$\mathscr{A} := (a_1 | \cdots | a_m), \tag{11}$$

containing the normal vectors of the constraints as columns, while the latter

$$\mathscr{C} := (c_1 | \cdots | c_k) \tag{12}$$

contains the data of the objective functionals as column vectors. In the following we will identify the set of feasible solutions $X = \{x \in \mathbb{R}^n | \mathscr{A}^T x \leqslant \mathbf{1}\}$, $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^m$, with $X(\mathscr{A})$ and the number v of efficient vertices of $X = X(\mathscr{A})$ with respect to (w.r.t.) the columns of $\mathscr{C}$ with $v(\mathscr{C}, \mathscr{A})$.

As we announced in Section 1 we are to analyze the expectation of $v(\mathscr{C}, \mathscr{A})$ for random $\mathscr{A}$ and $\mathscr{C}$ satisfying the (RIC) condition. An $\mathbb{R}^{n \times \ell}$ matrix is called *non-degenerate* if any choice of n different columns of it is linearly independent and if any choice of $n+1$ different columns of it is in the general position. Throughout this section we will assume that the matrix $(\mathscr{A}|\mathscr{C})$ is non-degenerate, which means in particular that random MOLPs of type (6)–(7) satisfy the (NC) condition with probability one. Aside from the number of efficient vertices we will study the number of vertices in the *contour* of X w.r.t. $\mathscr{C} = (c_1 | \cdots | c_k)$. By definition, the contour of X w.r.t. $\mathscr{C}$ consists of all $\tilde{x} \in X$ that *maximize or minimize some* objective functional $x \to c_j^T x$ for some $j \in \{1, \ldots, k\}$ over X while all others take fixed values. Under the assumption of non-degeneracy, in a parametric language, a vertex $\tilde{x} \in X$ lies in the contour of X w.r.t. $\mathscr{C}$ if and only if it maximizes some objective functional $x \to c_\alpha^T x$, where $c_\alpha := \sum_{i=1}^k \alpha_i c_i$, $\alpha := (\alpha_1, \ldots, \alpha_k) \neq 0$, is some non-zero linear combination of the column vectors of $\mathscr{C}$. So, like the efficient frontier the contour is a connected and closed subset of X's boundary. Figure 2 shows the efficient frontier and the contour of a polytope $X \subset \mathbb{R}^3$ for two objective functionals.

We denote the number of vertices of X that lie in the contour of X w.r.t. $\mathscr{C}$ with $V = V(\mathscr{C}, \mathscr{A})$.

It will be one major trick in our analysis to estimate the expectation $E(v)$ by $E(V)$, cf. Theorem 2.

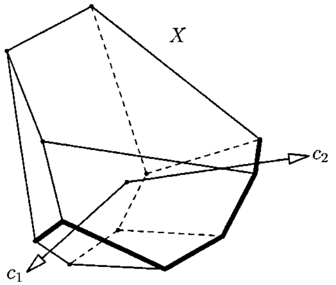

FIG. 2. The efficient frontier and the contour of X w.r.t. $\mathscr{C}$.

The vertices in the contour and in the efficient frontier can be characterized geometrically. For this purpose let

$$C := \operatorname{cone}(\{c_1, ..., c_k\}) = \operatorname{cone}(\mathscr{C}) \tag{13}$$

be the k-dimensional convex cone generated by $\mathscr{C}$ and

$$L := \operatorname{span}(\{c_1, ..., c_k\}) = \operatorname{span}(\mathscr{C}) \tag{14}$$

be the k-dimensional linear subspace generated by $\mathscr{C}$.

Note that we use the set-functionals conv, cone, aff and span as matrix-functionals as well. If the argument of the functionals is a matrix, the functionals are understood as functionals of the set of column vectors of this matrix.

Moreover, let $\tilde{x}$ be an arbitrary vertex of X and $J(\tilde{x}) \subset \{1, ..., m\}$ be the set of indices j of those constraints $a_j^T x \leqslant 1$ that are active in $\tilde{x}$. Then,

$$\operatorname{VC}(\tilde{x}) := \operatorname{cone}(\{a_j \mid j \in J(\tilde{x})\}), \tag{15}$$

is called *polar vertex cone* of $\tilde{x}$. It is well known that the following is true:

LEMMA 1. *Let* $(\mathscr{A} \mid \mathscr{C})$ *be non-degenerate*:

(i) *A vertex* $\tilde{x}$ *of* X *is efficient w.r.t.* $\mathscr{C}$ *if and only if the polar vertex cone of* $\tilde{x}$ *is intersected by* C.

(ii) *A vertex* $\tilde{x}$ *of* X *lies in the contour w.r.t.* $\mathscr{C}$ *if and only if the polar vertex cone of* $\tilde{x}$ *is intersected by* L.

Proof. We prove both items side by side writing the figures of item (ii) in brackets.

Necessity. If the vertex $\tilde{x}$ of X is efficient w.r.t. $\mathscr{C}$ (resp. lies in the contour of X w.r.t. $\mathscr{C}$), it maximizes some non-zero parametric objective functional $x \to c_\alpha^T x$ with $c_\alpha \in C$ (resp. $c_\alpha \in L$). This means c_α must lie in the polar vertex cone of $\tilde{x}$, as otherwise there exists a feasible direction along which $x \to c_\alpha^T x$ increases starting from $\tilde{x}$ contradicting the optimality of the vertex $\tilde{x}$.

Sufficiency. If the polar vertex cone $\mathrm{VC}(\tilde{x})$ of a vertex $\tilde{x} \in X$ is intersected by C (resp. L), there exists at least one ray $c \in C \cap \mathrm{VC}(\tilde{x})$ $(c \in L \cap \mathrm{VC}(\tilde{x}))$, which is a non-zero positive combination of $c_1, ..., c_k$ (resp. a non-zero linear combination of $c_1, ..., c_k$). Hence, the functional $x \to c^T x$ is maximized by $\tilde{x}$ and henceforth $\tilde{x}$ lies in the efficient frontier of X w.r.t. $\mathscr{C}$ (resp. lies in the contour of X w.r.t. $\mathscr{C}$). ∎

We learn from Lemma 1 that we can characterize efficiency (resp. membership in the contour) of a vertex $\tilde{x}$ of X w.r.t. $\mathscr{C}$ by intersections of C (resp. L) with the polar vertex cone of $\tilde{x}$. Hence, it will be useful to consider $\mathscr{A} \to v(\mathscr{C}, \mathscr{A})$ and $\mathscr{A} \to V(\mathscr{C}, \mathscr{A})$ for fixed $\mathscr{C}$ as functionals of the *polar polyhedron* $Y = Y(\mathscr{A})$ of $X(\mathscr{A})$, which is defined by

$$Y := X^p := \{ y \in \mathbb{R}^n | x^T y \leqslant 1, x \in X \}. \tag{16}$$

It is not hard to see that in our particular framework, Y is a polytope given by

$$Y = \mathrm{conv}(\{0, a_1, ..., a_m\}), \tag{17}$$

which is due to the restricted right hand sides of the constraints (7), cf. Borgwardt (1987, Lemma 1.5).

It is well known that for non-degenerate $\mathscr{A}$ the face lattices of X from (7) and Y from (16) are anti-isomorphic, cf. Stoer and Witzgall (1970, Chap. 1). That means, in particular, there is a one-to-one correspondence between the vertices of X and the facets of Y.

Moreover, by the assumption of non-degeneracy, the polyhedron X is *simple*, i.e., each j-face of X is a subset of exactly $n-j$ restrictive hyperplanes for $j = 0, ..., n-1$. In terms of Y, this means Y is *simplicial*, i.e., each j-face of Y is a simplex of dimension j.

It will be probably the most powerful structural tool in our analysis to exploit the one-to-one correspondences between the vertices in the efficient frontier (resp. in the contour) of X w.r.t $\mathscr{C}$ and the facets of Y that are intersected by C (resp. L).

For non-degenerate $\mathscr{A}$ we introduce the set $\mathscr{I}_{m,n}$ of all n-tuples $(i_1, ..., i_n)$ of indices $i_j \in \{1, ..., m\}$ with $1 \leqslant i_1 < i_2 < \cdots i_n \leqslant m$ and associate to each tuple $I \in \mathscr{I}_{m,n}$ the point x_I, which is defined as the unique intersection point of the hyperplanes $\{x | a_{i_j}^T x = 1\}$, $j = 1, ..., n$, i.e., x_I is the basic solution w.r.t. I. Furthermore, for $I = (i_1, ..., i_n) \in \mathscr{I}_{m,n}$ let

$$Y_I := \mathrm{conv}(\mathscr{A}_I) \qquad \text{with} \quad \mathscr{A}_I := (a_{i_1} | \cdots | a_{i_n}). \tag{18}$$

It is not hard to see and well known that x_I is a vertex of X if and only if Y_I is a facet of Y, cf. Borgwardt (1987, p. 69). This is a precise formulation of the above stated one-to-one correspondence of X-vertices and Y-facets. From Lemma 1 we deduce the following:

COROLLARY 2. *For non-degenerate* $(\mathscr{A}|\mathscr{C})$ *and* $I \in \mathscr{I}_{m,n}$ *holds*:

(i) x_I *is an efficient vertex of* X *w.r.t.* $\mathscr{C}$ *if and only if* Y_I *is a facet of* Y *that is intersected by* C.

(ii) x_I *is a vertex in the contour of* X *w.r.t.* $\mathscr{C}$ *if and only if* Y_I *is a facet of* Y *that is intersected by* L.

Proof. If x_I is a vertex of X, $\mathrm{VC}(x_I) = \mathrm{cone}(Y_I)$ for non-degenerated $\mathscr{A}$, i.e., the cone generated by Y_I is identical with the polar vertex cone of x_I, cf. (15). Hence, the claims (i) and (ii) follow from the corresponding items in Lemma 1. ▮

Items (i) and (ii) characterize efficiency and resp. membership in the contour of a vertex $\tilde{x}$ of X in terms of the polar polyhedron Y. For the latter case there is another interesting geometric characterization in terms of the primal polyhedron X: we call a vertex $\tilde{x}$ of X a *shadow-vertex* with respect to L, if the orthogonal projection of the vertex $\tilde{x}$ onto L is a vertex of the orthogonal projection of X onto L. In this notation, $\tilde{x}$ lies in the contour of X with respect to $\mathscr{C}$ if and only if $\tilde{x}$ is a shadow-vertex of X w.r.t. L.

The concept of shadow-vertices was introduced by Borgwardt (1980, 1982a, 1982b, 1987, 1996) in his famous average-case analysis of a parametric variant of the simplex algorithm solving linear prom gramming problems on a polyhedron X of type (7). The main step of the analysis was the investigation of the second phase of the simplex algorithm; i.e., starting with some vertex of X improve the objective functional until an optimal vertex is reached. Borgwardt proved that the expected number of pivot steps required to maximize the objective functional is just a fourth of the expected number of shadow-vertices of X with respect to a random plane, i.e., a two-dimensional linear subspace.

In the context of multiple objective linear programming we need to analyze shadow-vertices of X with respect to subspaces L of arbitrary dimension. So, in this sense our approach to the average-case complexity analysis of MOLPs can be considered as a generalization of Borgwardt's work on the average-case complexity analysis of the simplex algorithm. Figure 1 illustrates this generalization showing efficient frontiers for two and three objective functional. Figures 1 and 2—in particular the shape of the polytope and the examples with two objectives—are taken from Borgwardt (1987).

Corollary 2 gives that in case of non-degenerate $(\mathscr{A}|\mathscr{C})$ there is a one-to-one correspondence between the vertices of X in the efficient frontier (resp. in the contour) w.r.t. $\mathscr{C}$ and the facets of Y that are intersected by C (resp. L).

Thus, as an immediate consequence we obtain sum representations of the functionals $\mathscr{A} \to v(\mathscr{C}, \mathscr{A})$ and $\mathscr{A} \to V(\mathscr{C}, \mathscr{A})$ for fixed $\mathscr{C}$ if $(\mathscr{A}|\mathscr{C})$ is non-degenerate. Let χ denote an indicator functional of a Boolean argument, which equals one if the Boolean argument is true and zero otherwise. Then, it holds

$$v(\mathscr{C}, \mathscr{A}) = \sum_{\substack{I \in \mathscr{I}_{m,n} \\ Y_I \text{ facet of } Y(\mathscr{A})}} \chi(Y_I \cap C \neq \emptyset) \tag{19}$$

and

$$V(\mathscr{C}, \mathscr{A}) = \sum_{\substack{I \in \mathscr{I}_{m,n} \\ Y_I \text{ facet of } Y(\mathscr{A})}} \chi(Y_I \cap L \neq \emptyset) \tag{20}$$

Formulae (19) and (20) show that the functionals $\mathscr{A} \to v(\mathscr{C}, \mathscr{A})$ and $\mathscr{A} \to V(\mathscr{C}, \mathscr{A})$ can be represented as polytope functionals of Y that can be reduced via a sum of "facet-functionals" for any fixed $\mathscr{C}$. In former publications (Küfer, 1992, 1994) we analyzed such facet-additive polytope functionals systematically. Reduced to our needs we fix this structural property in the following way:

DEFINITION 1. For $m \geqslant n \geqslant 1$ let $\Phi: \mathscr{A} \to \mathbb{R}$, $\mathscr{A} \in \mathbb{R}^{n \times m}$, be a real functional that is invariant under column permutations of $\mathscr{A}$. If there exists a functional $\phi: \mathscr{B} \to \mathbb{R}$, $\mathscr{B} \in \mathbb{R}^{n \times n}$, that is invariant under column permutations of $\mathscr{B}$ such that for non-degenerate $\mathscr{A}$ holds

$$\Phi(\mathscr{A}) = \sum_{\substack{I \in \mathscr{I}_{m,n} \\ Y_I \text{ facet of } Y(\mathscr{A})}} \phi(\mathscr{A}_I), \tag{21}$$

the functional Φ is called a *facet-additive* polytope functional of Y with *facet-functional* ϕ.

For our particularly interesting examples $v(\mathscr{C}, \mathscr{A})$, cf. (19), and $V(\mathscr{C}, \mathscr{A})$, cf. (20), the facet-functionals $\mathscr{B} \to \chi(\text{conv}(\mathscr{B}) \cap C \neq \emptyset)$ and $\mathscr{B} \to \chi(\text{conv}(\mathscr{B}) \cap L \neq \emptyset)$ are Boolean functionals of $\mathscr{B}$ deciding whether $\text{conv}(\mathscr{B})$ does intersect C resp. L or not.

