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Commutativity in special unitary groups at odd primes

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A B S T R A C T

It is a classical result by Bott that $\mathrm{SU}(s)$ and $\mathrm{SU}(t)$ homotopy commute in $\mathrm{SU}(n)$ if and only if $s+t \leqslant n$. We consider the p-localization analog of this problem and give an answer at odd primes.

1. Introduction

Fix a positive integer $n \geqslant 2$, and let i^s denote the inclusion $\mathrm{SU}(s) \to \mathrm{SU}(n)$ for $s \leqslant n$. We say that for $s, t \leqslant n$, $\mathrm{SU}(s)$ and $\mathrm{SU}(t)$ commute in $\mathrm{SU}(n)$ up to homotopy if the Samelson product $\langle i^s, i^t \rangle$ is trivial. It is a naive question for which values of s, t, $\mathrm{SU}(s)$ and $\mathrm{SU}(t)$ homotopy commute in $\mathrm{SU}(n)$, and Bott [1] gave a complete answer to this:

Theorem 1.1. *(Bott [1]) $\mathrm{SU}(s)$ and $\mathrm{SU}(t)$ homotopy commute in $\mathrm{SU}(n)$ if and only if $s+t \leqslant n$.*

Localize $\mathrm{SU}(n)$ at a prime p in the sense of Hilton, Mislin and Roitberg [5], and denote the p-localization by $-_{(p)}$. We say that $\mathrm{SU}(s)_{(p)}$ and $\mathrm{SU}(t)_{(p)}$ homotopy commute in $\mathrm{SU}(n)_{(p)}$ if the Samelson product $\langle i^s_{(p)}, i^t_{(p)} \rangle$ is trivial as well as the usual case, where the multiplicative structure of $\mathrm{SU}(n)_{(p)}$ is inherited from $\mathrm{SU}(n)$. As is seen in [8,7,6], the above multiplicative structure on $\mathrm{SU}(n)_{(p)}$ depends on the prime p. Then it is worth considering the p-localization analog of the above question, that is, for which values of s, t, $\mathrm{SU}(s)_{(p)}$ and $\mathrm{SU}(t)_{(p)}$ homotopy commute in $\mathrm{SU}(n)_{(p)}$.

For a positive integer m, we define m' by $m \equiv m'$ (p) and $1 \leqslant m' \leqslant p$. Hereafter, let p denote an odd prime, and put $q = p-1$.

Theorem 1.2. *For positive integers s, t, n satisfying $2 \leqslant s, t \leqslant n \leqslant q^2+1$, we have:*

1. *Under the condition $s'+t' \geqslant p+1$, $\mathrm{SU}(s)_{(p)}$ and $\mathrm{SU}(t)_{(p)}$ homotopy commute in $\mathrm{SU}(n)_{(p)}$ if and only if $s+t \leqslant n$.*
2. *Under the condition $s'+t' \leqslant p$, $\mathrm{SU}(s)_{(p)}$ and $\mathrm{SU}(t)_{(p)}$ homotopy commute in $\mathrm{SU}(n)_{(p)}$ if and only if $s+t-\min\{s', t'\} \leqslant n$.*

Outline of the proof is as follows. We first decompose the Samelson product $\langle i^s_{(p)}, i^t_{(p)} \rangle$ into easier ones using the mod p decomposition of $\mathrm{SU}(n)$. We next calculate these Samelson products by applying unstable K-theory of Hamanaka and Kono [4,3], and determine the triviality of the Samelson product $\langle i^s_{(p)}, i^t_{(p)} \rangle$.

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2. Decomposition of the Samelson product $\langle i^s_{(p)}, i^t_{(p)}\rangle$

Hereafter, *everything will be localized at the odd prime p*. Then, in particular, the coefficients of cohomology will be $\mathbf{Z}_{(p)}$. We will often make no distinction between maps and their homotopy classes.

