Spectral properties of sums of certain Kronecker products

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\textbf{ABSTRACT}

We introduce two kinds of sums of Kronecker products, and their induced operators. We study the algebraic properties of these two kinds of matrices and their associated operators; the properties include their eigenvalues, their eigenvectors, and the relationships between their spectral radii or spectral abscissae. Furthermore, two projected matrices of these Kronecker products and their induced operators are also studied.

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\section{1. Introduction}

Kronecker product, also called direct product or tensor product, is an operation which owes its origin from group theory and has important applications in physics \cite{6}. Techniques involving Kronecker products have been successfully applied in many fields of matrix theory because of its computational and notational advantages, see \cite{1,2,5,7,8,12,13} and references therein. Some rank equalities and inequalities for matrix expressions of Kronecker products have been obtained in \cite{1,12}. Article \cite{2} has provided a set of maximal rank-deficient submatrices for a Kronecker product of Fourier matrices, while \cite{8} considered the approximation problem for dense block Toeplitz-plus-Hankel matrices by sums of Kronecker products of Toeplitz-plus-Hankel matrices. General explicit solutions of a class of Sylvester-polynomial matrix equations have been given by Kronecker maps in \cite{13}.
There are two kinds of sums of Kronecker products having basic forms $A \otimes A + C \otimes C$ and $A \otimes I + I \otimes A + C \otimes C$, which arise from the stability of a class of stochastic systems and the controllability/observability of bilinear systems. We refer the readers to the literature [3,4,9,10,14,15] and references therein for the motivation and applications of such matrices. Although these two kinds of Kronecker products are introduced in the above literature, no analysis has been devoted to studying their algebraic properties. This motivates us to investigate the spectral characteristics of these matrices. In order to carry out this task, an operator corresponding to each sum of Kronecker products is defined. For each operator, we consider two sets associated with two classes of eigenmatrices, namely, the complex symmetric and complex skew symmetric classes. Corresponding to these two sets of eigenmatrices, two induced operators are obtained. Two projected matrices for these two operators are introduced and their dimensions are characterized. Furthermore, exact relationships between these eigenmatrices, two induced operators are defined. The properties of their eigenvalues, eigenvectors or eigenmatrices are investigated.

This paper is organized as follows: Section 2 introduces the sum of Kronecker products $A \otimes A + C \otimes C$ and its operator. Two projected matrices and their corresponding operators are defined. The properties of their eigenvalues, eigenvectors or eigenmatrices are investigated for these matrices and their operators. Furthermore, the spectral radius relationships between these operators are discussed. Likewise, studies are carried out in Section 3 for the sum of Kronecker products $A \otimes I + I \otimes A + C \otimes C$ and its operator. Two projected matrices for these two operators are introduced and their dimensions are characterized. Furthermore, exact relationships between these eigenmatrices, two induced operators are obtained. Two projected matrices and their corresponding operators are defined. For each operator, we consider two sets associated with two classes of eigenmatrices, namely, the complex symmetric and complex skew symmetric classes. Corresponding to these two sets of eigenmatrices, two induced operators are obtained. Two projected matrices for these two operators are introduced and their dimensions are characterized. Furthermore, exact relationships between these eigenmatrices, two induced operators are obtained.

Definition 1. For real matrices $A$ and $C$, let $T_{AC} : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}$ be such that

$$T_{AC} : Z \mapsto AZA^T + CZC^T,$$

the spectrum of $T_{AC}$ is the set defined by $\sigma(T_{AC}) = \{ \lambda \in \mathbb{C} | T_{AC}(Z) = \lambda Z, Z \neq 0 \}$, and a nonzero matrix $Z$ is called an eigenmatrix associated with the eigenvalue $\lambda$ for operator $T_{AC}$, if $T_{AC}(Z) = \lambda Z$.

Using column operator, notice that

$$AZA^T + CZC^T = \lambda Z$$

is equivalent to

$$(A \otimes A + C \otimes C)\text{vec}(Z) = M_{AC}\text{vec}(Z) = \lambda \text{vec}(Z).$$
Therefore, \( \sigma(T_{AC}) = \sigma(M_{AC}) \). Furthermore, every eigenmatrix \( Z \) of operator \( T_{AC} \) for eigenvalue \( \lambda \) corresponds to one eigenvector \( \text{vec}(Z) \) of matrix \( M_{AC} \) for the same eigenvalue. 

First of all, we present the following basic property about operator \( T_{AC} \).

**Theorem 1.** For operator \( T_{AC} \), if \( T_{AC}(Z) = \lambda Z \), and \( Z = Z_1 + iZ_2 = Z^* \) with \( Z_i \in \mathbb{R}^{n \times n}, i = 1, 2 \), then \( \lambda \) is real, and \( T_{AC}(Z_i) = \lambda Z_i, i = 1, 2 \).