Unfortunately, the functionals (19) and (20) are not covered by Definition 1 yet, as the assumption that $\mathscr{A}$ is non-degenerate does not necessarily mean that $(\mathscr{A}|\mathscr{C})$ is non-degenerate as well. We handle this problem by averaging on the choice of $\mathscr{C}$ and we define the *average number of vertices in the efficient frontier* w.r.t. a random $\mathscr{C}$ by

$$\bar{v}_k(\mathscr{A}) := \mathrm{E}_{\mathscr{C}}(v(\mathscr{C}, \mathscr{A})). \tag{22}$$

Analogously, we define the *average number of vertices in the contour* w.r.t. a random $\mathscr{C}$ by

$$\bar{V}_k(\mathscr{A}) := \mathrm{E}_{\mathscr{C}}(V(\mathscr{C}, \mathscr{A})). \tag{23}$$

The subscript $\mathscr{C}$ at the expectation symbols in (22) and (23) simply indicates that we average on the choice of $\mathscr{C}$. In terms of the polar polyhedron, $\bar{v}_k(\mathscr{A})$ is the average number of Y-facets that are intersected by a randomly chosen k-dimensional convex cone C and $\bar{V}_k(\mathscr{A})$ is the average number of Y-facets that are intersected by a random k-dimensional linear subspace L. In terms of the primal polyhedron $X = X(\mathscr{A})$, $\bar{V}_k(\mathscr{A})$ is the average number of shadow-vertices w.r.t. to the random k-dimensional subspace L.

The functionals $\mathscr{A} \to \bar{v}_k(\mathscr{A})$ and $\mathscr{A} \to \bar{V}_k(\mathscr{A})$ are facet-additive polytope functionals of $Y = \mathrm{conv}(\{0, a_1, ..., a_m\})$ in the sense of Definition 1. It holds

$$\bar{v}_k(\mathscr{A}) = \sum_{\substack{I \in \mathscr{I}_{m,n} \\ Y_I \text{ facet of } Y(\mathscr{A})}} \mathrm{Pr}_{\mathscr{C}}(\mathrm{conv}(\mathscr{A}_I) \cap C \neq \emptyset), \tag{24}$$

i.e., $\bar{v}_k$ is a facet-additive functional with facet-functional $\mathscr{B} \to \mathrm{Pr}_{\mathscr{C}}(\mathrm{conv}(\mathscr{B}) \cap C \neq \emptyset)$, and the functional

$$\begin{gathered} \bar{V}_k(\mathscr{A}) = \sum_{\substack{I \in \mathscr{I}_{m,n} \\ Y_I \text{ facet of } Y(\mathscr{A})}} \phi_k(\mathscr{A}_I), \\ \phi_k(\mathscr{B}) := \mathrm{Pr}_{\mathscr{C}}(\mathrm{conv}(\mathscr{B}) \cap L \neq \emptyset), \end{gathered} \tag{25}$$

is facet-additive with facet-functional ϕ_k.

2.2. *The Relation between the Number of Vertices in the Efficient Frontier and in the Contour*

It seems plausible that there is some combinatorial relation between $\bar{v}_k(\mathscr{A})$ and $\bar{V}_k(\mathscr{A})$, as we can dissect the subspace L into 2^k convex cones of type $\mathrm{cone}(\{s_1 c_1, ..., s_k c_k\})$ with $s_i \in \{-1, 1\}$. Hence, the first guess is that $\bar{V}_k(\mathscr{A})$ should be something like $2^k \bar{v}_k(\mathscr{A})$. But it turns out that this rash conclusion is false as the convex cones $\mathrm{cone}(\{s_1 c_1, ..., s_k c_k\})$ have overlapping boundaries. So, certain facets of Y are intersected by two or more cones. Hence, at least $\bar{V}_k(\mathscr{A}) \leqslant 2^k \bar{v}_k(\mathscr{A})$ holds. But as we are particularly interested in upper bounds of the expectation of $\bar{v}_k(\mathscr{A})$, we are looking for estimates from the other side.

THEOREM 2. *For non-degenerate $\mathscr{A}$ holds*

$$2^{-k} \bar{V}_k(\mathscr{A}) \leqslant \bar{v}_k(\mathscr{A}) \leqslant \sum_{\ell=1}^{k} 2^{-\ell} \binom{k}{\ell} \bar{V}_\ell(\mathscr{A}). \tag{26}$$

Proof. As the left hand side is trivial we concentrate on proving the right. Let $c_1, ..., c_k$ be linearly independent in $\mathbb{R}^n$. We define for non-degenerate matrices $\mathscr{A} \in \mathbb{R}^{n \times m}$ and (closed) convex cones $C' = \text{cone}(\{c_{i_1}, ..., c_{i_\ell}\})$, $1 \leqslant i_1 < \cdots < i_\ell \leqslant k$, $\ell = 1, ..., k$,

$$v^0(C', \mathscr{A}) := \sum_{\substack{I \in \mathscr{I}_{m,n} \\ Y_I \text{ facet of } Y(\mathscr{A})}} \chi(C' \cap Y_I \neq \emptyset)\, \chi(\partial\, C' \cap Y_I \neq \emptyset) \tag{27}$$

with Y_I as in (18), $v^0(C', \mathscr{A})$ is the number of Y-facets that are intersected by C' but not by $\partial C'$, i.e., the number of Y-facets that are intersected exclusively by the relative interior of C' and not by its relative boundary. The relative topology of the closed convex cone $C' \subset \mathbb{R}^n$ consists of the intersections of the open sets of the canonical topology in $\mathbb{R}^n$ with the affine hull of C'.

For non-degenerate $\mathscr{A}$ and $C' = \text{cone}(\{c_{i_1}, ..., c_{i_\ell}\})$ let

$$\bar{v}_\ell^0(\mathscr{A}) := \mathrm{E}_{\mathscr{C}}(v^0(C', \mathscr{A})), \tag{28}$$

where the average is taken on the choice of the column vectors $c_1, ..., c_k$ of $\mathscr{C}$. The right hand side is obviously independent from the choice of the c_{i_j} as these vectors are identically distributed. $\bar{v}_\ell^0(\mathscr{A})$ denotes the average number of Y-facets that are intersected by the relative interior of the ℓ-dimensional cone C' but not by the relative boundary of C'.

First, we are going to overestimate $\bar{v}_\ell(\mathscr{A})$ in terms of $\bar{v}_\ell^0(\mathscr{A})$, $\ell = 1, ..., k$. From the definition of $v(\mathscr{C}, \mathscr{A})$ and $v^0(C', \mathscr{A})$ we are going to conclude that

$$v(\mathscr{C}, \mathscr{A}) \leqslant \sum_{\ell=1}^{k} \sum_{1 \leqslant i_1 < \cdots < i_\ell \leqslant k} v^0(\text{cone}(\{c_{i_1}, ..., c_{i_\ell}\}), \mathscr{A}). \tag{29}$$

In order to prove (29) we observe that C can be represented as a union of disjoint relatively open cones

$$C = \bigcup_{\ell=1}^{k} \bigcup_{1 \leqslant i_1 < \cdots < i_\ell \leqslant k} \text{int}(\text{cone}(\{c_{i_1}, ..., c_{i_\ell}\})) \cup \{0\}. \tag{30}$$

First of all, we remark that the zero-dimensional cone $\{0\}$ does not matter as Y_I cannot contain the origin if $\mathscr{A}$ is non-degenerate. For a given $\mathscr{A}$ we associate each cone $\operatorname{cone}(\{c_{i_1}, ..., c_{i_\ell}\})$ with the number $v^0(\operatorname{cone}(\{c_{i_1}, ..., c_{i_\ell}\}), \mathscr{A})$ of facets of $Y = \operatorname{conv}(\mathscr{A})$ that are intersected by the cone under consideration but not by its relative boundary. If a Y-facet Y_I is intersected by C there must be at least one subcone $\operatorname{cone}(\{c_{i_1}, ..., c_{i_\ell}\})$ intersecting Y_I with minimum number of generators c_{i_j}. The intersected facet Y_I will be counted by $v^0(\operatorname{cone}(\{c_{i_1}, ..., c_{i_\ell}\}), \mathscr{A})$. To prove this we have to check that Y_I does not intersect the relative boundary of $\operatorname{cone}(\{c_{i_1}, ..., c_{i_\ell}\})$. If we assume the opposite there will be a subcone of $\operatorname{cone}(\{c_{i_1}, ..., c_{i_\ell}\})$ that will intersect Y_I as well, contradicting the choice of our subcone. Hence, every Y-facet that intersects C is at least counted by one summand of the right hand side in (29) and (29) is proven. Unfortunately, we cannot avoid counting some facets of Y by different summands in (29). Hence, there are situations where $<$ holds in (29). Taking expectations on both sides of (29) we obtain

$$\bar{v}_k(\mathscr{A}) \leqslant \sum_{\ell=1}^{k} \binom{k}{\ell} \bar{v}^0_\ell(\mathscr{A}). \tag{31}$$

Next, we will estimate the number of Y-facets that are intersected by an ℓ-dimensional linear subspace from below. Let $\mathscr{C}^{(\ell)} := (c_1 | \cdots | c_\ell)$ and $L^{(\ell)} := \operatorname{span}(\mathscr{C}^{(\ell)})$ be the corresponding ℓ-dimensional linear subspace. We will prove that

$$V(\mathscr{C}^{(\ell)}, \mathscr{A}) \geqslant \sum_{s_1, ..., s_\ell \in \{-1, 1\}} v^0(\operatorname{cone}(\{s_1 c_1, ..., s_\ell c_\ell\}), \mathscr{A}). \tag{32}$$

It is obviously clear that

$$L^{(\ell)} \supset \bigcap_{s_1, ..., s_\ell \in \{-1, 1\}} \operatorname{int}(\operatorname{cone}(\{s_1 c_1, ..., s_\ell c_\ell\})). \tag{33}$$

But, in order to complete the proof of (32) we have to show that no Y_I is covered by two different summands on the right of (32): Suppose there are different cones $C' = \operatorname{cone}(\{s'_1 c_1, ..., s'_\ell c_\ell\})$ and $C'' = \operatorname{cone}(\{s''_1 c_1, ..., s''_\ell c_\ell\})$, a Y-facet Y_I and two points $y' \in \operatorname{int}(C') \cap Y_I$ and $y'' \in \operatorname{int}(C'') \cap Y_I$. Then, $\operatorname{conv}(\{y', y''\}) \subset Y_I$ as Y_I is convex. On the other hand, we know that $\partial C' \cap Y_I \neq \emptyset$ and $\partial C'' \cap Y_I \neq \emptyset$. Hence, by definition (27), Y_I is neither counted by $v^0(C', \mathscr{A})$ nor by $v^0(C'', \mathscr{A})$ and (32) is established.

If we average both sides of (32) on the choice of $\mathscr{C}$ we get

$$\bar{V}_\ell(\mathscr{A}) \geqslant 2^\ell \bar{v}^0_\ell(\mathscr{A}) \tag{34}$$

as the c_i are identically distributed spherically symmetric in $\mathbb{R}^n$. We insert (34) into (31) and obtain the desired estimate (26). ∎

2.3. *A Sum of Spherical Angles*

As we learned in (25) the polytope functional $\bar{V}_k$ is a facet-additive functional of Y with facet-functional $\phi_k: \mathscr{B} \to \Pr_{\mathscr{C}}(\operatorname{conv}(\mathscr{B}) \cap L \neq \varnothing)$. We now want to estimate this facet-functional by geometric figures—more precisely we will estimate it by a weighted sum of spherical angles spanned by submatrices of $\mathscr{B} = (b_1 | \cdots | b_n)$ that consist of $n-k+1$ columns. Therefore, we need some preliminary notation. We look at the face lattice of the $(n-1)$-simplex

$$S := \operatorname{conv}(\mathscr{B}), \qquad \mathscr{B} = (b_1 | \cdots | b_n), \tag{35}$$

for regular $\mathscr{B}$. For any permutation $\pi = (\pi(1), ..., \pi(n))$ of the integers $1, ..., n$ let

$$S_\pi^{(j)} := \operatorname{conv}(\{b_{\pi(1)}, ..., b_{\pi(j)}\}) \tag{36}$$

be the convex hull of the columns of $\mathscr{B}$ that correspond to the head $(\pi(1), ..., \pi(j))$ of the permutation π for $j = 1, ..., n$. Then, for regular $\mathscr{B}$, $S_\pi^{(j)}$ is a $(j-1)$-simplex and a face of S with dimension $j-1$ for all $\pi \in \Pi_n$, Π_n being the symmetric group of all n-permutations.

The mapping

$$\pi \to (S_\pi^{(n)}, S_\pi^{(n-1)}, ..., S_\pi^{(1)}, \varnothing) \tag{37}$$

is one-to-one between Π_n and the set of top-down paths in the face lattice of S going down dimension by dimension from $S = S_\pi^{(n)}$ to the empty set, cf. Fig. 3 for an illustration in $\mathbb{R}^3$.

As there are exactly $\binom{n}{j}$ different faces of S with dimension $j-1$ every $S_\pi^{(j)}$ is met by exactly $n!/\binom{n}{j}$ paths in the face lattice, cf. Fig. 3.