As in the introduction, we fix a positive integer $n \geqslant 2$. For $1 \leqslant m \leqslant q$, we set:

$$\rho(m,n) = \begin{cases} \lfloor \frac{n-2}{q} \rfloor + 1, & 1 \leqslant m \leqslant n - \lfloor \frac{n-2}{q} \rfloor q - 1, \\ \lfloor \frac{n-2}{q} \rfloor, & n - \lfloor \frac{n-2}{q} \rfloor q \leqslant m \leqslant q. \end{cases}$$

Let us recall the mod p decomposition of $\mathrm{SU}(n)$. Recall that the cohomology of $\mathrm{SU}(n)$ is given as

$$H^*\big(\mathrm{SU}(n)\big) = \Lambda(x_3, x_5, \ldots, x_{2n-1}),$$

where x_{2i-1} is the suspension of the universal Chern class c_i. For $1 \leqslant m \leqslant q$, Mimura, Nishida and Toda [9] constructed a simply connected finite complex B^k_m (or $B^k_m(p)$) having the following properties:

1. $H^*(B^k_m) = \Lambda(u_{2m+1}, u_{2m+2q+1}, \ldots, u_{2m+2(k-1)q+1})$, $|u_i| = i$.
2. There exists a map $\bar{e}_m(n) : B^{\rho(m,n)}_m \to \mathrm{SU}(n)$ such that $(\bar{e}_m(n))^*(x_{2m+2iq+1}) = u_{2m+2iq+1}$ for $i = 0, \ldots, \rho(m,n)$.

Then, in particular, the map

$$\varphi_n = \mu_q \circ \big(\bar{e}_1(n) \times \cdots \times \bar{e}_q(n)\big) : B^{\rho(1,n)}_1 \times \cdots \times B^{\rho(q,n)}_q \to \mathrm{SU}(n)$$

is a p-local homotopy equivalence, where μ_q is the q-fold multiplication of $\mathrm{SU}(n)$. This mod p decomposition of $\mathrm{SU}(n)$ corresponds to that of $\Sigma\mathbf{C}P^{n-1}$ via the inclusion $g : \Sigma\mathbf{C}P^{n-1} \to \mathrm{SU}(n)$ as follows. As is seen in [2] and [9], for $1 \leqslant m \leqslant q$, there is a simply connected finite complex A^k_m having the following properties:

1. $H^*(A^k_m) = \langle v_{2m+1}, v_{2m+2q+1}, \ldots, v_{2m+2(k-1)q+1}\rangle$, $|v_i| = i$.
2. There exists a map $g' : A^k_m \to B^k_m$ such that $g'^*(u_{2m+2iq+1}) = v_{2m+2iq+1}$ for $i = 0, \ldots, k-1$.
3. There exists a map $e_m(n) : A^{\rho(m,n)}_m \to \Sigma\mathbf{C}P^{n-1}$ satisfying a homotopy commutative diagram:

$$\begin{array}{ccc} A^{\rho(m,n)}_m & \xrightarrow{e_m(n)} & \Sigma\mathbf{C}P^{n-1} \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ B^{\rho(m,n)}_m & \xrightarrow[\bar{e}_m(n)]{} & \mathrm{SU}(n) \end{array}$$

We have an additional property of the map $g' : A^k_m \to B^k_m$.

Proposition 2.1. *(Cohen [2]) If $k \leqslant q$, then $\Sigma g' : \Sigma A^k_m \to \Sigma B^k_m$ admits a left homotopy inverse.*

For $s \leqslant n$, we put $\epsilon^s_m = i^s \circ g \circ e_m(s)$ and $\bar{\epsilon}^s_m = i^s \circ \bar{e}_m(s)$.

Theorem 2.1. *Suppose $s, t \leqslant q^2 + 1$. Then $\langle i^s, i^t\rangle$ is trivial if and only if so is $\langle \epsilon^s_i, \epsilon^t_j\rangle$ for each $1 \leqslant i, j \leqslant q$.*

Proof. For the pinch map $\pi : X \times Y \to X \wedge Y$, we notice that the induced map $\pi^* : [X \wedge Y, \mathrm{SU}(n)] \to [X \times Y, \mathrm{SU}(n)]$ is injective. Then the triviality of $\pi^*(\langle i^s, i^n\rangle)$ is equivalent to that of $\langle i^s, i^t\rangle$.