**Proof.** Assume that \( \lambda = \lambda_1 + i\lambda_2 \), then \( T_{AC}(Z) = \lambda Z \) means that
\[
A(Z_1 + iZ_2)A^T + C(Z_1 + iZ_2)C^T = (\lambda_1 + i\lambda_2)(Z_1 + iZ_2).
\]
Separating the real part from the imaginary part in the equality above, we have that
\[
AZ_1A^T + CZ_1C^T = \lambda_1 Z_1 - \lambda_2 Z_2, \tag{1}
\]
\[
AZ_2A^T + CZ_2C^T = \lambda_1 Z_2 + \lambda_2 Z_1. \tag{2}
\]
Since \( Z_1 = Z_1^T \) and \( Z_2 = -Z_2^T \), from (1) and (2), it is obtained that:
\[
\lambda_1 Z_1 - \lambda_2 Z_2 = \lambda_1 Z_1 + \lambda_2 Z_2,
\]
\[
- (\lambda_1 Z_2 + \lambda_2 Z_1) = - \lambda_1 Z_2 + \lambda_2 Z_1.
\]
Hence, we have that \( \lambda_2 Z_2 = 0 \) and \( \lambda_2 Z_1 = 0 \). Therefore, \( \lambda_2 = 0 \) since \( Z_1 + iZ_2 \neq 0 \), i.e., \( \lambda \) is real. \( T_{AC}(Z_i) = \lambda Z_i, i = 1, 2 \), can be directly obtained from that \( \lambda \) is real. This completes the proof. \( \square \)

There are two classes of eigenmatrices for operator \( T_{AC} \), which are described by the following theorem.

**Theorem 2.** For the operator \( T_{AC} \), there are \( \frac{n(n+1)}{2} \) eigenvalues whose eigenmatrices are complex symmetric; there are \( \frac{n(n-1)}{2} \) eigenvalues whose eigenmatrices are complex skew symmetric.

**Proof.** For any eigenvalue \( \lambda \) and its corresponding eigenmatrix \( Z \) of operator \( T_{AC} \), we have \( \lambda \left( Z + Z^T \right) = A \left( Z + Z^T \right)A^T + C \left( Z + Z^T \right)C^T \) and \( M_{AC} \text{vec}(Z) = \lambda \text{vec}(Z) \), where \( \left( Z + Z^T \right) \) is a complex symmetric matrix. Therefore, either the complex symmetric matrix \( \left( Z + Z^T \right) \) is an eigenmatrix corresponding to eigenvalue \( \lambda \), or eigenmatrix \( Z \) is complex skew symmetric matrix.

Furthermore, \( \text{vec} \left( AZA^T + CZC^T \right) = M_{AC} \text{vec}(Z) \). Write
\[
M_{AC} \equiv \begin{bmatrix} m_{11}, m_{21}, \ldots, m_{n1}, m_{12}, m_{22}, \ldots, m_{n2}, \ldots, m_{1n}, \ldots, m_{nn} \end{bmatrix}^T,
\]
where \( m_{ij} \in \mathbb{R}^{1 \times n^2}, i, j = 1, 2, \ldots, n \). It can be checked that \( m_{ij} \text{vec}(Z) = m_{ij} \text{vec}(Z) \) holds for any complex symmetric matrix \( Z \). Therefore, there exists a unique matrix \( L_{AC} \in \mathbb{R}^{\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}} \) such that \( M_{AC} \text{vec}(Z) \) can be rewritten as \( L_{AC} \eta \) for any complex symmetric matrix \( Z \), where
\[
\eta = \begin{bmatrix} z_{11}, z_{21}, \ldots, z_{n1}, z_{22}, \ldots, z_{n2}, \ldots, z_{n-1,n-1}, z_{n-1,1}, z_{n,1} \end{bmatrix}^T \equiv \Pi(Z).
\]
Here, \( \Pi(\cdot) \) is a projection operator which effectively selects the elements in the lower triangular part of \( Z \). It is obvious that \( L_{AC} \eta = \lambda \eta \) holds, if \( M_{AC} \text{vec}(Z) = \lambda \text{vec}(Z) \) for any complex symmetric matrix \( Z \). Conversely, \( M_{AC} \text{vec}(Z_\xi) = \lambda \text{vec}(Z_\xi) \) holds, if \( L_{AC} \xi = \lambda \xi \), where \( Z_\xi \) is a complex symmetric matrix satisfying \( \xi = \Pi(Z_\xi) \). Therefore, operator \( T_{AC} \) has \( \frac{n(n+1)}{2} \) eigenvalues whose eigenmatrices are complex symmetric, since \( L_{AC} \in \mathbb{R}^{\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}} \).

Similarly, it can be checked that \( m_{ij} \text{vec}(Z) = -m_{ij} \text{vec}(Z) \) holds for any complex skew symmetric matrix \( Z \). Therefore, there exists a unique matrix \( \bar{L}_{AC} \in \mathbb{R}^{\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}} \) such that \( M_{AC} \text{vec}(Z) \) can be rewritten as \( \bar{L}_{AC} \theta \) for any complex skew symmetric matrix \( Z \), where
\[ \theta = [z_{21}, \ldots, z_{n1}, z_{32}, \ldots, z_{n2}, \ldots, z_{n-1,n}]^T \triangleq \Pi(Z). \]

Here, \( \Pi(\cdot) \) is a projection operator which selects the elements in the strictly lower triangular part of \( Z \). It is obvious that \( \mathcal{L}(\cdot) = \lambda \theta \) holds, if \( M_{AC} \text{vec}(Z) = \lambda \text{vec}(Z) \) for any complex skew symmetric matrix \( Z \). Conversely, \( M_{AC} \text{vec}(Z_\xi) = \lambda \text{vec}(Z_\xi) \) holds, if \( \mathcal{L}(\cdot) = \lambda \xi \), where \( Z_\xi \) is a complex skew symmetric matrix satisfying \( \xi = \Pi(Z_\xi) \). Therefore the operator \( T_{AC} \) has \( \frac{n(n-1)}{2} \) eigenvalues whose eigenmatrices are complex skew symmetric, since \( \mathcal{L}(\cdot) \in \mathbb{R}^{n(n-1) \times n^2} \).