Now, we return to the facet-functional $\phi_k: \mathscr{B} \to \Pr_{\mathscr{C}}(\operatorname{conv}(\mathscr{B}) \cap L \neq \varnothing) = \Pr_{\mathscr{C}}(S \cap L \neq \varnothing)$, which we want to estimate. For matrices $\mathscr{B}$ and $\mathscr{C}$ with non-degenerate $(\mathscr{B}|\mathscr{C})$, geometric insight delivers that if S is intersected by L, it is intersected in *at least* k faces with dimension $n-k$. This is due to the observation that for non-degenerate $(\mathscr{B}|\mathscr{C})$ the intersection of the k-dimensional subspace L with the $(n-1)$-dimensional simplex S is a $(k-1)$-polytope, whose vertices lie on different $(n-k)$-faces of S. On the other hand it is trivial that L cannot intersect more than all $\binom{n}{k-1}$ different $(n-k)$-faces of S. Thus, observing the combinatorics of the face lattice established above, we have proven the following lower and upper bounds for the facet-functional $\phi_k: \mathscr{B} \to \Pr_{\mathscr{C}}(S \cap L \neq \varnothing)$ with $S = \operatorname{conv}(\mathscr{B})$:

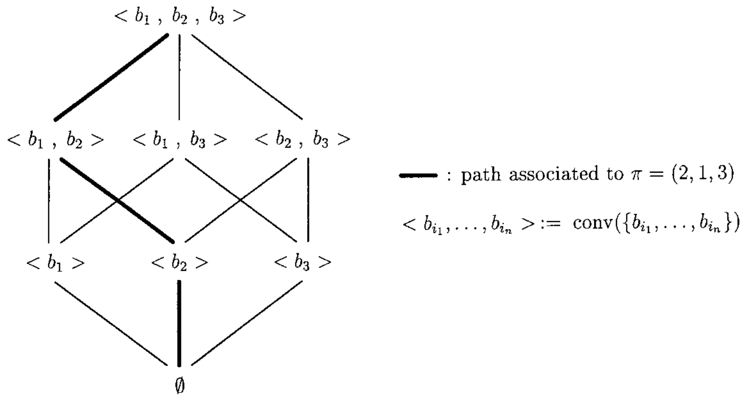

FIG. 3. The face lattice of S.

LEMMA 2. *For regular* $\mathscr{B} \in \mathbb{R}^{n \times n}$ *let*

$$\varphi_k(\mathscr{B}) := \frac{1}{k} \frac{\binom{n}{k-1}}{n!} \sum_{\pi \in \Pi_n} \Pr_{\mathscr{C}}(S_\pi^{(n-k+1)} \cap L \neq \varnothing), \tag{38}$$

where $S = \operatorname{conv}(\mathscr{B})$ *and* $S_\pi^{(j)}$ *as in* (36). *Then, it holds*

$$\frac{k}{\binom{n}{k-1}} \varphi_k(\mathscr{B}) \leqslant \phi_k(\mathscr{B}) \leqslant \varphi_k(\mathscr{B}). \tag{39}$$

We are aware that (39) is a rough worst-case estimate. It would be interesting to find out for random $\mathscr{B}$, which bound—the upper or the lower—is more likely.

In the rest of the work we will concentrate on the evaluation of the facet-additive polytope functional Φ_k defined by

$$\Phi_k(\mathscr{A}) := \sum_{\substack{I \in \mathscr{I}_{m,n} \\ Y_I \text{ facet of } Y(\mathscr{A})}} \varphi_k(\mathscr{A}_I) \tag{40}$$

with facet-functional φ_k given in (38).

In order to do that, we are going to express the probability $\Pr_{\mathscr{C}}(S_\pi^{(n-k+1)} \cap L \neq \varnothing)$ by a spherical angle, which is defined as follows.

Let U be an arbitrary j-dimensional subset of $\mathbb{R}^n$ with $0 \notin \text{int}(\text{aff}(U))$. Then, the j-dimensional spherical angle of U is defined by

$$W_j(U) := \frac{\lambda_j(\text{cone}(U) \cap \mathscr{S}^{n-1})}{\lambda_j(\mathscr{S}^j)}, \tag{41}$$

where λ_j denotes the j-dimensional Lebesgue-measure. Thus, the spherical angle of U is the Lebesgue-measure of the j-dimensional spherical projection of conv(U) onto $\mathscr{S}^{n-1}$ divided by the Lebesgue-measure of $\mathscr{S}^j$ and is henceforth a number in $[0,1]$.

For ease of notation we will abbreviate the j-dimensional Lebesgue-measure of the j-dimensional sphere with

$$\mu_{j+1} := \lambda_j(\mathscr{S}^j) = 2\,\frac{\pi^{(j+1)/2}}{\Gamma((j+1)/2)} \tag{42}$$

for $j \in \mathbb{N}_0$ in the following.

Lemma 3. *For regular $\mathscr{B} \in \mathbb{R}^{n \times n}$ holds*

$$\Pr_{\mathscr{C}}(S^{(n-k+1)} \cap L \neq \emptyset) = 2W_{n-k}(S^{(n-k+1)}) \tag{43}$$

with $S^{(j)} = S^{(j)}_{Id}$ as in (36).

Proof. As the columns c_i of $\mathscr{C}$ are spherically symmetrically distributed, we may assume without loss of generality that

$$b_i^{(n-k+2)} = b_i^{(n-k+3)} = \cdots = b_i^{(n)} = 0 \tag{44}$$

for $i = 1, ..., n$. We dissect L into two halfspaces L^+ and L^-, where we call

$$L^+ := \text{cone}(c_1, -c_1, ..., c_{k-1}, -c_{k-1}, c_k) \tag{45}$$

the upper halfspace and

$$L^- := \text{cone}(c_1, -c_1, ..., c_{k-1}, -c_{k-1}, -c_k) \tag{46}$$

the lower halfspace of L. For non-degenerate $(\mathscr{B}|\mathscr{C})$ the sets

$$L^+ \cap \mathscr{S}^{n-1} \cap \{x \in \mathbb{R}^n | x^{(\ell)} = 0, \ell = n-k+2, ..., n\} \tag{47}$$

and

$$L^- \cap \mathscr{S}^{n-1} \cap \{x \in \mathbb{R}^n | x^{(\ell)} = 0, \ell = n-k+2, ..., n\} \tag{48}$$

consist of exactly one point l^+ resp. l^-, each of which is uniformly distributed on

$$\mathscr{S}^{n-k} := \mathscr{S}^{n-1} \cap \{x \in \mathbb{R}^n | x^{(\ell)} = 0, \ell = n-k+2, ..., n\} \tag{49}$$

if the $c_1, ..., c_k$ are uniformly distributed on $\mathscr{S}^{n-1}$. Thus, the functionals

$$M \to \mu^+(M) := \Pr_{\mathscr{C}}(l^+ \in M) \quad \text{and} \quad M \to \mu^-(M) := \Pr_{\mathscr{C}}(l^- \cap M), \tag{50}$$

which are defined for Lebesgue-measurable $M \subset \mathscr{S}^{n-k}$ are spherically symmetric Haar-measures. Haar-measures that are invariant under the orthogonal group of transformations are uniquely determined up to a constant factor, cf. Nachbin (1965). Thus, as μ^+ and μ^- are probability measures, they are uniquely determined. It holds

$$\mu^+(M) = \mu^-(M) = \frac{\lambda_{n-k}(M)}{\mu_{n-k+1}}. \tag{51}$$

For regular $\mathscr{B}$ it turns out that

$$\Pr_{\mathscr{C}}(S^{(n-k+1)} \cap L^+ \cap L^- \neq \emptyset) = 0, \tag{52}$$

which yields

$$\begin{aligned}
&\Pr_{\mathscr{C}}(S^{(n-k+1)} \cap L \neq \emptyset) \\
&\quad = \Pr_{\mathscr{C}}(S^{(n-k+1)} \cap L^+ \neq \emptyset) + \Pr_{\mathscr{C}}(S^{(n-k+1)} \cap L^- \neq \emptyset) \\
&\quad = \Pr_{\mathscr{C}}(l^+ \in M') + \Pr_{\mathscr{C}}(l^- \in M') \\
&\quad = \mu^+(M') + \mu^-(M') \\
&\quad = 2W_{n-k}(M')
\end{aligned} \tag{53}$$

with $M' := \text{cone}(S^{(n-k+1)}) \cap \mathscr{S}^{n-1} \subset \mathscr{S}^{n-k}$ completing the proof of (43). ▮

Statement and proof of Lemma 3 are straightforward generalizations of Borgwardt's corresponding result for $k = 2$ given in Borgwardt (1987, pp. 126–129).

Now, the average number of vertices $\bar{V}_k(\mathscr{A})$ in the contour of X with respect to $\mathscr{C}$ has been reduced to the calculation of spherical angles. We summarize our results in the following corollary:

COROLLARY 3. *For non-degenerate $\mathscr{A}$ holds*

$$\frac{k}{\binom{n}{k-1}}\,\Phi_k(\mathscr{A}) \leqslant \bar{V}_k(\mathscr{A}) \leqslant \Phi_k(\mathscr{A}) \tag{54}$$

with

$$\Phi_k(\mathscr{A}) = \sum_{\substack{I \in \mathscr{I}_{m,n} \\ Y_I \text{ facet of } Y(\mathscr{A})}} \varphi_k(\mathscr{A}_I),$$

$$\varphi_k(\mathscr{B}) = \frac{2}{k}\,\frac{\binom{n}{k-1}}{n!} \sum_{\pi \in \Pi_n} W_{n-k}(S_\pi^{(n-k+1)}), \tag{55}$$

for $S = \mathrm{conv}(\mathscr{B})$ *and* $S_\pi^{(j)}$ *as in* (36).

2.4. *Rectangular Simplices and Standard-Reduction*

Suppose we are given the facet-additive polytope functional Φ_k with facet-functional φ_k from Corollary 3. We will show in this subsection that we can reduce φ_k in a sum of functionals of "rectangular simplices"—special simplicial pyramids pointed at the origin.

This further reduction will allow a simplified evaluation of expectations.

DEFINITION 2. Let $\mathscr{B} := (b_1 | \cdots | b_n) \in \mathbb{R}^{n \times n}$ be a regular matrix. We call the n-simplex $\mathrm{conv}(\{0, b_1, ..., b_n\})$ *rectangular simplex or r-simplex* if there exists a permutation $\pi \in \Pi_n$ such that the differences $d_\pi^{(i)} := b_{\pi(i)} - b_{\pi(i+1)}$, $i = 1, ..., n-1$, $d_\pi^{(n)} := b_{\pi(n)}$ are non-zero and pairwise orthogonal.

First, we are going to construct rectangular simplices associated to the regular matrix $\mathscr{B} = (b_1 | \cdots | b_n) \in \mathbb{R}^{n \times n}$ for all permutations π in the group Π_n of n-permutations. Let $S := \mathrm{conv}(\mathscr{B}) = \mathrm{conv}(\{b_1, ..., b_n\})$ be the $(n-1)$-simplex associated to $\mathscr{B}$. Furthermore, we recall the one-to-one correspondence (39) between Π_n and the top-down paths in the face lattice of S. We associate to every path

$$(S_\pi^{(n)}, S_\pi^{(n-1)}, ..., S_\pi^{(1)}, \varnothing) \tag{56}$$

with $S_\pi^{(j)} := \mathrm{conv}(\{b_{\pi(1)}, ..., b_{\pi(j)}\})$ a *decreasing chain* of affine subspaces

$$E_\pi^{(n)} \supset E_\pi^{(n-1)} \supset \cdots \supset E_\pi^{(1)} \tag{57}$$

by the affine hulls

$$E_\pi^{(j)} := \mathrm{aff}(S_\pi^{(j)}) \tag{58}$$

for $j = 1, ..., n$. $E_\pi^{(j)}$ is the $(j-1)$-dimensional affine subspace that supports $S_\pi^{(j)}$. Moreover, we equip every affine subspace $E_\pi^{(j)}$ with a (*relative*) *origin* $O_\pi^{(j)}$ by the following recursive construction:

$$\begin{aligned} O_\pi^{(n)} &= \mathrm{proj}_{E_\pi^{(n)}}(0), \\ O_\pi^{(j)} &= \mathrm{proj}_{E_\pi^{(j)}}(O_\pi^{(j+1)}), \qquad j = n-1, ..., 1. \end{aligned} \tag{59}$$

Here, proj_E denotes the ortho-projector onto E. The set of relative origins corresponding to π is denoted by

$$O_\pi := \{ O_\pi^{(1)}, O_\pi^{(2)}, ..., O_\pi^{(n)} \}. \tag{60}$$

The Euclidian distances between consecutive relative origins will be denoted by

$$h_\pi^{(j)} := \| O_\pi^{(j)} - O_\pi^{(j+1)} \|_2 \tag{61}$$

for $j := 1, ..., n-1$ and with

$$h_\pi^{(n)} := \| O_\pi^{(n)} \|_2. \tag{62}$$

We introduce the *vector of distances* corresponding to π by

$$h_\pi := (h_\pi^{(1)}, ..., h_\pi^{(n)}). \tag{63}$$

Definition 3. Let $\mathscr{B} = (b_1 | \cdots | b_n) \in \mathbb{R}^{n \times n}$, $\pi \in \Pi_n$, and O_π as in (60). If O_π is linearly independent, the simplex

$$R_\pi := \mathrm{conv}(\{0\} \cup O_\pi) = \mathrm{conv}(\{ O_\pi^{(1)}, ..., O_\pi^{(n)}, 0\}) \tag{64}$$

is called the r-*simplex associated to* $\mathscr{B}$ *with respect to* π.

Figure 4 shows the r-simplex associated to $\mathscr{B} = (b_1 | b_2 | b_3) \in \mathbb{R}^3$ with respect to the permutation $\pi = (2, 3, 1)$.

The assumption that the matrix $\mathscr{B}$ is regular does not guarantee that O_π is linearly independent for all $\pi \in \Pi_n$.

Definition 4. The matrix $\mathscr{B} = (b_1 | \cdots | b_n) \in \mathbb{R}^{n \times n}$ is called *strongly regular* or *strongly non-singular* if O_π is linearly independent for all $\pi \in \Pi_n$.

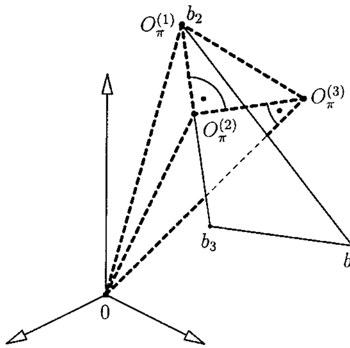

FIG. 4. The rectangular simplex associated to $\mathscr{B} = (b_1 | b_2 | b_3)$ w.r.t. $\pi = (2, 3, 1)$.

Clearly, strong regularity implies regularity. The converse is not generally true bus it is not hard to prove that the following holds:

Remark. In the Rotation Symmetry Model a random $(n \times n)$-matrix is strongly non-degenerate with probability one.

Henceforth, we may assume without loss of generality that $\mathscr{B}$ is strongly regular. The following characterizations of strong regularity are immediate from Definitions 3 and 4:

Remark. The following items are equivalent for $\mathscr{B} \in \mathbb{R}^{n \times n}$:

(i) $\mathscr{B}$ is strongly regular.