Denote the projection $\prod_{i=1}^q B^{\rho(i,r)}_i \to B^{\rho(m,r)}_m$ and the diagonal map $X \to X^n$ by p_m and Δ^n_X, respectively. Note that the composite of maps

$$\mathrm{SU}(r) \xrightarrow{\Delta^q_{\mathrm{SU}(r)}} \prod_{i=1}^q \mathrm{SU}(r) \xrightarrow{\prod_{i=1}^q \varphi_r^{-1}} \prod_{i=1}^q \prod_{j=1}^q B^{\rho(j,r)}_j \xrightarrow{\prod_{i=1}^q p_i} \prod_{i=1}^q B^{\rho(i,r)}_i \xrightarrow{\varphi_r} \mathrm{SU}(r)$$

is equal to the identity of $\mathrm{SU}(r)$. Then, in particular, the product $(\bar{\epsilon}^r_1 \circ p_1 \circ \varphi_r^{-1}) \cdots (\bar{\epsilon}^r_q \circ p_q \circ \varphi_r^{-1})$ in the group $[\mathrm{SU}(s), \mathrm{SU}(n)]$ is the map i^s. Let $q_r : \mathrm{SU}(s) \times \mathrm{SU}(t) \to \mathrm{SU}(r)$ be the projection for $r = s, t$. We put $\lambda^r_i = \bar{\epsilon}^r_i \circ p_i \circ \varphi_r^{-1} \circ q_r$ for $r = s, t$. Then, for $(q_s \times q_t) \circ \Delta^2_{\mathrm{SU}(s)\times\mathrm{SU}(t)} = 1_{\mathrm{SU}(s)\times\mathrm{SU}(t)}$, we obtain an equality

$$\big[\lambda^s_1 \cdots \lambda^s_q, \lambda^t_1 \cdots \lambda^t_q\big] = \pi^*\big(\langle i^s, i^t\rangle\big)$$

where the left-hand side is the commutator in the group $[\mathrm{SU}(s) \times \mathrm{SU}(t), \mathrm{SU}(n)]$.

Let G be a group, and let x^y stand for yxy^{-1} for $x, y \in G$. Then obviously we have

$$[x, yz] = [x, y][x, z]^y$$

for $x, y, z \in G$. It follows that $[\lambda_1^s \cdots \lambda_q^s, \lambda_1^t \cdots \lambda_q^t]$ is equal to a product of $[\lambda_i^s, \lambda_j^t]^{\alpha_{ij}}$ for some $\alpha_{ij} \in [\mathrm{SU}(s) \times \mathrm{SU}(t), \mathrm{SU}(n)]$. Then we get $\langle i^s, i^t \rangle$ is trivial if so is $[\lambda_i^s, \lambda_j^t]$ for each i, j. On the other hand, we have

$$\left[\lambda_i^s, \lambda_j^t\right] = (q_s \times q_t)^* \circ \pi^*\left(\langle \bar{\epsilon}_i^s, \bar{\epsilon}_j^t \rangle\right)$$

as well as above. Thus we obtain $\langle i^s, i^t \rangle$ is trivial if so is $\langle \bar{\epsilon}_i^s, \bar{\epsilon}_j^t \rangle$ for each i, j.

Assume $r \leqslant q^2 + 1$. Let $\kappa_i : \Sigma B_i^{\rho(i,r)} \to \Sigma A_i^{\rho(i,r)}$ be a left homotopy inverse in Proposition 2.1. We denote the composite

$$\Sigma \mathrm{SU}(r) \xrightarrow[\simeq]{\Sigma \varphi_r^{-1}} \Sigma\left(\prod_{i=1}^{q} \left(B_i^{\rho(i,r)}\right)\right) \xrightarrow{\text{proj}} \bigvee_{i=1}^{q} \Sigma B_i^{\rho(i,r)} \xrightarrow{\bigvee_{i=1}^{q} \kappa_i} \bigvee_{i=1}^{q} \Sigma A_i^{\rho(i,r)}$$

by κ. Then, as in [8], we deduce that there is a self homotopy equivalence $\alpha : \mathrm{SU}(r) \to \mathrm{SU}(r)$ satisfying the following homotopy commutative diagram by looking at cohomology.