**Remark 1.** For the matrix \( M_{AC} \in \mathbb{R}^{n^2 \times n^2} \), there are \( \frac{n(n+1)}{2} \) eigenvalues, for any eigenvector \( \xi \) of which, there is a complex symmetric \( Z \) such that \( \text{vec}(Z) = \xi \); there are \( \frac{n(n-1)}{2} \) eigenvalues, for any eigenvector \( \xi \) of which, there exits a complex skew symmetric \( Z \) such that \( \text{vec}(Z) = \xi \).

**Remark 2.** For \( A \in \mathbb{R}^{n \times n} \), there are \( n \) linearly independent eigenvectors if and only if \( A \) is a non-defective matrix. When \( A \) is defective, the number of linearly independent eigenvectors of \( A \) is strictly less than \( n \). However, one can supplement with generalized eigenvectors in order to obtain a basis of \( \mathbb{R}^n \) [7]. Notice that every eigenmatrix of operator \( T_{AC} \) corresponds to an eigenvector of matrix \( M_{AC} \), so all \( \frac{n(n+1)}{2} \) complex symmetric matrices (or \( \frac{n(n-1)}{2} \) complex skew symmetric matrices) of operator \( T_{AC} \) are not linearly independent when \( M_{AC} \) is defective. In this case, one can make use of the concept of generalized eigenmatrices for operator \( T_{AC} \) similar to that of generalized eigenvectors of matrix \( M_{AC} \).

The following example gives as an illustration of the idea of generalized eigenmatrices in Remark 2.

**Example 1.** Take

\[ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

which gives

\[ M_{AC} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]

There are four eigenvalues and they are all equal to 1. By computing we only obtain two linearly independent eigenvectors

\[ \xi_1 = [0 \quad 1 \quad -1 \quad 0]^T, \quad \xi_2 = [1 \quad 0 \quad 0 \quad 0]^T \]

since \( M_{AC} \) is similar to matrix

\[ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]

Via the chain of generalized eigenvectors:

\[ (M_{AC} - I)\xi_3 = \xi_2, \]  \hspace{1cm} (3)
\[ (M_{AC} - I)\xi_4 = \xi_3, \]  \hspace{1cm} (4)

two generalized eigenvectors are obtained

\[ \xi_3 = [0 \quad 0.5 \quad 0.5 \quad 0]^T, \quad \xi_4 = [0 \quad -0.25 \quad -0.25 \quad 0.5]^T. \]
Therefore, for operator $T_{A,C}$, we have two eigenmatrices
\[
Z_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
\]
and two generalized eigenmatrices
\[
Z_3 = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}, \quad Z_4 = \begin{bmatrix} 0 & -0.25 \\ -0.25 & 0.5 \end{bmatrix}.
\]
It is clear to see that for operator $T_{A,C}$, there are 1 skew symmetric eigenmatrix and 3 symmetric eigenmatrices together when generalized eigenmatrices are counted.

Now, one may ask what are the two matrices $Z_3$ and $Z_4$ in the original matrix operator $T_{A,C}$. From (3) and (4), we have
\[
AZ_3 A^T + CZ_3 C^T - Z_3 = Z_2, \quad (5)
\]
\[
AZ_4 A^T + CZ_4 C^T - Z_4 = Z_3, \quad (6)
\]
which correspond to the characterization of the chain of generalized eigenmatrices.

The matrices $L_{A,C}$ and $T_{A,C}$ in the proof of Theorem 2 can be regarded as two projected matrices induced by matrix $M_{A,C}$. In fact, there are two operators corresponding to matrices $L_{A,C}$ and $T_{A,C}$.

**Definition 2.** For real matrices $A$ and $C$, let $\mathcal{L}_{A,C} : \mathbb{S}^{n \times n} \to \mathbb{S}^{n \times n}$ such that
\[
\mathcal{L}_{A,C} : Z \mapsto AZA^T + CZC^T,
\]
the spectrum of $\mathcal{L}_{A,C}$ is the set defined by $\sigma(\mathcal{L}_{A,C}) = \{ \lambda \in \mathbb{C} \mid \mathcal{L}_{A,C}(Z) = \lambda Z, Z \neq 0 \}$, and a nonzero complex symmetric matrix $Z$ is called an eigenmatrix associated with the eigenvalue $\lambda$ for operator $\mathcal{L}_{A,C}$, if $\mathcal{L}_{A,C}(Z) = \lambda Z$.

**Definition 3.** For real matrices $A$ and $C$, let $\mathcal{T}_{A,C} : \mathbb{S}^{n \times n} \to \mathbb{S}^{n \times n}$ such that
\[
\mathcal{T}_{A,C} : Z \mapsto AZA^T + CZC^T,
\]
the spectrum of $\mathcal{T}_{A,C}$ is the set defined by $\sigma(\mathcal{T}_{A,C}) = \{ \lambda \in \mathbb{C} \mid \mathcal{T}_{A,C}(Z) = \lambda Z, Z \neq 0 \}$, and a nonzero complex skew symmetric matrix $Z$ is called an eigenmatrix associated with the eigenvalue $\lambda$ for operator $\mathcal{T}_{A,C}$, if $\mathcal{T}_{A,C}(Z) = \lambda Z$.