(ii) h_π is strictly positive for all $\pi \in \Pi_n$.

(iii) For all $\pi \in \Pi_n$ there exists an r-simplex with respect to π.

Next, we are going to introduce a sign-functional

$$\mathrm{sig}_\sigma \colon (\mathscr{B}, \pi) \to \mathrm{sig}_\sigma(\mathscr{B}, \pi) \tag{65}$$

for matrices $\mathscr{B} = (b_1 | \cdots | b_n) \in \mathbb{R}^{n \times n}$ and permutations $\pi \in \Pi_n$ which is controlled by a fixed signature

$$\sigma = (\sigma^{(1)}, ..., \sigma^{(n-1)})^T \in \{-1, 1\}^{n-1}. \tag{66}$$

For polytopes $Z', Z'' \subset \mathbb{R}^n$ and arbitrary $O \in \mathbb{R}^n$ we define

$$\chi_1(Z', Z'', O) := \begin{cases} 1 & \text{if} \quad Z'' \text{ is a facet of } Z' \text{ and of } \mathrm{conv}(\{O\} \cup Z') \\ 0 & \text{else} \end{cases} \tag{67}$$

and

$$\chi_2(Z', Z'', O) := \begin{cases} 1 & \text{if} \quad Z'' \text{ is a facet of } Z' \text{ but not of } \operatorname{conv}(\{O\} \cup Z') \\ 0 & \text{else} \end{cases} \tag{68}$$

If $\chi_i(Z', Z'', O) = 1$ we call Z'' a *Z'-facet of the ith kind for $i = 1, 2$ with respect to O*. Furthermore, the *jth sign of the matrix $\mathscr{B}$ under the permutation π with respect to the signature σ* is given by

$$\operatorname{sig}_\sigma^{(j)}(\mathscr{B}, \pi) := [\chi_1 - \sigma^{(j)} \chi_2](S_\pi^{(j+1)}, S_\pi^{(j)}, O_\pi^{(j+1)}) \tag{69}$$

for $j = n-1, n-2, ..., 1$. Finally, the *sign of the matrix $\mathscr{B}$ under the permutation π with respect to the signature σ* is defined by

$$\operatorname{sig}_\sigma(\mathscr{B}, \pi) := \prod_{j=1}^{n-1} \operatorname{sig}_\sigma^{(j)}(\mathscr{B}, \pi). \tag{70}$$

Now, we are ready to state the central definition of this subsection:

Definition 5.

(i) The functional $\phi: \mathscr{B} \to \mathbb{R}$ for $\mathscr{B} = (b_1 | \cdots | b_n)$ is called *standard-reducible* if there exists a signature $\sigma \in \{-1, 1\}^{n-1}$ and a functional $\psi: \mathbb{R}^n \to \mathbb{R}$ such that for all strongly regular $\mathscr{B}$ holds

$$\phi(\mathscr{B}) = \sum_{\pi \in \Pi_n} \operatorname{sig}_\sigma(\mathscr{B}, \pi) \, \psi(h_\pi) \tag{71}$$

with h_π as in (63). ψ is called the r-*simplex-functional of* ϕ.

(ii) A facet-additive polytope functional Φ is called *standard-reducible* if the associated facet-functional ϕ is standard-reducible.

Roughly speaking, a functional ϕ is standard-reducible if it can be represented as a certain signed sum of functionals of the vectors of distances associated to the r-simplices.

In this work, we focus on the facet-additive functionals Φ_k, cf. Corollary 3, whose facet-functionals φ_k are given by

$$\varphi_k(\mathscr{B}) = \frac{2}{k} \frac{\binom{n}{k-1}}{n!} \sum_{\pi \in \Pi_n} W_{n-k}(S_\pi^{(n-k+1)}). \tag{72}$$

We will prove in the next theorem that φ_k is standard-reducible for all $k \in \{1, ..., n\}$. Before we can state the result, we need two more symbols. For $j = 1, ..., n$ let

$$r_j(h) := ((h^{(j)})^2 + (h^{(j+1)})^2 + \cdots + (h^{(n)})^2)^{1/2} \tag{73}$$

be the norms of the tails of the vector h. Finally, for $\ell = 1, ..., n$ let

$$\Sigma^{(\ell)}(h) := \operatorname{conv}\left(\left\{\begin{pmatrix}0\\ \vdots\\ \vdots\\ 0\end{pmatrix}, \begin{pmatrix}0\\ \vdots\\ 0\\ h^{(\ell-1)}\end{pmatrix}, ..., \begin{pmatrix}h^{(1)}\\ \vdots\\ \vdots\\ h^{(\ell-1)}\end{pmatrix}\right\}\right). \tag{74}$$

$\Sigma^{(\ell)}(h)$ represents the $(\ell-1)$-simplex $\operatorname{conv}(\{O_{Id}^{(1)}, ..., O_{Id}^{(\ell)}\})$ written in coordinates of $E^{(\ell)} = E_{Id}^{(\ell)} = \operatorname{aff}(\{O_{Id}^{(1)}, ..., O_{Id}^{(\ell)}\})$.

THEOREM 3. *For $k \in \{1, ..., n\}$, the facet functional φ_k from (72) is standard-reducible with the signature $\sigma_k \in \{-1, 1\}^{n-1}$ given by*

$$\sigma_k^{(j)} = \begin{cases} -1 & \text{if } n-k+1 \leqslant j \leqslant n-1 \\ 1 & \text{if } 1 \leqslant j \leqslant n-k \end{cases} \tag{75}$$

and the r*-simplex-functional $\psi_k \colon \mathbb{R}^n \to \mathbb{R}_0^+$ defined by*

$$\psi_k(h) := \frac{2}{k!} \frac{1}{\mu_{n-k+1}} \int_{\Sigma^{(n-k+1)}(h)} \frac{r_{n-k+1}(h)}{(\|z\|_2^2 + r_{n-k+1}^2(h))^{(n-k+1)/2}} \, dz. \tag{76}$$

Proof. We have to prove that the functional φ_k,

$$\varphi_k(\mathscr{B}) = \frac{2}{k} \frac{\binom{n}{k-1}}{n!} \sum_{\pi \in \Pi_n} W_{n-k}(S_\pi^{(n-k+1)}), \tag{77}$$

with $S_\pi^{(j)}$ as in (36) is standard-reducible. The proof is divided into three stages:

Stage 1. For $j = 0, ..., n$ let

$$\Pi_{n,j} := \{\tilde{\pi} \in \Pi_n \mid \tilde{\pi}(i) = i, i = j+1, ..., n\}. \tag{78}$$

We will show that

$$W_{n-k}(S_\pi^{(n-k+1)}) = \sum_{\tau \in \Pi_{n,\, n-k+1}} \operatorname{sig}_{\sigma_k}(\mathscr{B}, \tau \circ \pi)\, W_{n-k}(\operatorname{conv}(\{O_{\tau\circ\pi}^{(1)}, \ldots, O_{\tau\circ\pi}^{(n-k+1)}\})) \tag{79}$$

with σ_k as in (75). Without loss of generality let $\pi = \mathrm{Id}$. We will prove by induction that for all $\ell = n-k+1, n-k, \ldots, 2$ holds

$$W_{n-k}(S_{Id}^{(n-k+1)}) = \sum_{\substack{\tau \in \Pi_{n,\, n-k+1} \\ \tau(1) \leqslant \cdots \leqslant \tau(\ell-1)}} W_{n-k}(\operatorname{conv}(S_\tau^{(\ell-1)} \cup \{O_\tau^{(\ell)}, \ldots, O_\tau^{(n-k+1)}\})) \prod_{j=\ell-1}^{n-k} \operatorname{sig}_{\sigma_k}^{(j)}(\mathscr{B}, \tau) \tag{80}$$

with $S^{(j)} = S_{Id}^{(j)}$.

Equation (79) follows from (80) with $\ell = 2$ as $S_\tau^{(1)} = \{O_\tau^{(1)}\}$ and $\operatorname{sig}_{\sigma_k}^{(j)}(\mathscr{B}, \tau) = 1$ for $j = n-k+1, \ldots, n-1$ and regular $\mathscr{B}$.

We remark that $S_\tau^{(\ell-1)} = \operatorname{conv}(\{b_i \mid i = 1, \ldots, n-k+1, \quad i \neq \tau(\ell), \ldots, \tau(n-k+1)\})$ is well defined even if only the tail $(\tau(\ell), \ldots, \tau(n-k+1))$ of the permutation τ is known. The same is true for $O_\tau^{(\ell)}, \ldots, O_\tau^{(n-k+1)}$ and $\operatorname{sig}_{\sigma_k}^{(\ell-1)}(\mathscr{B}, \tau), \ldots, \operatorname{sig}_{\sigma_k}^{(n-k)}(\mathscr{B}, \tau)$, which depend exclusively on the choice of the tail $(\tau(\ell), \ldots, \tau(n-k+1))$ as well. Hence, the right hand side of (8) can be interpreted as a cumulation on the choice of $\tau(\ell), \ldots, \tau(n-k+1)$, where the order of the other indices $\tau(1), \ldots, \tau(\ell-1)$ is preserved.

The induction for proving (80) is based on the following dissection of the spherical angle $W_{n-k}(\operatorname{conv}(S_\tau^{(j)} \cup \{O_\tau^{(j+1)}, \ldots, O_\tau^{(n-k+1)}\}))$. For $j = n-k+1, \ldots, 2$ holds

$$\begin{aligned} W_{n-k}&(\operatorname{conv}(S_\tau^{(j)} \cup \{O_\tau^{(j+1)}, \ldots, O_\tau^{(n-k+1)}\})) \\ &= \sum_{\substack{\tilde\tau \in \Pi_{n,n-k+1} \\ \tilde\tau(j+1) = \tau(j+1), \ldots, \tilde\tau(n) = \tau(n) \\ \tilde\tau(1) \leqslant \cdots \leqslant \tilde\tau(j-1)}} [\chi_1 - \chi_2](S_{\tilde\tau}^{(j)}, S_{\tilde\tau}^{(j-1)}, O_{\tilde\tau}^{(j)}) \\ &\quad \times W_{n-k}(\operatorname{conv}(S_{\tilde\tau}^{(j-1)} \cup \{O_{\tilde\tau}^{(j)}, \ldots, O_{\tilde\tau}^{(n-k+1)}\})) \\ &= \sum_{\substack{\tilde\tau \in \Pi_{n,n-k+1} \\ \tilde\tau(j+1) = \tau(j+1), \ldots, \tilde\tau(n) = \tau(n) \\ \tilde\tau(1) \leqslant \cdots \leqslant \tilde\tau(j-1)}} \operatorname{sig}_{\sigma_k}^{(j-1)}(\mathscr{B}, \tilde\tau) \\ &\quad \times W_{n-k}(\operatorname{conv}(S_{\tilde\tau}^{(j-1)} \cup \{O_{\tilde\tau}^{(j)}, \ldots, O_{\tilde\tau}^{(n-k+1)}\})). \end{aligned} \tag{81}$$

We remark that $O^{(j)}_{\tilde{\tau}}$ does only depend on τ and not on the particular choice of $\tilde{\tau}$. By definition, $O^{(j)}_{\tilde{\tau}}$ is the ortho-projection of $O^{(j+1)}_{\tau}$ onto $E^{(j)}_{\tau} = \operatorname{aff}(S^{(j)}_{\tau})$.

We will explain formula (81) geometrically rather than to give a formal proof.

Before we state the general principle let us look at two particular cases with $k=1$, $j=2$, and $n=3$ with $\tau=(3,2,1)$. Figure 5 shows the dissection of $\operatorname{conv}(S^{(2)}_{\tau} \cup \{O^{(3)}_{\tau}\})$ in the $E^{(3)}_{\tau}$-plane, which reflects the dissection of $W_2(\operatorname{conv}(S^{(2)}_{\tau} \cup \{O^{(3)}_{\tau}\}))$. On the left hand side we have $\operatorname{conv}(S^{(2)}_{\tau} \cup \{O^{(3)}_{\tau}\}) = \operatorname{conv}(S^{(1)}_{\tilde{\tau}_1} \cup \{O^{(2)}_{\tilde{\tau}_1}, O^{(3)}_{\tilde{\tau}_1}\}) \setminus \operatorname{conv}(S^{(1)}_{\tilde{\tau}_2} \cup \{O^{(2)}_{\tilde{\tau}_2}, O^{(3)}_{\tilde{\tau}_2}\})$ with $\tilde{\tau}_1 = (2,3,1)$ and $\tilde{\tau}_2 = \tau$, which yields $W_2(\operatorname{conv}(S^{(2)}_{\tau} \cup \{O^{(3)}_{\tau}\})) = W_2(\operatorname{conv}(S^{(1)}_{\tilde{\tau}_1} \cup \{O^{(2)}_{\tilde{\tau}_1}, O^{(3)}_{\tilde{\tau}_1}\})) - W_2(\operatorname{conv}(S^{(1)}_{\tilde{\tau}_2} \cup \{O^{(2)}_{\tilde{\tau}_2}, O^{(3)}_{\tilde{\tau}_2}\}))$. On the right hand side we have $\operatorname{conv}(S^{(2)}_{\tau} \cup \{O^{(3)}_{\tau}\}) = \operatorname{conv}(S^{(1)}_{\tilde{\tau}_1} \cup \{O^{(2)}_{\tilde{\tau}_1}, O^{(3)}_{\tilde{\tau}_1}\}) \cup \operatorname{conv}(S^{(1)}_{\tilde{\tau}_2} \cup \{O^{(2)}_{\tilde{\tau}_2}, O^{(3)}_{\tilde{\tau}_2}\})$ with $\tilde{\tau}_1, \tilde{\tau}_2$ as above. Thus, it holds $W_2(\operatorname{conv}(S^{(2)}_{\tau} \cup \{O^{(3)}_{\tau}\})) = W_2(\operatorname{conv}(S^{(1)}_{\tilde{\tau}_1} \cup \{O^{(2)}_{\tilde{\tau}_1}, O^{(3)}_{\tilde{\tau}_1}\})) + W_2\operatorname{conv}(S^{(1)}_{\tilde{\tau}_2} \cup \{O^{(2)}_{\tilde{\tau}_2}, O^{(3)}_{\tilde{\tau}_2}\}))$.