$$\begin{array}{ccc}
\Sigma \mathrm{SU}(r) & \xrightarrow{\mathrm{ad}\, 1_{\mathrm{SU}(r)}} & B\mathrm{SU}(r) \\
\downarrow{\scriptstyle \kappa \circ \Sigma \alpha} & & \| \\
\bigvee_{i=1}^{q} \Sigma A_i^{\rho(i,r)} & \xrightarrow[\bigvee_{i=1}^{q} \mathrm{ad}\, \epsilon_i^r]{} & B\mathrm{SU}(r)
\end{array}$$

Here, $\mathrm{ad} : [X, \Omega Y] \approx [\Sigma X, Y]$ means the adjoint congruence. Since $\bigvee_{i=1}^{q} \Sigma A_i^{\rho(i,r)}$ is a homotopy retract of $\Sigma \mathrm{SU}(r)$, by rearranging the map $g' : A_i^{\rho(i,r)} \to B_i^{\rho(i,r)}$, we may assume α is the identity. Then we get

$$\left(\mathrm{ad}\, \epsilon_i^r\right) \circ \kappa_i = \mathrm{ad}\, \bar{\epsilon}_i^r. \tag{2.1}$$

Now we suppose that $\langle \epsilon_i^s, \epsilon_j^t \rangle$ is trivial, where $s, t \leqslant q^2 + 1$. Equivalently, we suppose the Whitehead product $[\mathrm{ad}\, \epsilon_i^s, \mathrm{ad}\, \epsilon_j^t]$ is trivial. Then there is an extension $\Sigma A_i^{\rho(i,s)} \times \Sigma A_j^{\rho(j,t)} \to B\mathrm{SU}(n)$ of $\mathrm{ad}\, \epsilon_i^s \vee \mathrm{ad}\, \epsilon_j^t$. Then it follows from (2.1) that there exists an extension $\Sigma B_i^{\rho(i,s)} \times \Sigma B_j^{\rho(j,t)} \to B\mathrm{SU}(n)$ of $\mathrm{ad}\, \bar{\epsilon}_i^s \vee \mathrm{ad}\, \bar{\epsilon}_j^t$. This shows that the Whitehead product $[\mathrm{ad}\, \bar{\epsilon}_i^s, \mathrm{ad}\, \bar{\epsilon}_j^t]$ is trivial, which is equivalent to that the Samelson product $\langle \bar{\epsilon}_i^s, \bar{\epsilon}_j^t \rangle$ is trivial. Thus we complete the proof of the if part. The only if part is trivial. □

3. Review of unstable *K*-theory

We give a brief review of p-local unstable K-theory which is a group $[X, \mathrm{U}(n)]$. See [3] and [4] for details. Apply the functor $[X, -]$ to a fibre sequence

$$\Omega \mathrm{U}(\infty) \xrightarrow{\Omega \pi} \Omega W_n \xrightarrow{\delta} \mathrm{U}(n) \xrightarrow{i} \mathrm{U}(\infty) \xrightarrow{\pi} (W_n),$$

where $W_n = \mathrm{U}(\infty)/\mathrm{U}(n)$. Then we get an exact sequence of groups

$$\widetilde{K}^0(X) \xrightarrow{\Theta} [X, \Omega W_n] \xrightarrow{\delta_*} \left[X, \mathrm{U}(n)\right] \xrightarrow{i_*} \widetilde{K}^{-1}(X),$$

where Θ is the composite of $(\Omega q)_*$ and the Bott map $\beta : \widetilde{K}^0(X) \xrightarrow{\cong} \widetilde{K}^{-2}(X)$. In order to compute $[X, \mathrm{U}(n)]$, Hamanaka and Kono [4,3] make use of the above exact sequence by comparing the group $[X, \Omega W_n]$ with the cohomology of X as follows. Looking at the p-component of the homotopy groups of spheres, we see that there is a p-local homotopy equivalence:

$$W_n \simeq_{(p)} \left(S^{2n+1} \vee S^{2n+3} \vee \cdots \vee S^{2n+2p-3}\right) \cup (\text{higher dimensional cells}).$$