From the proof of Theorem 2 and Definitions 2 and 3, we have $\sigma(\mathcal{L}_{A,C}) = \sigma(L_{A,C}), \sigma(\mathcal{T}_{A,C}) = \sigma(T_{A,C}), \sigma(\mathcal{L}_{A,C}) \cup \sigma(\mathcal{T}_{A,C}) = \sigma(M_{A,C})$, and $\sigma(L_{A,C}) \cup \sigma(T_{A,C}) = \sigma(M_{A,C})$. Furthermore, every eigenmatrix $Z$ of operator $\mathcal{L}_{A,C}$ or $\mathcal{T}_{A,C}$ for eigenvalue $\lambda$ corresponds to one eigenvector $\text{vec}(Z)$ of matrix $L_{A,C}$ or $T_{A,C}$ for the same eigenvalue.

First, we focus on the relationship between $\sigma(\mathcal{L}_{A,C})$ and $\sigma(M_{A,C})$, or equivalently, between $\sigma(L_{A,C})$ and $\sigma(M_{A,C})$. From the construction of matrix $L_{A,C}$ and relationship $\sigma(L_{A,C}) \cup \sigma(T_{A,C}) = \sigma(M_{A,C})$, we know that there exist two matrices $W_1$ and $W_2$, obtained from products of some elementary matrices and $W_1 \cdot W_2 = I$, such that
\[
W_1 M_{A,C} W_2 = \begin{bmatrix} L_{A,C} & \ast \\ 0 & L_{A,C}^* \end{bmatrix}. \quad (7)
\]

**Example 2.** If matrices $A \in \mathbb{R}^{2 \times 2}$ and $C \in \mathbb{R}^{2 \times 2}$, then
\[
W_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]
and
\[
W_2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}.
\]

For matrices \(A\) and \(C\) given in Example 1, via formula (7) one can get
\[
L_{AC} = \begin{bmatrix}
1 & 2 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}, \quad \sigma(L_{AC}) = [1].
\]

Since \(\sigma(L_{AC}) \subset \sigma(M_{AC})\), we have \(\rho(M_{AC}) \geq \rho(L_{AC})\). In the following we prove that \(\rho(M_{AC}) = \rho(L_{AC})\) holds. The following lemmas are useful for proving of our main results.

**Lemma 1** [11]. For any matrix \(L \in \mathbb{C}^{n \times n}\), there exist matrices \(L_i \in \mathbb{C}^{n^+}, i = 1, 2, 3, 4\) such that \(L = (L_1 - L_2) + i(L_3 - L_4)\). Moreover, \(\|L_i\| \leq \|L\|\) for \(i = 1, 2, 3, 4\).

**Lemma 2** [3]. For any matrix \(P \in \mathbb{C}^{n^+}\), there exist matrices \(P_i = x_i x_i^*, i = 1, 2, \ldots, q\) with \(x_i \in \mathbb{C}^n\) such that \(P = \sum_{i=1}^{q} P_i\). Moreover, \(\|P_i\| \leq \|P\|\) for \(i = 1, 2, \ldots, q\).

**Lemma 3.** For matrix iteration
\[
X(k + 1) = AX(k)A^T + CX(k)C^T, \quad X(0) = X_0,
\]
where \(A\) and \(C\) are all real \(n \times n\) matrices, the following two assertions are equivalent:

(i) \(\lim_{k \to \infty} X(k) = 0\) holds for any \(X_0\) satisfying \(X_0 = x_0 x_0^*\), where \(x_0 \in \mathbb{C}^n\).

(ii) \(\lim_{k \to \infty} X(k) = 0\) holds for any \(X_0 \in \mathbb{C}^{n \times n}\).

**Proof.** It is obvious that (ii) \(\Rightarrow\) (i). It suffices to prove that (i) \(\Rightarrow\) (ii). Assume that (i) holds. For any \(X_0 \in \mathbb{C}^{n \times n}\), using Lemma 1, there are matrices \(L_i \in \mathbb{C}^{n^+}, i = 1, 2, 3, 4\) such that \(X_0 = (L_1 - L_2) + i(L_3 - L_4)\), with \(\|L_i\| \leq \|X_0\|\) for \(i = 1, 2, 3, 4\). Furthermore, for any \(L_i, i = 1, 2, 3, 4\), according to Lemma 2, for some \(q_i\), there exist \(L_{i1}, L_{i2}, \ldots, L_{iq_i}\) with \(L_{ij} = x_j x_j^*\) and \(\|L_{ij}\| \leq \|L_i\|, i = 1, 2, 3, 4, j = 1, 2, \ldots, q_i\). Thus, for the initial condition \(X_0 \in \mathbb{C}^{n \times n}\), we have
\[
X_0 = (L_1 - L_2) + i(L_3 - L_4)
\]
\[
= \left( \sum_{j=1}^{q_1} x_{1j} x_{1j}^* - \sum_{j=1}^{q_2} x_{2j} x_{2j}^* \right) + i \left( \sum_{j=1}^{q_3} x_{3j} x_{3j}^* - \sum_{j=1}^{q_4} x_{4j} x_{4j}^* \right).
\]
Since \(A\) and \(C\) are real in iteration (8), we obtain that
\[
X(k + 1) = \left( \sum_{j=1}^{q_1} x_{1j}(k + 1) - \sum_{j=1}^{q_2} x_{2j}(k + 1) \right) + i \left( \sum_{j=1}^{q_3} x_{3j}(k + 1) - \sum_{j=1}^{q_4} x_{4j}(k + 1) \right),
\]
where \(X(k + 1)\) and \(X_{ij}(k + 1)\) are the \((k + 1)\)th iterates of (8), respectively, for initial conditions \(X(0) = X_0\) and \(X(0) = x_{ij} x_{ij}^*, i = 1, 2, 3, 4, j = 1, 2, \ldots, q_i\). From (i), we know that \(\lim_{k \to \infty} X_{ij}(k) = 0, i = 1, 2, 3, 4, j = 1, 2, \ldots, q_i\). Therefore, \(\lim_{k \to \infty} X(k) = 0\) for any initial condition \(X(0) = X_0\). This completes the proof. \(\square\)
Now we are ready to prove the following theorem, which deals with the exact relationship between \( \rho(M_{AC}) \) and \( \rho(L_{AC}) \), or equivalently, between \( \rho(M_{AC}) \) and \( \rho(L_{AC}) \).