Taking the two examples of the figure as a guide, we can interpret formula (81) in the following way. The spherical measure $W_{n-k}(\operatorname{conv}(S^{(j)}_{\tau} \cup \{O^{(j+1)}_{\tau}, ..., O^{(n-k+1)}_{\tau}\}))$ can be written as a sum of the spherical measures $W_{n-k}(\operatorname{conv}(S^{(j-1)}_{\tilde{\tau}} \cup \{O^{(j)}_{\tilde{\tau}}, ..., O^{(n-k+1)}_{\tilde{\tau}}\}))$ that are associated to the j different facets $S^{(j-1)}_{\tilde{\tau}}$ of $S^{(j)}_{\tau}$. The contribution $W_{n-k}(\operatorname{conv}(S^{(j-1)}_{\tilde{\tau}} \cup \{O^{(j)}_{\tau}, ..., O^{(n-k+1)}_{\tilde{\tau}}\}))$ is added if $S^{(j-1)}_{\tilde{\tau}}$ is a $S^{(j)}_{\tau}$-facet of the first kind with respect to $O^{(j)}_{\tau}$ and is subtracted if $S^{(j-1)}_{\tilde{\tau}}$ is a $S^{(j)}_{\tau}$-facet of the second kind with respect to $O^{(j)}_{\tau}$.

We start the induction for proving (80) with $\ell = n-k+1$. Here, (81) is obviously identical with (80) and we are done. Now, we assume that (80) is true for an $\ell \in \{3, ..., n-k+1\}$ and we will show that then (80) will hold for $\ell - 1$ as well. We dissect $W_{n-k}(\operatorname{conv}(S^{(\ell-1)}_{\tau} \cup \{O^{(\ell)}_{\tau}, ..., O^{(n-k+1)}_{\tilde{\tau}}\}))$ using (81) and obtain

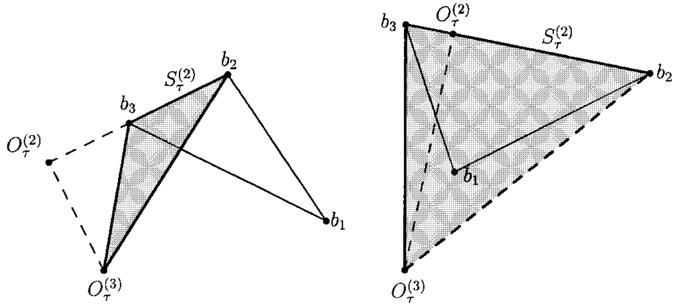

FIG. 5. The dissection of $W_2(\operatorname{conv}(S^{(2)}_{\tau} \cup \{O^{(3)}_{\tau}\}))$ for $\tau = (3, 2, 1)$.

$$
\begin{aligned}
W_{n-k}(S^{(n-k+1)}) &= \sum_{\substack{\tau \in \Pi_{n,\,n-k+1} \\ \tau(1) \leqslant \cdots \leqslant \tau(\ell-1)}} \prod_{j=\ell-1}^{n-k} \operatorname{sig}_{\sigma_k}^{(j)}(\mathscr{B}, \tau) \\
&\quad \times \sum_{\substack{\tilde{\tau} \in \Pi_{n,\,n-k+1} \\ \tilde{\tau}(\ell)=\tau(\ell), \ldots, \tilde{\tau}(n)=\tau(n) \\ \tilde{\tau}(1) \leqslant \cdots \leqslant \tilde{\tau}(\ell-2)}} \operatorname{sig}_{\sigma_k}^{(\ell-2)}(\mathscr{B}, \tilde{\tau}) \\
&\quad \times W_{n-k}(\operatorname{conv}(S_{\tilde{\tau}}^{(\ell-2)} \cup \{O_{\tilde{\tau}}^{(\ell-1)}, \ldots, O_{\tilde{\tau}}^{(n-k+1)}\})) \\
&= \sum_{\substack{\tilde{\tau} \in \Pi_{n,\,n-k+1} \\ \tilde{\tau}(1) \leqslant \cdots \leqslant \tilde{\tau}(\ell-2)}} W_{n-k}(\operatorname{conv}(S_{\tilde{\tau}}^{(\ell-2)} \cup \{O_{\tilde{\tau}}^{(\ell-1)}, \ldots, O_{\tilde{\tau}}^{(n-k+1)}\})) \\
&\quad \times \prod_{j=\ell-2}^{n-k} \operatorname{sig}_{\sigma_k}^{(j)}(\mathscr{B}, \tilde{\tau}). \qquad (82)
\end{aligned}
$$

Now, the induction is complete and (79) is proven.

Stage 2. The objective of this stage is to conclude from (79) that

$$
\sum_{\pi \in \Pi_n} W_{n-k}(S_\pi^{(n-k+1)}) = (n-k+1)! \sum_{\rho \in \Pi_n} \operatorname{sig}_{\sigma_k}(\mathscr{B}, \rho)\, W_{n-k}(\operatorname{conv}(\{O_\rho^{(1)}, \ldots, O_\rho^{(n-k+1)}\})). \qquad (83)
$$

We observe that the summands on the right hand side of (79) depend only from $\tau \circ \pi$. Thus, it seems natural to substitute $\rho = \tau \circ \pi$ and to ask how often each ρ will arise in the sum. We will show for every $\rho \in \Pi_n$ that there exist exactly $(n-k+1)!$ combinations of $\pi \in \Pi_n$ and $\tau \in \Pi_{n,\,n-k+1}$ such that $\rho = \tau \circ \pi$. These combinations can be encountered as follows: given a permutation $\rho \in \Pi_n$, then, there exist exactly $(n-k+1)!$ permutations $\pi \in \Pi_n$ with $\rho(j) = \pi(j)$ for $j = n-k+2, \ldots, n$. We fix such a choice of π and choose $\tau \in \Pi_{n,\,n-k+1}$ in such a way that $\tau \circ \pi(j) = \rho(j)$ for $j = 1, \ldots, n-k+1$. This choice of τ is uniquely determined. It is easily checked that π runs through Π_n and τ through $\Pi_{n,\,n-k+1}$ if ρ runs through Π_n. Thus, (83) is fulfilled.

Stage 3. Finally, we will represent $W_{n-k}(\operatorname{conv}(\{O_\rho^{(1)}, \ldots, O_\rho^{(n-k+1)}\}))$ from (83) by an integral, where we assume without loss of generality that $\rho = \mathrm{Id}$. The spherical angle $W_{n-k}(\operatorname{conv}(\{O^{(1)}, \ldots, O^{(n-k+1)}\}))$ with $O^{(j)} = O_{Id}^{(j)}$ is invariant under rotations. With the aid of definition (59), we conclude that the $(n-k)$-dimensional spherical angle of $\operatorname{conv}(\{O^{(1)}, \ldots, O^{(n-k+1)}\})$ equals the $(n-k)$-dimensional spherical angle generated by the $(n-k)$-dimensional simplex

$$\tilde{\Sigma}^{(n-k+1)}(h) := \operatorname{conv}\left(\left\{\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ h^{(n-k+1)} \\ \vdots \\ h^{(n)} \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ h^{(n-k)} \\ h^{(n-k+1)} \\ \vdots \\ h^{(n)} \end{pmatrix}, \ldots, \begin{pmatrix} h^{(1)} \\ \vdots \\ \vdots \\ h^{(n-k)} \\ h^{(n-k+1)} \\ \vdots \\ h^{(n)} \end{pmatrix}\right\}\right). \quad (84)$$

Hence, we have $W_{n-k}(\operatorname{conv}(\{O^{(1)}, \ldots, O^{(n-k+1)}\})) = W_{n-k}(\tilde{\Sigma}^{(n-k+1)}(h))$, where

$$\tilde{\Sigma}^{(n-k+1)}(h) = h^{(n-k+1)}e_{n-k+1} + \cdots + h^{(n)}e_n + \begin{pmatrix} \Sigma^{(n-k+1)}(h) \\ 0 \end{pmatrix} \quad (85)$$

with $\Sigma^{(n-k+1)}(h)$ as in (74). By definition (41) of the spherical angle we get

$$\begin{aligned} W_{n-k}&(\operatorname{conv}(\{O^{(1)}, \ldots, O^{(n-k+1)}\})) \\ &= W_{n-k}(\tilde{\Sigma}^{(n-k+1)}(h)) \\ &= \frac{\lambda_{n-k}(\operatorname{cone}(\tilde{\Sigma}^{(n-k+1)}(h)) \cap \mathscr{S}^{n-1})}{\mu_{n-k+1}} \\ &= \frac{r_{n-k+1}(h)}{\mu_{n-k+1}} \int_{\Sigma^{(n-k+1)}(h)} \frac{dz}{(\|z\|_2^2 + r_{n-k+1}^2(h))^{(n-k+1)/2}} \end{aligned} \quad (86)$$

and Stage 3 is complete.

Now, we insert (79) and (83) into (77) and obtain

$$\varphi_k(\mathscr{B}) = \sum_{\rho \in \Pi_n} \operatorname{sig}_{\sigma k}(B, \rho) \frac{2}{k!} W_{n-k}(\operatorname{conv}(\{O_\rho^{(1)}, \ldots, O_\rho^{(n-k+1)}\})), \quad (87)$$

from which the claim follows if we replace $W_{n-k}(\operatorname{conv}(\{O_\rho^{(1)}, \ldots, O_\rho^{(n-k+1)}\}))$ in (79) by the integral representation given in (86). ∎

From Definition 5(ii) we conclude:

Corollary 4. *The functional Φ_k is standard-reducible.*

In the next section we will profit from the standard-reducibility of Φ_k when we are going to calculate the asymptotic expectation of Φ_k.

3. THE ASYMPTOTIC EXPECTED NUMBER OF EFFICIENT VERTICES

In this section we want to calculate the asymptotic expectation for the average number Φ_k from (40) asymptotically for distributions with algebraic tail, cf. Section 1. As we learned from Theorem 2 and Corollary 3, this will be sufficient for proving Theorem 1.

In Subsection 3.1 we will provide an integral formula for standard-reducible polytope functionals under spherically symmetric distributions with algebraic tail in the unit ball, which we will use in the second subsection for estimating $\mathrm{E}(\Phi_k)$.

3.1. *Expectations under Distributions with Algebraic Tail*

Let Φ be a standard-reducible polytope functional with facet-functional ϕ, cf. Definitions 1 and 5. In Küfer (1997), we provided an integral formula for expectations of standard-reducible polytope functionals in the Rotation Symmetry Model under the additional assumption that the underlying distribution has a density function: Let a be spherically symmetrically distributed in $\mathbb{R}^n$ with radial distribution function F. Then, the distribution of $a \in \mathbb{R}^n$ has a density function if there exists a positive function $f \in \mathscr{L}^1(\mathbb{R}^n)$ such that $F(t) = \int_{t\mathscr{U}^n} f(a)\, da$ for all $t \in \mathbb{R}_0^+$, where $\mathscr{U}^n$ denotes the unit ball in $\mathbb{R}^n$.

We report the result of Küfer (1997) reduced to our needs:

THEOREM 4 (Küfer, 1997). *Let Φ be a facet-additive polytope functional with standard-reducible facet-functional ϕ and $m \geqslant n \geqslant 1$. For distributions satisfying the (RIC) with density f holds*

$$\mathrm{E}(\Phi) = (m)_n \int_{[0,\,\infty)} G^{m-n}(h^{(n)})\, \Lambda_\phi(h^{(n)})\, dh^{(n)} \tag{88}$$

with

$$G(\eta) := 1 - \frac{\mu_{n-1}}{\mu_n} \int_\eta^\infty \int_{\eta/t}^1 (1-\zeta^2)^{(n-3)/2}\, d\zeta\, dF(t) \tag{89}$$

and

$$\Lambda_\phi(h^{(n)}) := \int_{[0,\,\infty)^{n-1}} p_\sigma(h)\, \psi(h)\, dh^{(1)} \cdots dh^{(n-1)}, \tag{90}$$

where

$$p_\sigma(h) := \mu_n \prod_{j=1}^{n-1} p_\sigma^{(j)}(h)\, f(h),$$

$$p_\sigma^{(j)}(h) = \mu_j \int_{\mathbb{R}^j} [\chi(\xi_j^{(j)} \leqslant h^{(j)}) - \sigma^{(j)} \chi(\xi_j^{(j)} \geqslant h^{(j)})]\, |\xi_j^{(j)} - h^{(j)}|^{n-j}$$

$$\times f(\xi_j \,|\, h^{(j+1)}, ..., h^{(j)})\, d\xi_j \tag{91}$$

This subsection will give a formula for $\mathrm{E}(\Phi)$ in case of spherically symmetric distributions in the unit ball with algebraic tail.

Let F be a radial distribution function with algebraic tail in the unit ball, i.e.,

$$F(t) = 1 \qquad \text{for} \quad t \geqslant 1 \tag{92}$$

and $\bar{F} = 1 - F$ satisfies

$$\bar{F}(t) \sim N(1-t)^\gamma \tag{93}$$

for $t \to 1$. The relation $\sim$ in (93) means that the quotient of both sides approaches 1 if $t \to 1$. As a first step, we are going to analyse the asymptotic behaviour of $G(\eta)$ for $\eta \to 1$, which can be easily derived from the definition of G:

LEMMA 4 (Küfer, 1988). *If the radial distribution function F satisfies* (92)–(93) *we have for $\eta \to 1$*

$$1 - G(\eta) \sim \frac{\mu_{n-1}}{\mu_n} \frac{N}{2^{\gamma+1}} B\left(\frac{n-1}{2}, \gamma + 1\right) (1-\eta^2)^{(n-1+2\gamma)/2}. \tag{94}$$

The asymptotic evaluation of Λ_ϕ is a bit more complicated. We begin with the functions $p_\sigma^{(j)}$ given in (91). In order to present the result we need some additional notation. Like in (73), let $r_j = r_j(h) = ((h^{(j)})^2 + \cdots + (h^{(n)})^2)^{1/2}$ be the tail norm of the vector h. Moreover, for distributions supported by the unit ball it will be advantageous to have normalized coordinates besides $h^{(1)}, ..., h^{(n)}$. We define η_i by

$$h^{(i)} = (1 - r_{i+1}^2)^{1/2}\, \eta_i \tag{95}$$

for $i = 1, ..., n$. Then, if $h^{(i)}$ ranges in $[0, (1-r_{i+1}^2)^{1/2}]$, η_i ranges in $[0,1]$. In this notation we have the following representation of $p_\sigma^{(j)}$:

LEMMA 5. *Under spherically symmetric distributions with algebraic tail in the unit ball holds for* $j=1, ..., n$

$$p_\sigma^{(j)}(h) \sim \frac{\mu_j}{\mu_n} \frac{N\gamma}{2^{\gamma-1}} (1-r_{j+1}^2)^{(n-2+2\gamma)/2}\, g_{\sigma^{(j)}}^{(n-j,\, j)}(\eta_j), \tag{96}$$

with $h^{(n)} \to 1$ *and* $h^{(\ell)} \to 0$, $\ell = 1, ..., n-1$. *The functions* $g_\alpha^{(\ell,\, j)}$ *are given by*

$$g_\alpha^{(\ell,\, j)}(\eta) := \int_0^1 \int_{\mathscr{S}^{j-1}} [\chi(\zeta\omega^{(j)} \leqslant \eta) - \alpha\chi(\zeta\omega^{(j)} \geqslant \eta)]\, |\eta - \zeta\omega^{(j)}|^{\ell}\, d_\mu(\omega)$$
$$\times \zeta^{j-1}(1-\zeta^2)^{\gamma-1}\, d\zeta \tag{97}$$

for $\ell \geqslant 0$ *and* $\alpha \in \{-1, 1\}$.