The cohomology of W_n is given as

$$H^*(W_n) = \Lambda(y_{2n+1}, y_{2n+3}, \ldots), \quad \pi^*(y_i) = x_i.$$

Then the map

$$\prod_{i=0}^{q-1} y_{2n+2i+1} : W_n \to \prod_{i=0}^{q-1} K(\mathbf{Z}_{(p)}, 2n + 2i + 1)$$

is a $(2n+2q)$-equivalence. Put $a_{i-1} = \sigma(y_i)$ for the cohomology suspension σ. It follows that the map

$$\prod_{i=0}^{q-1} a_{2n+2i} : \Omega W_n \to \prod_{i=0}^{q-1} K(\mathbf{Z}_{(p)}, 2n+2i)$$

is a loop map which is a $(2n+2q-1)$-equivalence. Then if X is a CW-complex of dimension $\leqslant 2n+2q-2$, we get a natural isomorphism of groups

$$[X, \Omega W_n] \cong \bigoplus_{i=0}^{q-1} H^{2n+2i}(X)$$

by assigning $\alpha \in [X, \Omega W_n]$ to $(\alpha^*(a_{2n}), \dots, \alpha^*(a_{2n+2q}))$. Moreover, we can easily describe the map Θ via this isomorphism using Chern character. Summarizing, we have established:

Theorem 3.1. *(Hamanaka and Kono [4], Hamanaka [3]) For a CW-complex X of dimension $\leqslant 2n+2q-2$, there is an exact sequence of groups*

$$\widetilde{K}^0(X) \xrightarrow{\Theta} \bigoplus_{i=0}^{q-1} H^{2n+2i}(X) \to \big[X, \mathrm{U}(n)\big] \xrightarrow{i_*} \widetilde{K}^1(X) \tag{3.1}$$

in which Θ is given by

$$\Theta(\xi) = \big(n!\mathrm{ch}_n(\xi), \dots, (n+q-1)!\mathrm{ch}_{n+q-1}(\xi)\big)$$

for $\xi \in \widetilde{K}^0(X)$, where ch_k means the $2k$-dimensional part of the Chern character.

In order to see the group structure of $[X, \mathrm{U}(n)]$, we look at commutators in $[X, \mathrm{U}(n)]$. Let $\bar{\gamma} : \mathrm{U}(n) \wedge \mathrm{U}(n) \to \mathrm{U}(n)$ be the reduced commutator map. Since $\mathrm{U}(\infty)$ is homotopy abelian, $i \circ \bar{\gamma}$ is null homotopic so that it lifts to ΩW_n through $\delta : \Omega W_n \to \mathrm{U}(n)$. By looking carefully at the Whitehead product of the inclusion $\Sigma \mathrm{U}(n) \to B\mathrm{U}(n)$, we can choose a good lift as:

Theorem 3.2. *(Hamanaka and Kono [4], Hamanaka [3]) There is a lift $\tilde{\gamma} : \mathrm{U}(n) \wedge \mathrm{U}(n) \to \Omega W_n$ of $\bar{\gamma}$ such that, for $k \geqslant n$,*

$$\tilde{\gamma}(a_{2k}) = \sum_{\substack{i+j+1=k \\ 0 \leqslant i, j \leqslant n-1}} x_{2i+1} \otimes x_{2j+1}.$$

Corollary 3.1. *Let X be a CW-complex of dimension $\leqslant 2n+2q-2$. For $\alpha, \beta \in [X, \mathrm{U}(n)]$, we put*

$$\theta_k = \sum_{\substack{i+j+1=k \\ 0 \leqslant i, j \leqslant n-1}} \alpha^*(x_{2i+1}) \beta^*(x_{2j+1}).$$

Then the commutator $[\alpha, \beta]$ comes from

$$(\theta_n, \dots, \theta_{n+q-1})$$

in the exact sequence (3.1).

4. Calculation of the Samelson product $\langle \epsilon_i^s, \epsilon_j^t \rangle$

We calculate the Samelson product $\langle \epsilon_i^s, \epsilon_j^t \rangle$ by using results in the previous section. When X is simply connected, we may identify $[X, \mathrm{SU}(n)]$ and $[X, \mathrm{U}(n)]$ since they are naturally isomorphic. By Corollary 3.1, we have:

Proposition 4.1. *Put*

$$\chi_k = \sum_{\substack{0 \leqslant a \leqslant \rho(\ell,s)-1 \\ 0 \leqslant b \leqslant \rho(m,t)-1 \\ \ell+m+(a+b)q+1=k}} v_{2\ell+2aq+1} \otimes v_{2m+2bq+1}.$$