**Theorem 3.** For any real matrices \( A \) and \( C \), \( \rho(M_{AC}) = \rho(L_{AC}) \), or \( \rho(M_{AC}) = \rho(L_{AC}) \).

**Proof.** Study the following matrix iteration:

\[
X(k + 1) = AX(k)A^T + CX(k)C^T, \quad X(0) = X_0 = x_0x_0^*.
\]

where \( x_0 \in \mathbb{C}^n \). Using column operator, (9) is equivalent to

\[
\text{vec}(X(k + 1)) = (A \otimes A + C \otimes C)\text{vec}(X(k)) = M_{AC}\text{vec}(X(k)),
\]

with initial value \( \text{vec}(X(0)) = \text{vec}(X_0) \). From Lemma 3, we have

\[
\lim_{k \to \infty} X(k) = 0 \iff \lim_{k \to \infty} \text{vec}(X(k)) = 0 \iff \rho(M_{AC}) < 1.
\]

On the other hand, from Theorem 2 and noticing that the symmetry of matrix \( X(k) \), (9) can also be equivalently written as

\[
Y(k + 1) = L_{AC}Y(k)
\]

where \( Y(k) = [x_{11}(k), x_{21}(k), \ldots, x_{n1}(k), x_{22}(k), \ldots, x_{n2}(k), \ldots, x_{n-n,n-1}(k), x_{n-1,n}(k), x_{nn}(k)]^T \), and \( X(k) = (x_0(k))_{n \times n} \). It is clear that

\[
\lim_{k \to \infty} X(k) = 0 \iff \lim_{k \to \infty} Y(k) = 0 \iff \rho(L_{AC}) < 1.
\]

Thus we have

\[
\rho(M_{AC}) < 1 \iff \rho(L_{AC}) < 1. \tag{12}
\]

Since \( \sigma(L_{AC}) \subset \sigma(M_{AC}) \), then \( \rho(L_{AC}) \leq \rho(M_{AC}) \). Assume that \( \rho(L_{AC}) = \rho_0 < \rho(M_{AC}) \), then there exists \( \varepsilon > 0 \) such that \( \rho_0 + \varepsilon < \rho(M_{AC}) \). Note that \( M = \frac{1}{\sqrt{\rho_0 + \varepsilon}}A_{\sqrt{\rho_0 + \varepsilon}}^{-1}C \) and that

\[
L = \frac{1}{\sqrt{\rho_0 + \varepsilon}}A_{\sqrt{\rho_0 + \varepsilon}}^{-1}C = \frac{1}{\rho_0}L_{AC}, \text{ so we have } \rho \left( M \frac{1}{\sqrt{\rho_0 + \varepsilon}}A_{\sqrt{\rho_0 + \varepsilon}}^{-1}C \right) > 1, \text{ and } \rho \left( L \frac{1}{\sqrt{\rho_0 + \varepsilon}}A_{\sqrt{\rho_0 + \varepsilon}}^{-1}C \right) < 1, \text{ which conflicts with (12)}. \therefore, \rho(L_{AC}) = \rho(M_{AC}) \) holds. \( \square \)

For matrices \( M_{AC} \) and \( L_{AC} \), Theorem 3 discusses the property of maximal modulus of their eigenvalues. Motivated by Theorem 3, one may ask whether \( \sigma(M_{AC}) = \sigma(L_{AC}) \) holds. Next we show that \( \sigma(M_{AC}) = \sigma(L_{AC}) \) holds for \( C = 0 \).

**Theorem 4.** For real matrix \( A \), \( \sigma(M_{A0}) = \sigma(L_{A0}) \).

**Proof.** The spectrum of matrix \( M_{A0} \) is \( \{ \lambda_i : i, j = 1, 2, \ldots, n \} \), where \( \{ \lambda_i : i = 1, 2, \ldots, n \} \) is the spectrum of matrix \( A \). It is clear that \( \sigma(M_{A0}) = \min(\lambda_i^2 : \lambda_i \in \sigma(A)) \). For \( \lambda_i^2, i = 1, 2, \ldots, n \), the corresponding eigenvector is \( \text{vec}(\xi_i\xi_i^T) \), where \( \xi_i \) is an eigenvector of \( \lambda_i \) of matrix \( A \). Thus \( \lambda_i^2, i = 1, 2, \ldots, n \) are all eigenvalues of matrix \( L_{A0} \) since \( \xi_i\xi_i^T \) is complex symmetric. Therefore \( \sigma(M_{A0}) = \sigma(L_{A0}) \). \( \square \)

However, for a general matrix \( C \), we have \( \sigma(M_{AC}) \leq \sigma(L_{AC}) \), but \( \sigma(M_{AC}) = \sigma(L_{AC}) \) does not always hold, which we illustrate by the following example.