Proof. For a distribution with density f holds

$$p_\sigma^{(j)}(h) = \mu_j \int_{\mathbb{R}^j} [\chi(\xi^{(j)} \leqslant h^{(j)}) - \sigma^{(j)}\chi(\xi^{(j)} \geqslant h^{(j)})]\, |\xi^{(j)} - h^{(j)}|^{n-j}$$
$$\times f(\xi | h^{j+1)}, ..., h^{(n)})\, d\xi, \tag{98}$$

cf. (91). First, we will represent (98) in polar coordinates: let $\xi = r\omega$ with $r \in \mathbb{R}_0^+$ and an $\omega \in \mathscr{S}^{j-1}$. Moreover, let $\hat{f}\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$\hat{f}(t) := f(h') \qquad \text{for} \quad h' \in \mathbb{R}^n \quad \text{with} \quad \|h'\|_2 = t. \tag{99}$$

Finally, let $\tilde{g}_\alpha$ be the distribution-independent function given by

$$\tilde{g}_\alpha(r) := \int_{\mathscr{S}^{j-1}} [\chi(r\omega^{(j)} \leqslant h^{(j)}) - \alpha\chi(r\omega^{(j)} \geqslant h^{(j)})]\, |r\omega^{(j)} - h^{(j)}|^{n-j}\, d\mu(\omega), \tag{100}$$

for $r \geqslant 0$. With this notation, we obtain from (98)

$$p_\sigma^{(j)}(h) = \mu_j \int_0^\infty \tilde{g}_{\sigma^{(j)}}(r)\, r^{j-1} \hat{f}((r^2 + r_{j+1}^2)^{1/2})\, dr. \tag{101}$$

We substitute $r^2 + r_{j+1}^2 = t^2$ and get with the aid of

$$\hat{f}((r^2 + r_{j+1}^2)^{1/2})\, dr = \frac{1}{\mu_n} \frac{1}{(t^2 - r_{j+1}^2)^{1/2}} \frac{dF(t)}{t^{n-2}} \tag{102}$$

the following representation of $p_\sigma^{(j)}(h)$:

$$p_\sigma^{(j)}(h) = \frac{\mu_j}{\mu_n} \int_{r_{j+1}}^{\infty} \tilde{g}_{\sigma^{(j)}}((t^2 - r_{j+1}^2)^{1/2})(t^2 - r_{j+1}^2)^{(j-2)/2} \frac{dF(t)}{t^{n-2}}. \tag{103}$$

The representation (103) holds for all distributions in the Rotation Symmetry Model and hence in particular for those satisfying (92)–(93). We integrate by parts and obtain with (93) for $h^{(n)} \to 1$ and $h^{(\ell)} \to 0$, $\ell = 1, ..., n-1$,

$$\begin{aligned} p_\sigma^{(j)}(h) &\sim \frac{\mu_j}{\mu_n} N\gamma \int_{r_{j+1}}^{1} \tilde{g}_{\sigma^{(j)}}((t^2 - r_{j+1}^2)^{1/2})(t^2 - r_{j+1}^2)^{(j-2)/2}(1-t)^{\gamma-1}\, dt \\ &\sim \frac{\mu_j}{\mu_n} \frac{N\gamma}{2^{\gamma-1}} (1 - r_{j+1}^2)^{(j-2+2\gamma)/2} \int_0^1 \tilde{g}_{\sigma^{(j)}}((1 - r_{j+1}^2)^{1/2}\, \zeta)\, \zeta^{j-1}(1-\zeta^2)^{\gamma-1}\, d\zeta \\ &= \frac{\mu_j}{\mu_n} \frac{N\gamma}{2^{\gamma-1}} (1 - r_{j+1}^2)^{(n-2+2\gamma)/2}\, g_{\sigma^{(j)}}^{(n-j,\, j)}(\eta_j). \end{aligned} \tag{104}$$

The second asymptotic equation in (104) results from the substitution $t^2 - r_{j+1}^2 = \zeta^2(1 - r_{j+1}^2)$, while the last equation simply exploits (95) and the definitions of $g_\alpha^{(\ell,\, j)}$ and $\tilde{g}_\gamma$, cf. (97) and (100). ∎

It is interesting that the functions $p_\sigma^{(j)}$ can be asymptotically represented as a simple product of two factors, where the latter depends only on η_j. This will be very useful for the following. Though the functions $g_\alpha^{(\ell,\, j)}$ in (97) look a bit complicated at first sight, they have some nice properties, which we list without the elementary proofs:

Remark. For $j \geqslant 2$ and $\ell > 0$ holds:

(i) $g_\alpha^{(\ell,\, j)}$ is an odd function for $\alpha = +1$ and an even function for $\alpha = -1$.

(ii) For $\alpha \in \{-1, 1\}$,

$$\frac{\partial}{\partial \eta} g_\alpha^{(\ell+1,\, j)}(\eta) = (\ell + 1)\, g_{-\alpha}^{(\ell,\, j)}(\eta). \tag{105}$$

Having calculated the functions $p_\sigma^{(j)}$, we can now calculate $\Lambda_\phi(h^{(n)})$ asymptotically for $h^{(n)} \to 1$:

LEMMA 6. *For $h^{(n)} \to 1$ holds*

$$\Lambda_\phi(h^{(n)}) \sim \frac{N\gamma}{2^{\gamma-1}} \int_0^{\sqrt{1-r_n^2}} \cdots \int_0^{\sqrt{1-r_2^2}} \psi(h) \times \prod_{j=1}^{n-1} p_\sigma^{(j)}(h)(1-r_1^2)^{\gamma-1}\, dh^{(1)} \cdots dh^{(n-1)}. \tag{106}$$

Proof. Like in the proof of Lemma 10 we start in the class of distributions with density f. We know from (90) that for any distribution with density holds

$$\Lambda_\phi(h^{(n)}) = \mu_n \int_{[0,\infty)^{n-1}} \psi(h) \prod_{j=1}^{n-1} p_\sigma^{(j)}(h)\, f(h)\, dh^{(1)} \cdots dh^{(n-1)}. \tag{107}$$

Observing the connection between the radial distribution function F and the density function f it is not hard to see that with $t = ((h^{(1)})^2 + r_2^2)^{1/2}$ we have

$$f(h)\, dh^{(1)} = \frac{1}{\mu_n} \frac{1}{\sqrt{t^2 - r_2^2}} \frac{dF(t)}{t^{n-2}}. \tag{108}$$

We insert this into (107) and get

$$\Lambda_\phi(h^{(n)}) = \int_{[0,\infty)^{n-2}} \int_{r_2}^{\infty} \prod_{j=1}^{n-1} p_\sigma^{(j)}(h) \frac{\psi(h)}{\sqrt{t^2 - r_2^2}} \frac{dF(t)}{t^{n-2}}\, dh^{(2)} \cdots dh^{(n-1)}, \tag{109}$$

which holds for all distributions of our model. Now, we assume that F fulfills (92)–(93), integrate by parts, and obtain for $h^{(n)} \to 1$,

$$\begin{aligned} &\Lambda_\phi(h^{(n)}) \\ &\sim N\gamma \int_0^{\sqrt{1-r_n^2}} \cdots \int_0^{\sqrt{1-r_3^2}} \int_{r_2}^{1} \psi(h) \prod_{j=1}^{n-1} p_\sigma^{(j)}(h) \frac{dh^{(1)} \cdots dh^{(n-1)}}{\sqrt{t^2 - r_2^2}} \\ &\sim \frac{N\gamma}{2^{\gamma-1}} \int_0^{\sqrt{1-r_n^2}} \cdots \int_0^{\sqrt{1-r_3^2}} \int_{r^2}^{1} \psi(h) \prod_{j=1}^{n-1} p_\sigma^{(j)}(h)(1-r_1^2)^{\gamma-1}\, dh^{(1)} \cdots dh^{(n-1)} \end{aligned} \tag{110}$$

and the proof is complete. ▮

For ease of notation, we normalize the domain of integration in (106). For this purpose we substitute $h^{(i)}$ by η_i using (95) for $i=1, ..., n$. This means, in particular,

$$\sqrt{1-r_j^2}=\prod_{\ell=j}^{n}\sqrt{1-\eta_\ell^2} \tag{111}$$

for $j=1, ..., n$ and

$$(1-r_1^2)^{\gamma-1}\,dh^{(1)}\cdots dh^{(n-1)}=\prod_{j=1}^{n}(1-\eta_j^2)^{(j-3+2\gamma)/2}\,d\eta_1\cdots d\eta_{n-1}. \tag{112}$$

We insert this into (107) and obtain with $\eta_n=h^{(n)}\to 1$,

$$\begin{aligned}\Lambda_\phi(h^{(n)})\sim(1-(h^{(n)})^2)^{(n-3+2\gamma)/2}\,\frac{N\gamma}{2^{\gamma-1}}\int_{[0,1]^{n-1}}\psi(h)\\ \times\prod_{j=1}^{n-1}p_\sigma^{(j)}(h)(1-\eta_j^2)^{(j-3+2\gamma)/2}\,d\eta_j\end{aligned} \tag{113}$$

with $h=h(\eta_1, ..., \eta_{n-1})$. We recall the explicit formulae for $p_\sigma^{(j)}$ from Lemma 5 and get the following representation of Λ_ϕ if we substitute $(1-r_j^2)^{1/2}$ by the factorization (111):

THEOREM 5. *Let Φ be a standard-reducible polytope functional with facet-functional ϕ and $m\geqslant n\geqslant 1$. Then, for spherically symmetric distributions in the unit ball with algebraic tail holds*

$$\mathrm{E}(\Phi)=(m)_n\int_0^1 G^{m-n}(h^{(n)})\,\Lambda_\phi(h^{(n)})\,dh^{(n)} \tag{114}$$

with

$$\begin{aligned}\Lambda_\phi(h^{(n)})\sim\frac{N\gamma}{2^{\gamma-1}}(1-(h^{(n)})^2)^{[(n-2+2\gamma)\,n-1]/2}\int_{[0,1]^{n-1}}\psi(h)\\ \times\prod_{j=1}^{n-1}q_\sigma^{(j)}(\eta_j)\,d\eta_1...d\eta_{n-1},\end{aligned} \tag{115}$$

for $h^{(n)}\to 1$, where $h=h(\eta_1, ..., \eta_{n-1})$ and

$$q_\sigma^{(j)}(\eta):=\frac{N\gamma}{2^{\gamma-1}}\frac{\mu_j}{\mu_n}(1-\eta^2)^{[(n-2+2\gamma)\,j-n+j-1]/2}\,g_{\sigma(j)}^{(n-j,\,j)}(\eta) \tag{116}$$

with functions $g_\alpha^{(\ell,\,j)}$ as in (97) for $j=1, ..., n-1$.

3.2. *The Proof of Theorem* 1

As an application of Theorem 5 we will now study the asymptotic behaviour of the expectation $\mathrm{E}(\Phi_k)$ for fixed n and $m \to \infty$ for distributions with algebraic tail in the unit ball. We will obtain the asymptotic behaviour of the expected number of vertices in the contour and the expected number of efficient vertices of X with respect to $\mathscr{C}$ for the same distributions as corollaries.

By Corollary 4 and Theorem 4, the expectation $\mathrm{E}(\Phi_k)$ fulfills in case of distributions in the unit ball with algebraic tail

$$\mathrm{E}(\Phi_k) = (m)_n \int_0^1 G^{m-n}(h^{(n)})\, \Lambda_{\varphi_k}(h^{(n)})\, dh^{(n)} \tag{117}$$

with Λ_{φ_k} as in (90) setting $\phi = \varphi_k$ and G as in (89).