Then if $\ell+m+(\rho(\ell,s)+\rho(m,t)-2)q+1 \leqslant n+q-1$, the commutator $[\epsilon_\ell^s \circ \pi_\ell^s, \epsilon_m^t \circ \pi_m^t]$ comes from $(\chi_n, \dots, \chi_{n+q-2})$ in the exact sequence of Theorem 3.1 with $X = A_\ell^{\rho(\ell,s)} \times A_m^{\rho(m,t)}$, where π_k^r is the projection $A_\ell^{\rho(\ell,s)} \times A_m^{\rho(m,t)} \to A_k^{\rho(k,r)}$ for $(k,r) = (\ell,s), (m,t)$.

We denote the canonical line bundle over $\mathbf{C}P^r$ by η. Put $\xi_{2k+1} = e_m(r)^*(\beta(\eta^k - 1)) \in \widetilde{K}^{-1}(A_m^{\rho(m,r)})$, where β is the Bott map as above and 1 means the rank one trivial line bundle. Then it is easy to see that

$$\widetilde{K}^{-1}\big(A_m^{\rho(m,r)}\big) = \langle \xi_{2m+1}, \xi_{2m+2q+1}, \ldots, \xi_{2m+2(\rho(m,r)-1)q+1} \rangle$$

and

$$\mathrm{ch}(\xi_{2m+2iq+1}) = \sum_{a=0}^{\rho(m,r)-1} \frac{(m+iq)^{m+aq}}{(m+aq)!} \Sigma v_{2m+2aq+1}.$$

In particular, we obtain:

Proposition 4.2. *$\widetilde{K}^0(A_\ell^{\rho(\ell,s)} \wedge A_m^{\rho(m,t)})$ is a free $\mathbf{Z}_{(p)}$-module generated by $\beta^{-1}(\xi_{2\ell+2iq+1} \wedge \xi_{2m+2jq+1})$ for $0 \leqslant i \leqslant \rho(\ell,s)-1$ and $0 \leqslant j \leqslant \rho(m,t)-1$, and we have*

$$\mathrm{ch}_k\big(\beta^{-1}(\xi_{2\ell+2iq+1} \wedge \xi_{2m+2jq+1})\big) = \sum_{\substack{0\leqslant a\leqslant \rho(\ell,s)-1\\ 0\leqslant b\leqslant \rho(m,t)-1\\ \ell+m+1+(a+b)q=k}} \frac{(\ell+iq)^{\ell+aq}(m+jq)^{m+bq}}{(\ell+aq)!(m+bq)!} v_{2\ell+2aq+1} \otimes v_{2m+2bq+1}.$$

Applying the above results, we obtain a criterion for the triviality of $\langle i^s, i^t \rangle$.

Theorem 4.1. *Suppose that $s, t \leqslant q^2+1$ and $n+1 \leqslant s+t \leqslant n+q$. The Samelson product $\langle i^s, i^t\rangle$ is trivial if and only if for all (c,d) satisfying conditions*

$$1 \leqslant c \leqslant s-1, \qquad 1 \leqslant d \leqslant t-1, \qquad n \leqslant c+d+1 \leqslant n+q-1, \tag{4.1}$$

it holds that

$$(c+d+1)\binom{c+d}{c} \not\equiv 0 \quad (p).$$

Proof. As in the proof of Theorem 2.1, we have

$$\big[\epsilon_i^s \circ \pi_i^s, \epsilon_j^t \circ \pi_j^t\big] = \pi^*\big(\langle \epsilon_i^s, \epsilon_j^t \rangle\big)$$

in the group $[A_i^{\rho(i,s)} \times A_j^{\rho(j,t)}, \mathrm{SU}(n)]$ and π^* is injective, where π is as in the proof of Theorem 2.1 and π_k^r is as in Proposition 4.1. Then we check the triviality of the commutator $[\epsilon_i^s \circ \pi_i^s, \epsilon_j^t \circ \pi_j^t]$.