**Example 3.** Take

\[
A = \begin{bmatrix} 0.5245 & 0.4820 \\ 1.3643 & -0.7871 \end{bmatrix}, \quad C = \begin{bmatrix} 0.7520 & -0.8162 \\ -0.1669 & 2.0941 \end{bmatrix}
\]

then
Definition 4. Operators can be similarly defined, but are omitted here for brevity.

Example 3, it is easy to see that

\[ \sigma(A) \subset \sigma(D) \]

\[ \sigma(A) \subset \sigma(D) \]

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Pre- and post-multiplying \( M_{AC} \), respectively, by matrices \( W_1 \) and \( W_2 \) given in Example 2, we have that

\[
W_1 M_{AC} W_2 = \begin{bmatrix}
0.8406 & -0.7219 & 0.8985 & -0.3610 \\
0.5901 & 1.9557 & -2.0886 & 0.7938 \\
1.8892 & -2.8467 & 5.0048 & -1.4233 \\
0 & 0 & 0 & 0.3681
\end{bmatrix}.
\]

Thus, we obtain

\[
L_{AC} = \begin{bmatrix}
0.8406 & -0.7219 & 0.8985 \\
0.5901 & 1.9557 & -2.0886 \\
1.8892 & -2.8467 & 5.0048 \\
0 & 0 & 0 & 0.3681
\end{bmatrix}, ~ T_{AC} = [0.3681].
\]

Computing directly gives \( \sigma(M_{AC}) = \{0.5989 + 0.3916i, 0.5989 - 0.3916i, 6.6031, 0.3681\} \) and \( \sigma(L_{AC}) = \{0.5989 + 0.3916i, 0.5989 - 0.3916i, 6.6031\} \). It is easy to check that \( \sigma(M_{AC}) = 0.3681 \), while \( \sigma(L_{AC}) = \sqrt{(0.5989^2 + 0.3916^2)} = 0.7156 \). Hence, we obtain \( \sigma(M_{AC}) \neq \sigma(L_{AC}) \).

From the discussions above, there are clear spectral relationships between \( \sigma(L_{AC}) \) and \( \sigma(M_{AC}) \).

As for \( \sigma(L_{AC}) \) and \( \sigma(M_{AC}) \), from the proof of Theorem 2, we know \( \sigma(L_{AC}) \subset \sigma(M_{AC}) \). In addition, in Example 3, it is easy to see that \( \rho(M_{AC}) = 6.6031 \), while \( \rho(L_{AC}) = 0.3681 \). Thus, \( \rho(M_{AC}) = \rho(L_{AC}) \) does not hold in general. The following example illustrates that \( \sigma(M_{AC}) = \sigma(L_{AC}) \) does not hold either.

Example 4. Take

\[
A = \begin{bmatrix}
-0.4326 & 0.1253 \\
-1.6656 & 0.2877
\end{bmatrix}, ~ C = \begin{bmatrix}
-1.1465 & 1.8927 \\
1.1909 & -0.0376
\end{bmatrix}.
\]

We obtain that \( \sigma(M_{AC}) = \{3.8621, -1.9198, 0.7696, -1.2887\} \), and \( \sigma(L_{AC}) = \{-1.2887\} \). Hence \( \sigma(M_{AC}) \neq \sigma(L_{AC}) \), while \( \sigma(M_{AC}) \neq \sigma(L_{AC}) \).

In the end of this section, we introduce the adjoint operator of \( T_{AC} \). As for \( L_{AC} \) and \( T_{AC} \), adjoint operators can be similarly defined, but are omitted here for brevity.

Definition 4. Define \( T_{AC}^* : C^{n \times n} \to C^{n \times n} \), be the adjoint operator of \( T_{AC} \) such that

\[
T_{AC}^* : Z \mapsto A^T Z A + C^T Z C.
\]

The spectrum of \( T_{AC}^* \) is the set

\[
\sigma(T_{AC}^*) = \{ \lambda \in C \mid T_{AC}^* (Z) = \lambda Z, Z \neq 0 \}.
\]

From Definitions 1 and 4, it is clear that \( T_{AC}^* = T_{AC}^{*T} \). Furthermore, for operator \( T_{AC} \) and its adjoint operator \( T_{AC}^* \), the following relationships hold:

1. \( \sigma(T_{AC}^*) = \sigma(T_{AC}) = \sigma(M_{AC}) = \sigma(M_{AC}^{T}) \).
2. \( \rho(M_{AC}) = \rho(M_{AC}^{T}) \).

3. Kronecker product \( A \otimes I + I \otimes A + C \otimes C \)

All conclusions in this section can be generalized to the general case \( \sum_{i=1}^{m_1} (A_i \otimes I + I \otimes A_i) + \sum_{i=1}^{m_2} C_i \otimes C_i \). For simplicity, we only consider the case \( m_1 = m_2 = 1 \). Denote \( A \otimes I + I \otimes A + C \otimes C \)

\[
M_{AC} = \begin{bmatrix}
0.8406 & -0.3610 & -0.3610 & 0.8985 \\
0.5901 & 1.1619 & 0.7938 & -2.0886 \\
0.5901 & 0.7938 & 1.1619 & -2.0886 \\
1.8892 & -1.4233 & -1.4233 & 5.0048
\end{bmatrix}.
\]
\( C \triangleq \tilde{M}_{AC} \), where the matrices \( A \) and \( C \) belong to \( \mathbb{R}^{n \times n} \), \( n \geq 1 \). In this section, we develop results parallel to those obtained in Section 2 to matrix \( \tilde{M}_{AC} \). We introduce the following operator corresponding to \( \tilde{M}_{AC} \).