The integral on the right hand side of (117) can be considered as a Laplacian integral. For fixed n and big m the integrand in (117) is dominated by its behaviour near $h^{(n)} = 1$ as the function G takes its maximum in $h^{(n)} = 1$. Hence, for the evaluation of $\mathrm{E}(\Phi_k)$ for fixed n and big m it is sufficient to know the asymptotic behaviour of $\Lambda_{\varphi_k}(h^{(n)})$ near $h^{(n)} = 1$. By (115), $\Lambda_{\varphi_k}(h^{(n)})$ is a weighted integral of the r-simplex-functional ψ_k. Thus, in order to calculate the asymptotic behaviour of Λ_{φ_k} near $h^{(n)} = 1$ we need to know the asymptotic behaviour of $\psi_k(h)$ for $h^{(n)}$ near 1 and $h^{(j)}$ near 0 for $j = 1, ..., n-1$:

LEMMA 7. *For $h^{(n)} \to 1$ and $h^{(j)} \to 0$, $j = 1, ..., n-1$, holds*

$$\begin{aligned}\psi_k(h) &\sim \frac{2}{n!}\,\frac{1}{\mu_{n-k+1}} \binom{n}{k} \prod_{j=1}^{n-k} h^{(j)} \\ &= \frac{2}{n!}\,\frac{1}{\mu_{n-k+1}} \binom{n}{k} \prod_{j=n-k+1}^{n} (1-\eta_j^2)^{(n-k)/2} \prod_{j=2}^{n-k} (1-\eta_j^2)^{(j-1)/2}\, \eta_j.\end{aligned} \tag{118}$$

Proof. We consider ψ_k in the representation (76) from Theorem 3,

$$\psi_k(h) = \frac{2}{n!}\,\frac{1}{\mu_{n-k+1}} \int_{\Sigma^{(n-k+1)}(h)} \frac{r_{n-k+1}(h)}{(\|z\|_2^2 + r_{n-k+1}^2(h))^{(n-k+1)/2}}\, dz, \tag{119}$$

where $r_j(h) = ((h^{(j)})^2 + \cdots + (h^{(n)})^2)^{1/2}$ and

$$\Sigma^{(n-k+1)}(h) = \mathrm{conv}\left(\left\{\begin{pmatrix}0\\ \vdots\\ \vdots\\ 0\end{pmatrix}, \begin{pmatrix}0\\ \vdots\\ 0\\ h^{(n-k)}\end{pmatrix}, ..., \begin{pmatrix}h^{(1)}\\ \vdots\\ h^{(n-k-1)}\\ h^{(n-k)}\end{pmatrix}\right\}\right). \tag{120}$$

If $h^{(n)} \to 1$ and $h^{(j)} \to 0$, $j = 1, \ldots, n-1$, we know that $r_{n-k+1}(h) \to 1$ and hence

$$\psi_k(h) \sim \frac{2}{k!} \frac{1}{\mu_{n-k+1}} \lambda_{n-k}(\Sigma^{(n-k+1)}(h)). \tag{121}$$

With $\lambda_{n-k}(\Sigma^{(n-k+1)}(h)) = 1/(n-k)! \prod_{j=1}^{n-k} h^{(j)}$ we obtain the desired formula

$$\psi_k(h) \sim \frac{2}{k!(n-k)!} \frac{1}{\mu_{n-k+1}} \prod_{j=1}^{n-k} h^{(j)}. \tag{122}$$

The second equation of (118) follows by invoking the substitution $h^{(i)} = (1 - r_{i+1}^2)^{1/2} \eta_i$ and observing (111). ▮

Next, we will exploit Lemma 7 for the calculation of $\Lambda_{\varphi_k}(h^{(n)})$ near $h^{(n)} = 1$. More precisely, we will calculate the asymptotic behaviour of the quotient $\Lambda_{\varphi_k}(h^{(n)})/\Lambda_{\varphi_1}(h^{(n)})$ near $h^{(n)} = 1$. This will be sufficient for us as $\Lambda_{\varphi_1}(h^{(n)})$ is explicitly known in terms of G, cf. Theorem 6. Immediately from Theorem 5 we obtain the asymptotic behaviour of $\Lambda_{\varphi_k}(h^{(n)})/\Lambda_{\varphi_1}(h^{(n)})$ for $h^{(n)}$ near 1 if we invoke the factorization of ψ_k given in Lemma 7:

Corollary 5. *For $k = 1, \ldots, n$ and $h^{(n)} \to 1$ holds*

$$\frac{\Lambda_{\varphi_k}(h^{(n)})}{\Lambda_{\varphi_1}(h^{(n)})} \sim D_k(n)(1 - h^{(n)})^2)^{-(k-1)/2} \tag{123}$$

with

$$D_k(n) := \frac{1}{n} \frac{\mu_n}{\mu_{n-k+1}} \binom{n}{k} \prod_{\ell=1}^{k-1} d_{\ell,k}(n), \tag{124}$$

where

$$d_{\ell,k}(n) := \frac{\int_0^1 g_{-1}^{(\ell, n-\ell)}(\eta)(1-\eta^2)^{[(n-1+2\gamma)(n-\ell)-k-1]/2}\, d\eta}{\int_0^1 g_{+1}^{(\ell, n-\ell)}(\eta)(1-\eta^2)^{[(n-1+2\gamma)(n-\ell)-\ell-2]/2}\, \eta\, d\eta}. \tag{125}$$

The function Λ_{φ_1} can be given in a surprisingly simple way. Our proof presented here is strongly based on earlier work Küfer (1988) of the author, where the nontrivial part of work was done:

THEOREM 6. *For $\eta \in [0, 1]$ holds*

$$\Lambda_{\varphi_1}(\eta) = \frac{2}{(n-1)!}(1 - G(\eta))^{n-1} \frac{\partial}{\partial \eta} G(\eta). \tag{126}$$

Proof. In Küfer (1988) we analyzed the function $\Lambda_{\tilde{V}}$ defined by

$$\Lambda_{\tilde{V}}(\eta) := \mathrm{E}(W_{n-1}(\mathrm{conv}(\mathscr{B}))|h^{(n)} = \eta) \frac{d\Pr(h^{(n)} \leqslant \eta)}{d\eta} \tag{127}$$

for $\eta \in [0, 1]$ with $h^{(n)} = h^{(n)}_{Id}$ as in (62). Corollary 3 delivers in the particular case $k = 1$ for regular $\mathscr{B}$

$$\varphi_1(\mathscr{B}) = 2W_{n-1}(\mathrm{conv}(\mathscr{B})) \tag{128}$$

which yields inserted into (127) that

$$\Lambda_{\tilde{V}}(\eta) = \frac{1}{2} \mathrm{E}(\varphi_1(\mathscr{B})|h^{(n)} = \eta) \frac{d\Pr(h^{(n)} \leqslant \eta)}{d\eta} \tag{129}$$

for $\eta \in [0, 1]$. If we reduce φ_1 via rectangular simplices, we get observing the definition of Λ_{φ_1}

$$\Lambda_{\tilde{V}}(\eta) = \frac{n!}{2} \Lambda_{\varphi_1}(\eta) \tag{130}$$

for $\eta \in [0, 1]$. In Küfer (1988) we proved that $\Lambda_{\tilde{V}}$ fulfills

$$\Lambda_{\tilde{V}}(\eta) = n(1 - G(\eta))^{n-1} \frac{\partial}{\partial \eta} G(\eta) \tag{131}$$

for $\eta \in [0, 1]$, which yields the claim if we insert (131) into (130). ∎

Now, we are ready for the asymptotic evaluation of $\mathrm{E}(\Phi_k)$ for fixed n and k and $m \to \infty$. We apply Watson's Lemma for Laplacian integrals and replace Λ_{φ_k} in (117) by two factors, the first of which is given by the right hand side of (123) and the second by the right hand side of (126). If we replace, in addition, $h^{(n)}$ by η, we obtain for $m \to \infty$

$$\mathrm{E}(\Phi_k) \sim 2\, D_k(n)\, n \binom{m}{n} \int_0^1 G^{m-n}(\eta)(1 - G(\eta))^{n-1}(1 - \eta^2)^{-(k-1)/2}\, dG(\eta). \tag{132}$$

We express $(1-\eta^2)^{-(k-1)/2}$ in terms of G: From (94), we obtain for $\eta \to 1$

$$(1-\eta^2)^{-(k-1)/2} \sim \left(\frac{\mu_{n-1}}{\mu_n} \frac{N}{2^{\gamma+1}} B\left(\frac{n-1}{2}, \gamma+1\right)\right)^{(k-1)/(n-1+2\gamma)} \times (1-G(\eta))^{-(k-1)/(n-1+2\gamma)}. \tag{133}$$

If we insert this into (132) and substitute $G(\eta) = 1-\zeta$, we get for $m \to \infty$

$$\begin{aligned} \mathrm{E}(\Phi_k) &\sim 2\, D_k(n) \left(\frac{\mu_{n-1}}{\mu_n} \frac{N}{2^{\gamma+1}} B\left(\frac{n-1}{2}, \gamma+1\right)\right)^{(k-1)/(n-1+2\gamma)} \\ &\quad \times n \binom{m}{n} \int_0^{1/2} (1-\zeta)^{m-n} \zeta^{n-1-(k-1)/(n-1+2\gamma)}\, d\zeta \\ &\sim 2\, D_k(n) \left(\frac{\mu_{n-1}}{\mu_n} \frac{N}{2^{\gamma+1}} B\left(\frac{n-1}{2}, \gamma+1\right)\right)^{(k-1)/(n-1+2\gamma)} \\ &\quad \times \frac{\Gamma(n-(k-1)/(n-1+2\gamma))}{\Gamma(n)}\, m^{(k-1)/(n-1+2\gamma)}. \end{aligned} \tag{134}$$

Thus, if we define

$$\begin{aligned} \bar{K}_\gamma(n,k) &:= 2^{-k+1} D_k(n) \left(\frac{\mu_{n-1}}{\mu_n} \frac{N}{2^{\gamma+1}} B\left(\frac{n-1}{2}, \gamma+1\right)\right)^{(k-1)/(n-1+2\gamma)} \\ &\quad \times \frac{\Gamma(n-(k-1)/(n-1+2\gamma))}{\Gamma(n)}, \\ \underline{K}_\gamma(n,k) &:= \frac{k}{\binom{n}{k-1}} \bar{K}_\gamma(n,k), \end{aligned} \tag{135}$$

we have the following result:

COROLLARY 6. *Given a spherically symmetric distribution whose radial distribution function satisfies* (92)–(93), *it holds for fixed* n, k *and* $m \to \infty$

(i) $$\mathrm{E}(\Phi_k) \sim 2^k \bar{K}_\gamma(n,k)\, m^{(k-1)/(n-1+2\gamma)} \tag{136}$$

(ii) $$\underline{K}_\gamma(n,k)\, m^{(k-1)/(n-1+2\gamma)}(1+o(1)) \leqslant 2^{-k}\mathrm{E}(\bar{V}_k) \leqslant \bar{K}_\gamma(n,k)\, m^{(k-1)/(n-1+2\gamma)}(1+o(1)) \tag{137}$$

(iii) $\underline{K}_\gamma(n, k)\, m^{(k-1)/(n-1+2\gamma)}(1+o(1))$

$$\leqslant 2^{-k}\mathrm{E}(\bar{v}_k) \leqslant \bar{K}_\gamma(n, k)\, m^{(k-1)/(n-1+2\gamma)}(1+o(1)) \tag{138}$$

Proof. While the first item has been proven above, the second item is immediate from (i) and Corollary 3. The last item exploits Theorem 2, which yields

$$2^{-k}\mathrm{E}(\bar{V}_k) \leqslant \mathrm{E}(\bar{v}_k) \leqslant 2^{-k}\mathrm{E}(\bar{V}_k) + \sum_{\ell=1}^{k-1} 2^{-\ell} \binom{k}{\ell} \mathrm{E}(\bar{V}_\ell). \tag{139}$$

From (ii) we know that $\mathrm{E}(\bar{V}_\ell) = O(m^{(\ell-1)/(n-1+2\gamma)})$ for fixed n, $m \to \infty$ and $\ell = 1, ..., k-1$. Hence, for fixed n and $m \to \infty$ holds

$$\mathrm{E}(\bar{v}_k) \sim 2^{-k}\mathrm{E}(\bar{V}_k), \tag{140}$$

from which item (iii) follows.

We remark that (140) can be proven by simple asymptotic arguments without knowledge of the more involved Theorem 2. But, for further publications concerning estimates of $\mathrm{E}(\bar{v}_k)$ it will be useful to have proper bounds of $\bar{v}_k$ in terms of $\bar{V}_k$ as well. ∎

Item (iii) of Corollary 6 corresponds to Eq. (9) of Theorem 1. In order to complete the proof of Theorem 1 it remains to establish the asymptotic behaviour of $\bar{K}_\gamma(n, k)$ that was stated in (10) for $n \to \infty$ and fixed k.

If we look a bit closer at the definition of $\bar{K}_\gamma(n, k)$ in (135), we see that for fixed k and $n \to \infty$ holds

$$\bar{K}_\gamma(n, k) \sim 2^{-k+1} D_k(n), \tag{141}$$

as we may conclude with the aid of Stirling's formula and from the definition of μ_j that for $n \to \infty$ holds

$$\frac{\Gamma(n-(k-1)/(n-1+2\gamma))}{\Gamma(n)} \to 1$$

and

$$\left(\frac{\mu_{n-1}}{\mu_n} \frac{N}{2^{\gamma+1}} B\left(\frac{n-1}{2}, \gamma+1\right)\right)^{(k-1)/(n-1+2\gamma)} \to 1. \tag{142}$$

Hence, we concentrate our attention to $D_k(n)$ from (124). The most interesting factors of $D_k(n)$ are the integral quotients $d_{\ell,k}(n)$ from (125). We observe the following asymptotic behaviour of $d_{\ell,k}(n)$ for large n:

LEMMA 8. *For fixed k, $\ell = 1, ..., k-1$ and $n \to \infty$ holds*

$$d_{\ell,k}(n) \sim \frac{n^{3/2}}{\sqrt{2}} \frac{\Gamma((\ell+1)/2)}{\Gamma((\ell+2)/2)}. \tag{143}$$

Proof. If we want to estimate $d_{\ell,k}(n)$ we need first of all upper and lower bounds for the functions $g_\alpha^{(\ell,j)}$. From (105), we conclude that $g_\alpha^{(\ell,j)}$ has non-negative derivatives up to the degree ℓ. Moreover, we know that, for odd j the jth derivative of $g_{+1}^{(\ell,j)}$ vanishes in 0, as $g_{+1}^{(\ell,j)}$ is an odd function. Hence, we obtain the following bounds for $g_\alpha^{(\ell,j)}(\eta)$ by Taylor expansions in 0

$$g_{-1}^{(\ell,j)}(\eta) \geqslant g_{-1}^{(\ell,j)}(0) \tag{144}$$

and

$$\ell g_{-1}^{(\ell-1,j)}(0)\, \eta \leqslant g_{+1}^{(\ell,j)}(\eta) \leqslant \ell g_{-1}^{(\ell-1,j)}(0)\, \eta + \binom{\ell}{3} g_{-1}^{(\ell-3,j)}(\eta)\, \eta^3. \tag{145}$$

On the other hand, if we estimate $|\eta-\zeta|^\ell \leqslant \sum_{i=0}^{\ell} \binom{\ell}{i} |\eta|^i |\zeta|^{\ell-i}$ in the definition (97) of $g_{-1}^{(\ell,j)}(\eta)$, we get the upper bound

$$g_{-1}^{(\ell,j)}(\eta) \leqslant \sum_{i=0}^{\ell} \binom{\ell}{i} g_{-1}^{(\ell-i,j)}(0)\, \eta^i. \tag{146}$$