Put $c_i = i + (\rho(i,s)-1)q$ and $d_j = j + (\rho(j,t)-1)q$ for $1 \leqslant i, j \leqslant q$. If c_i and d_j satisfy $n \leqslant c_i + d_j \leqslant n+q-1$, the commutator $[\epsilon_i^s \circ \pi_i^s, \epsilon_j^t \circ \pi_j^t]$ comes from $H^{2c_i+2d_j}(A_i^{\rho(i,s)} \times A_j^{\rho(j,t)})$ in the exact sequence of 3.1. Then by Theorem 3.1, Propositions 4.1 and 4.2, $[\epsilon_i^s \circ \pi_i^s, \epsilon_j^t \circ \pi_j^t]$ is trivial if and only if $\frac{(c_i+d_j+1)!}{c_i! d_j!} \not\equiv 0$ (p). Since the set of all pairs (c_i, d_j) for $1 \leqslant i, j \leqslant q$ is exactly the same as that of pairs (c,d) satisfying the above condition (4.1), the proof is completed. □

Proof of Theorem 1.2. We assume $s+t \leqslant n+q$ from which together with the fact that the nontriviality of $\langle i^s, i^t\rangle$ implies that of $\langle i^u, i^v \rangle$ for $s \leqslant u$ and $t \leqslant v$, we can deduce the result for $s+t \geqslant n+q+1$. Note, in particular, that we have $s+t < p^2$ under this assumption.

First, we suppose $s'+t' \geqslant p+1$. If $s+t \leqslant n$, then $\langle i^s, i^t\rangle$ is trivial, obviously. Then we suppose further that $s+t \geqslant n+1$. Put $c = s-1$ and $d = t-1$. Then (c,d) satisfies the condition (4.1). When $s'+t' = p+1$, we have $c+d+1 \equiv 0$ (p) and then $\langle i^s, i^t\rangle$ is nontrivial by Theorem 4.1. Suppose $s'+t' \geqslant p+2$. Then we have $c+d+1 \not\equiv 0$ (p). By Lucas' theorem and $c+d < p^2$, we get $\binom{c+d}{c} \equiv 0$ (p) if and only if $\binom{\overline{c+d}}{\bar{c}} = 0$, where $\overline{m}$ is the remainder of a positive integer m divided by p. By definition, we have $\overline{c+d} = s'+t'-2-p$ and $\bar{c} = s'-1$, and thus $\overline{c+d} < \bar{c}$ which implies $\binom{\overline{c+d}}{\bar{c}} = 0$. Therefore we have established the first part of Theorem 1.2.

Next, we suppose $s'+t' \leqslant p$. Choose (c,d) to satisfy the condition (4.1). Then, as above, we have $\binom{c+d}{c} \equiv 0$ (p) if and only if $\overline{c+d} < \bar{c}$. One can easily see that $\overline{c+d} < \bar{c}$ if and only if $\bar{c} + \bar{d} \geqslant p$. Suppose $s - s' \leqslant c$ and $t - t' \leqslant d$. Then we have $\bar{c} + \bar{d} \leqslant p-2$ since $\bar{c} \leqslant s'-1$, $\bar{d} \leqslant t'-1$ and $s'+t' \leqslant p$. Thus we have obtained that $\binom{c+d}{c} \equiv 0$ (p) implies

$$c+d+1 \leqslant \max\big\{s+t-s'-1, s+t-t'-1\big\} = s+t-\min\big\{s',t'\big\}-1.$$

Now, for $n \leqslant c+d+1$, we get $\binom{c+d}{c} \equiv 0$ (p) implies $n+1 \leqslant s+t-\min\{s',t'\}$. We also have that $c+d+1 \equiv 0$ (p) implies $n \leqslant s+t-s'-t' \leqslant s+t-\min\{s',t'\}-1$. Then it follows from Theorem 4.1 that if $\langle i^s, i^t\rangle$ is nontrivial, then $n+1 \leqslant$

$\underline{s+t} - \min\{s', t'\}$. Conversely, if $n+1 \leqslant s+t-\min\{s', t'\}$, we may put $(c, d) = (s-s'-1, t-1)$ or $(s-1, t-t'-1)$ to get $\overline{c+d} < \bar{c}$, where (c, d) satisfies the condition (4.1). Thus, by Theorem 4.1, we obtain if $n+1 \leqslant s+t-\min\{s', t'\}$, then $\langle i^s, i^t \rangle$ is nontrivial. This completes the proof. □

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