**Definition 5.** For real matrices \( A \) and \( C \), let \( \tilde{T}_{AC} : C^{n \times n} \to C^{n \times n} \) such that

\[
\tilde{T}_{AC} : Z \mapsto AZ + ZA^T + CZC^T,
\]

the spectrum of \( \tilde{T}_{AC} \) is the set defined by \( \sigma(\tilde{T}_{AC}) = \{ \lambda \in \mathbb{C} | \tilde{T}_{AC}(Z) = \lambda Z, Z \neq 0 \} \), and a nonzero matrix \( Z \) is called an eigenmatrix associated with the eigenvalue \( \lambda \) for operator \( \tilde{T}_{AC} \). If \( \tilde{T}_{AC}(Z) = \lambda Z \).

Similarly using column operator, notice that

\[
AZ + ZA^T + CZC^T = \lambda Z
\]

is equivalent to

\[
(A \otimes I + I \otimes A + C \otimes C)vec(Z) = \tilde{M}_{AC}vec(Z) = \lambda vec(Z).
\]

Therefore, \( \sigma(\tilde{T}_{AC}) = \sigma(\tilde{M}_{AC}) \). Furthermore, every eigenmatrix \( Z \) of operator \( \tilde{T}_{AC} \) corresponds to one eigenvector \( vec(Z) \) of matrix \( \tilde{M}_{AC} \) for the same eigenvalue \( \lambda \).

It is easy to check that Theorem 1 and Theorem 2 hold for the operator \( \tilde{T}_{AC} \). Similarly, we can also introduce projected matrices \( \tilde{I}_{AC} \in \mathbb{R}^{n(n+1) \times n(n+1)} \) and \( \tilde{I}_{AC} \in \mathbb{R}^{n(n-1) \times n(n-1)} \) for the matrix \( \tilde{M}_{AC} \).

Furthermore, there are also operators corresponding to matrices \( \tilde{I}_{AC} \) and \( \tilde{I}_{AC} \).

**Definition 6.** For real matrices \( A \) and \( C \), let \( \tilde{E}_{AC} : \mathbb{S}^{n \times n} \to \mathbb{S}^{n \times n} \) such that

\[
\tilde{E}_{AC} : Z \mapsto AZ + ZA^T + CZC^T,
\]

the spectrum of \( \tilde{E}_{AC} \) is the set defined by \( \sigma(\tilde{E}_{AC}) = \{ \lambda \in \mathbb{C} | \tilde{E}_{AC}(Z) = \lambda Z, Z \neq 0 \} \), and a nonzero complex symmetric matrix \( Z \) is called an eigenmatrix associated with the eigenvalue \( \lambda \) for operator \( \tilde{E}_{AC} \), if \( \tilde{E}_{AC}(Z) = \lambda Z \).

**Definition 7.** For real matrices \( A \) and \( C \), let \( \tilde{E}_{AC} : \mathbb{S}^{n \times n} \to \mathbb{S}^{n \times n} \) such that

\[
\tilde{E}_{AC} : Z \mapsto AZ + ZA^T + CZC^T,
\]

the spectrum of \( \tilde{E}_{AC} \) is the set defined by \( \sigma(\tilde{E}_{AC}) = \{ \lambda \in \mathbb{C} | \tilde{E}_{AC}(Z) = \lambda Z, Z \neq 0 \} \), and a nonzero complex skew symmetric matrix \( Z \) is called an eigenmatrix associated with the eigenvalue \( \lambda \) for operator \( \tilde{E}_{AC} \), if \( \tilde{E}_{AC}(Z) = \lambda Z \).

As for relationships of \( \sigma(\tilde{T}_{AC}), \sigma(\tilde{M}_{AC}), \sigma(\tilde{E}_{AC}), \sigma(\tilde{I}_{AC}) \) and \( \sigma(\tilde{I}_{AC}) \), we similarly have \( \sigma(\tilde{E}_{AC}) = \sigma(\tilde{I}_{AC}) \subset \sigma(\tilde{E}_{AC}) = \sigma(\tilde{I}_{AC}) = \sigma(\tilde{E}_{AC}) = \sigma(\tilde{I}_{AC}) \subset \sigma(\tilde{E}_{AC}) = \sigma(\tilde{I}_{AC}) \). Corresponding to Theorem 3, further relationships between \( \tilde{M}_{AC} \) and \( \tilde{I}_{AC} \) can be established. To arrive at the results, we extend Lemma 3 to the matrix differential equation case.

**Lemma 4.** For matrix differential equation \( \dot{X}(t) = AX(t) + X(t)A^T + CX(t)C^T \), \( X(0) = X_0 \), where \( A \) and \( C \) are all real \( n \times n \) matrices, the following two assertions are equivalent:

(i) \( \lim_{t \to \infty} X(t) = 0 \) holds for any \( X_0 \) satisfying \( X_0 = x_0 x_0^* \), where \( x_0 \in \mathbb{C}^n \).

(ii) \( \lim_{t \to \infty} X(t) = 0 \) holds for any \( X_0 \in \mathbb{C}^{n \times n} \).