We insert this into (144) and (145) and obtain the inclusions

$$g_{-1}^{(\ell,j)}(0) \leqslant g_{-1}^{(\ell,j)}(\eta) \leqslant \sum_{i=0}^{\ell} \binom{\ell}{i} g_{-1}^{(\ell-i,j)}(0)\, \eta^i \tag{147}$$

and

$$\begin{aligned} &\ell g_{-1}^{(\ell-1,j)}(0)\, \eta \\ &\quad \leqslant g_{+1}^{(\ell,j)}(\eta) \\ &\quad \leqslant \ell g_{-1}^{(\ell-1,j)}(0)\, \eta + \binom{\ell}{3} \sum_{i=0}^{\ell-3} \binom{\ell-3}{i} g_{-1}^{(\ell-3-i,j)}(0)\, \eta^{i+3}. \end{aligned} \tag{148}$$

We use these inclusions for estimating $d_{\ell,k}(n)$ from above and from below and obtain the following bounds:

Upper bound of $d_{\ell,k}(n)$:

$$d_{\ell,k}(n) \leqslant \tilde{d}_{\ell,k}(n) \cdot (1 + \Delta_{\ell,k}(n)) \tag{149}$$

with

$$\tilde{d}_{\ell,k}(n) := \frac{g_{-1}^{(\ell,n-\ell)}(0)}{\ell g_{-1}^{(\ell-1,n-\ell)}(0)} \frac{\int_0^1 (1-\eta^2)^{[(n-1+2\gamma)(n-\ell)-1-k]/2}\, d\eta}{\int_0^1 (1-\eta^2)^{[(n-1+2\gamma)(n-\ell)-\ell-2]/2}\, \eta^2\, d\eta} \tag{150}$$

and

$$\Delta_{\ell,k}(n) := \sum_{i=1}^{\ell} \binom{\ell}{i} \frac{g_{-1}^{(\ell-i,n-\ell)}(0)}{g_{-1}^{(\ell,n-\ell)}(0)} \frac{\int_0^1 (1-\eta^2)^{[(n-1+2\gamma)(n-\ell)-1-k]/2}\, \eta^i\, d\eta}{\int_0^1 (1-\eta^2)^{[(n-1+2\gamma)(n-\ell)-\ell-2]/2}\, d\eta} \tag{151}$$

Lower bound of $d_{\ell,k}(n)$:

$$d_{\ell,k}(n) \geqslant \tilde{d}_{\ell,k}(n) \cdot \frac{1}{1+\delta_{\ell,k}(n)} \tag{152}$$

with $\tilde{d}_{\ell,k}(n)$ as in (150) and

$$\delta_{\ell,k}(n) := \binom{\ell}{3} \sum_{i=0}^{\ell-3} \binom{\ell-3}{i} \frac{g_{-1}^{(\ell-3-i,n-\ell)}(0)}{\ell g_{-1}^{(\ell-1,n-\ell)}(0)} \times \frac{\int_0^1 (1-\eta^2)^{[(n-1+2\gamma)(n-\ell)-\ell-2]/2}\, \eta^{i+4}\, d\eta}{\int_0^1 (1-\eta^2)^{[(n-1+2\gamma)(n-\ell)-\ell-2]/2}\, \eta^2\, d\eta}. \tag{153}$$

We will show that $\tilde{d}_{\ell,k}(n)$ is the dominating part in each bound for large n. More, precisely we want to show that for $n \to \infty$ holds

$$\delta_{\ell,k}(n) \to 0 \qquad \text{and} \qquad \Delta_{\ell,k}(n) \to 0. \tag{154}$$

For this purpose we use two asymptotic equations, which are easily verified with the help of Stirling's formula:

(i) *For fixed* $i, j, \ell \in \mathbb{N}_0$ *and* $n \to \infty$,

$$\frac{g_{-1}^{(i,\, n-\ell)}(0)}{g_{-1}^{(j,\, n-\ell)}(0)} \sim \left(\frac{n}{2}\right)^{(j-i)/2} \frac{\Gamma((i+1)/2)}{\Gamma((j+1)/2)}. \tag{155}$$

(ii) *For fixed* $i, j \in \mathbb{N}_0$ *and* $n \to \infty$,

$$\frac{\int_0^1 (1-\eta^2)^{n^2/2(1+o(1))}\, \eta^i \, d\eta}{\int_0^1 (1-\eta^2)^{n^2/2(1+o(1))}\, \eta^j \, d\eta} \sim \left(\frac{n}{\sqrt{2}}\right)^{j-i} \frac{\Gamma((i+1)/2)}{\Gamma((j+1)/2)}. \tag{156}$$

Using (i) and (ii) we obtain for $n \to \infty$ and fixed k, ℓ

$$\Delta_{\ell,\, k}(n) = \sum_{i=1}^{\ell} O(n^{i/2} n^{-i}) = O(n^{-1/2}) \tag{157}$$

and

$$\delta_{\ell,\, k}(n) = \sum_{i=0}^{\ell-3} O(n^{(2+i)/2} n^{-2-i}) = O(n^{-3/2}). \tag{158}$$

Thus, we have for $n \to \infty$ and fixed ℓ, k

$$d_{\ell,\, k}(n) = \tilde{d}_{\ell,\, k}(n)(1 + O(n^{-1/2})). \tag{159}$$

It remains to study the asymptotic behaviour of $\tilde{d}_{\ell,\, k}(n)$, which we do again with the help of (i) and (ii). It holds for $n \to \infty$

$$\tilde{d}_{\ell,\, k}(n) \sim \left(\frac{n}{2}\right)^{-1/2} \frac{\Gamma((\ell+1)/2)}{\ell\,\Gamma(\ell/2)} \frac{n^2}{2} \frac{\Gamma(1/2)}{\Gamma(3/2)} \sim \frac{n^{3/2}}{\sqrt{2}} \frac{\Gamma((\ell+1)/2)}{\Gamma((\ell+2)/2)}. \tag{160}$$

and the proof of (143) is complete. ∎

We now are ready to establish line (10) of Theorem 1. An immediate consequence of Lemma 8 is the following asymptotic equivalent for the product of the functions $d_{\ell,k}(n)$ for $n \to \infty$ and fixed k

$$\prod_{\ell=1}^{k-1} d_{\ell,k}(n) \sim \left(\frac{n^3}{2}\right)^{(k-1)/2} \frac{1}{\Gamma((k+1)/2)}. \tag{161}$$

Moreover, from the definition of μ_j we conclude that for $n \to \infty$ and fixed k holds

$$\frac{1}{n}\frac{\mu_n}{\mu_{n-k+1}}\binom{n}{k} \sim \left(\frac{2\pi}{n}\right)^{(k-1)/2} \frac{n^{k-1}}{k!}. \tag{162}$$

If we put (161) and (162) together, we obtain the following asymptotics for $D_k(n)$ if k is fixed and $n \to \infty$:

$$D_k(n) \sim \frac{\pi^{(k-1)/2}}{\Gamma((k+1)/2)k!}\, n^{2(k-1)}. \tag{163}$$

From this we obtain with (135) the desired asymptotic behaviour of $\bar{K}_\gamma(n,k)$:

COROLLARY 7. *For fixed k and $n \to \infty$:*

$$\bar{K}_\gamma(n,k) \sim \left(\frac{\sqrt{\pi}}{2}\right)^{k-1} \frac{1}{k!\Gamma((k+1)/2)} \cdot n^{2(k-1)}. \tag{164}$$

3.3. *Concluding Remarks and Open Problems*

At the end of our presentation we report about some related results and some open problems in the probabilistic analysis of the number of efficient vertices within the Rotation Symmetry Model.

It is the outcome of Theorem 1 that the order of growth in m on the right hand side of (9) becomes smaller for growing γ, which is due to the fact that the tail of the distribution becomes thinner if γ is growing.

Theorem 1 remains true within the bigger class of *regularly varying distributions in the unit ball*, cf. Carnal (1970), and Küfer (1992, 1994).

It is not satisfying that there is a gap between lower and upper bound of $\mathrm{E}(v)$ and $\mathrm{E}(V)$ with respect to the growth of n, which is caused by the estimate in Lemma 2. We conjecture that for large n the expected value is very close to the upper bound.

Another interesting question is the question for the reliability of the expectations $\mathrm{E}(v)$ or $\mathrm{E}(V)$: Is it possible to prove that large deviations from the expectation become rare for large m and fixed n, k? This is the question for a tailbound of the distribution of the number of efficient vertices resp. the number of vertices in the contour.

In order to estimate tailbounds we need some more information about the random variable under consideration: one natural way to estimate tailbounds is Chebychev's inequality, which requires information about the variance:

$$\Pr\left(\left|\frac{\Phi}{\mathrm{E}(\Phi)}-1\right|\geqslant\eta\right)\leqslant\frac{1}{\eta^2}\cdot\frac{\mathrm{Var}(\Phi)}{\mathrm{E}^2(\Phi)},\qquad \eta>0. \tag{165}$$

We have some experience in estimating variances of facet-additive polytope functionals from above: in Küfer (1992, 1995) we estimated the variance for the number of pivot steps required by Phase II of Borgwardt's parametric variant of the simplex method for fixed n and $m\to\infty$. With a similar technique it is possible to obtain asymptotic upper bounds for the variance of the number of efficient vertices in the contour:

Under the assumptions of Theorem 1 *it holds for fixed* k, n *and* $m\to\infty$,

$$\frac{\mathrm{Var}(V)}{\mathrm{E}^2(V)}=O(m^{-(k-1)/(n-1+2\gamma)}). \tag{166}$$

In the light of Chebychev's inequality (166) means that even small deviations from the expectated number of vertices in the contour are unlikely if n, k are fixed and m is large. But, unfortunately this is not true for the number of efficient vertices: here, the quotient $\mathrm{Var}(v)/\mathrm{E}^2(v)$ does not tend to zero for fixed k, n and distributions with γ-algebraic tail in the ball, if m gets large. The quotient converges to a constant depending on γ, k, and n, which tends to zero for fixed k and $n\to\infty$. So, an interesting open question is whether there exist distributions in the Rotation Symmetry Model such that the quotient $\mathrm{Var}(v)/\mathrm{E}^2(v)$ is bounded from above by a function in n, which tends to zero for fixed k if $n\to\infty$.

So far, we have reported exclusively about asymptotic results for $m\to\infty$ and fixed n, k. Of great interest in optimization theory are proper distribution independent estimates of $\mathrm{E}(v)$ and $\mathrm{E}(V)$ respectively for fixed k and arbitrary n and m, which are meaningful even for moderate choices of n and m. We conjecture that for fixed k the expectation of $\mathrm{E}(V)$ and henceforth $\mathrm{E}(v)$ is polynomially bounded in terms of n and m. More precisely, we believe that the following holds:

Conjecture. There exists a constant K depending exclusively on k such that for all distributions in the Rotation Symmetry Model and all n, m and fixed k holds

$$\mathrm{E}(V)\leqslant Kn^{2(k-1)}m^{(k-1)/(n-1)}. \tag{167}$$

REFERENCES

Adler, I., and Berenguer, S. E. (1981), "Random Linear Programs," Operations Research Center Report, No. 81–4, University of California, Berkeley.

Adler, I., Karp, R. M., and Shamir, R. (1987), A simplex variant solving an $m \times d$ linear program in $O(\min(m^2, d^2))$ expected number of pivot steps, *J. Complexity* **3**, 372–378.

Adler, I., and Meggido, N. (1985), A simplex algorithm whose average number of steps is bounded between two quadratic functions of the smaller dimension, *J. Assoc. Comput. Mach.* **32**, 871–895.

Borgwardt, K. H. (1980), Die asymptotische Ordnung der mittleren Schrittzahl von Simplexverfahren, *Methods Oper. Res.* **37**, 81–95.

Borgwardt, K. H. (1982a), Some distribution-independent results about the asymptotic order of the average number of pivot steps of the simplex method, *Methods Oper. Res.* **7**, 441–462.

Borgwardt, K. H. (1982b), The average number of pivot steps required by the simplex method is polynomial, *Z. Oper. Res.* **26**, 157–177.

Borgwardt, K. H. (1987), "The Simplex Method—A Probabilistic Analysis," Springer-Verlag, Berlin.

Borgwardt, K. H. (1996), A sharp upper bound for the expected number of shadow vertices in the rotation symmetry model, preprint, University of Augsburg

Carnal, H. (1970), Die konvexe Hülle von n rotationssymmetrisch verteilten Punkten, *Z. Wahrsch. Verw. Gebiete* **15**, 168–176.

Haimovich, M. (in press), The simplex algorithm is very good—On the expected number of pivot steps and related properties of random linear programs, *Math. Programming*.

Küfer, K.-H. (1988), "Zur Asymptotik in der stochastischen Polyedertheorie," Diplomarbeit, Universität Kaiserslautern.

Küfer, K.-H. (1992), "Asymptotische Varianzanalysen in der stochastischen Polyedertheorie," Dissertation, Universität Kaiserslautern.

Küfer, K.-H. (1994), Unified probabilistic analysis of polyhedral functionals: A survey, *J. Comp. Inf.* **4**, 145–161.

Küfer, K.-H. (1995), On the variance of the number of pivot steps required by the simplex algorithm, *Z. Oper. Res.* **42**, 1–24.

Küfer, K.-H. (1997), Reduction of polytope functionals via rectangular simplices, preprint, Universität Kaiserslautern.

May, J. H., and Smith, R. L. (1982), Random polytopes: Their definition, generation and aggregate properties, *Math. Programming* **24**, 39–54.

Nachbin, L. (1965), "The Haar Integral," Van Nostrand, Princeton, NJ.

Steuer, R. (1985), "Multicriteria Optimization: Theory, Computations and Applications," Wiley, New York.

Stoer, J., and Witzgall, C. (1970), "Convexity and Optimization in Finite Dimensions," Springer-Verlag, Berlin.

Rényi, A., and Sulanke, R. (1963), Über die konvexe Hülle von n zufällig gewählten Punkten I, *Z. Wahrsch. Verw. Gebiete* **2**, 75–84.

Zeleny, M. (1974), "Linear Multiobjective Programming," Lecture Notes in Economics and Mathematical Systems, Vol. 95, Springer-Verlag, New York/Berlin.