The proof of Lemma 4 can be carried out by following similar lines as in the proof of Lemma 3 and thus omitted here.

Likewise, utilizing Lemma 4, we can deal with the exact spectral abscissa relationship for matrices \( \tilde{M}_{AC} \) and \( \tilde{I}_{AC} \), or equivalently, for operators \( \tilde{M}_{AC} \) and \( \tilde{E}_{AC} \), which is presented in the following theorem.
Theorem 5. For any real matrices A and C, \(\alpha(\tilde{M}_{A,C}) = \alpha(\tilde{L}_{A,C})\) or \(\alpha(\tilde{M}_{A,C}) = \alpha(\tilde{L}_{A,C})\).

Proof. Study the following matrix differential equation
\[
\dot{X}(t) = AX(t) + X(t)A^T + CX(t)C^T, \quad X(0) = X_0 = x_0x_0^T,
\] (13)
where \(x_0 \in \mathbb{C}^n\). Using column operator, (13) is equivalent to
\[
\text{vec}(\dot{X}(t)) = (A \otimes I + I \otimes A + C \otimes C)\text{vec}(X(t)) = \tilde{M}_{A,C}\text{vec}(X(t)),
\] (14)
with initial value \(\text{vec}(X(0)) = \text{vec}(X_0)\). According to Lemma 4, \(\lim_{t \to \infty} \text{vec}(X(t)) = 0\) holds for any initial value \(X_0\) satisfying \(X_0 = x_0x_0^T\), \(x_0 \in \mathbb{C}^n\), if and only if, for any initial value \(X_0 \in \mathbb{C}^{n \times n}\), \(\lim_{t \to \infty} \text{vec}(X(t)) = 0\) holds. The latter is equivalent to \(\alpha(\tilde{M}_{A,C}) < 0\). Hence, we have that
\[
\lim_{t \to \infty} X(t) = 0 \iff \lim_{t \to \infty} \text{vec}(X(t)) = 0 \iff \alpha(\tilde{M}_{A,C}) < 0.
\]

Similar to Theorem 3, from the symmetry of matrix X(t), (13) can also be equivalently written as
\[
\dot{Y}(t) = \tilde{L}_{A,C}Y(t),
\] (15)
where \(Y(t) = [x_{11}(t), x_{21}(t), \ldots, x_{n1}(t), x_{22}(t), \ldots, x_{n2}(t), \ldots, x_{n-1,n-1}(t), x_{n-1,n}(t), x_{nn}(t)]^T\), and \(X(t) = (x_{ij}(t))_{n \times n}\). It is clear that
\[
\lim_{t \to \infty} X(t) = 0 \iff \lim_{t \to \infty} Y(t) = 0 \iff \alpha(\tilde{L}_{A,C}) < 0.
\]
Thus we have that
\[
\alpha(\tilde{M}_{A,C}) < 0 \iff \alpha(\tilde{L}_{A,C}) < 0.
\] (16)

Since \(\sigma(\tilde{L}_{A,C}) \subset \sigma(\tilde{M}_{A,C})\), it is clear that \(\alpha(\tilde{L}_{A,C}) \leq \alpha(\tilde{M}_{A,C})\). Assume \(\alpha(\tilde{L}_{A,C}) = \alpha_0 < \alpha(\tilde{M}_{A,C})\), then there exists a small scalar \(\varepsilon > 0\) such that \(\alpha_0 + \varepsilon < \alpha(\tilde{M}_{A,C})\). Noting that \(L_{2(\alpha_0 + \varepsilon)}L = (\alpha_0 + \varepsilon)I\), \(\tilde{M}_{A - \frac{1}{2}(\alpha_0 + \varepsilon)I, C} = (\tilde{M}_{A,C}) - (\alpha_0 + \varepsilon)I\), \(\tilde{L}_{A - \frac{1}{2}(\alpha_0 + \varepsilon)I, C} = (\tilde{L}_{A,C}) - (\alpha_0 + \varepsilon)I\), we have that \(\alpha(\tilde{M}_{A - \frac{1}{2}(\alpha_0 + \varepsilon)I, C}) > 0\), and \(\alpha(\tilde{L}_{A - \frac{1}{2}(\alpha_0 + \varepsilon)I, C}) < 0\), which conflicts with the conclusion in (16). Therefore, \(\alpha(\tilde{L}_{A,C}) = \alpha(\tilde{M}_{A,C})\) holds. This completes the proof. \(\square\)

Theorem 5 describes the property of the farthest right eigenvalue of \(\tilde{M}_{A,C}\) and \(\tilde{L}_{A,C}\). For the farthest left one, similar to Theorem 4, we can prove that \(\varphi(\tilde{M}_{A,D}) = \varphi(\tilde{L}_{A,0})\) for any real matrix A. On the other hand, we can show that \(\varphi(\tilde{M}_{A,D}) = \varphi(\tilde{L}_{A,0})\) does not always hold for a general matrix C; and neither \(\alpha(\tilde{M}_{A,C}) = \alpha(\tilde{L}_{A,C})\) nor \(\varphi(\tilde{M}_{A,C}) = \varphi(\tilde{L}_{A,C})\) holds for general matrices A and C. Furthermore, we can also similarly define adjoint operators for \(\tilde{T}_{A,C}, \tilde{L}_{A,C}\) and \(\tilde{Z}_{A,C}\), and the corresponding spectral results can be obtained.

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