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A Sublinear Oscillation Theorem

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An oscillation criterion is given for the second order nonlinear differential equation $y'' + a(t) f(y) = 0$, where $a(t)$ is continuous but is not assumed to be non-negative for all large values of t and $f(y)$ is non-decreasing in y and satisfies $yf(y) > 0$ for $y \neq 0$. This result generalizes a recent extension of Belohorec's theorem due to Kwong and Wong and also incorporates an earlier result of Coles. © 1989 Academic Press, Inc.

We consider the second order nonlinear differential equation

$$y'' + a(t) f(y) = 0, \qquad t \in [0, \infty), \tag{1}$$

where $a(t) \in C[0, \infty)$ and $f(y) \in C(-\infty, \infty)$, non-decreasing in y, and satisfies

$$yf(y) > 0, \qquad \text{if} \quad y \neq 0. \tag{2}$$

We are here concerned with those solutions of (1) which exists and can be continued on some ray $[t_0, \infty)$ where $t_0 \geqslant 0$ may depend on the particular solution. Such a solution is said to be *oscillatory* if it has arbitrarily large zeros. Equation (1) is called *oscillatory* if all continuable solutions are oscillatory. For a general discussion of nonlinear oscillation problems concerning Eq. (1), we refer the reader to [8]. The prototype of Eq. (1) is the so-called Emden–Fowler equation

$$y'' + a(t) \, |y|^{\gamma} \operatorname{sgn} y = 0, \qquad \gamma > 0, \tag{3}$$

and we refer to [9] for further details and references. We are here interested in the following well known result of Belohorec for the *sublinear* case of Eq. (3).

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THEOREM A (Belohorec [2]). *Let $0<\gamma<1$. Then Eq. (3) is oscillatory if*

$$\lim_{T\to\infty}\int_0^T t^\gamma a(t)\,dt=+\infty. \tag{4}$$

Here $a(t)$ is not assumed to be non-negative in (4). If in addition that $a(t)\geqslant 0$ on $[0,\infty)$, then in an earlier paper [1], Belohorec showed that condition (4) is both necessary and sufficient for Eq. (3) to be oscillatory. The corresponding condition for *sublinearity* of the more general Eq. (1) is known to be

$$0<\int_0^\varepsilon \frac{dy}{f(y)},\int_{-\varepsilon}^0 \frac{dy}{f(y)}<\infty, \qquad \text{for all} \quad \varepsilon>0. \tag{5}$$

Now assume that the function $f(y)$ satisfies (5), it is natural to seek an extension of Theorem A for the more general Eq. (1). Denote $F(y)=\int_0^y (dv/f(v))$. An analogue in the sublinear case was given in the following result:

THEOREM B (Coles [3]). *Assume that there exists a positieve function $\rho\in C^2[0,\infty)$ and $\rho'\geqslant 0$ and a positive constant $c>0$ such that $F(y)$ satisfies*

$$\frac{F''(y)\,F(y)}{F'^2(y)}\leqslant -\frac{1}{c}, \qquad \textit{for all } y \tag{6}$$

(note: $F'(y)$ denotes differentiation of $F(y)$ with respect to its independent variable y), *and at the same time $\rho(t)$ satisfies*

$$\frac{\rho''(t)\,\rho(t)}{\rho'^2(t)}\leqslant -c. \tag{7}$$

Furthermore, $a(t)$ and $\rho(t)$ also satisfy

$$\lim_{T\to\infty}\int_0^T \rho(t)\,a(t)\,dt=\lim_{T\to\infty}\int_0^T \frac{1}{\rho(t)}\int_0^t \rho(s)\,a(s)\,ds\,dt=\infty. \tag{8}$$

Then, Eq. (1) is oscillatory.

Coles [3] noted that condition (8) was implied by the following condition which might be easier to verify:

$$\lim_{T\to\infty}\int_0^T \rho(t)\,a(t)\,dt=\int_0^\infty \frac{1}{\rho(t)}\,dt=\infty. \tag{9}$$

However, if $\rho(t)$ is concave then $\int_0^\infty (1/\rho) = \infty$, so the second part of condition (9) is automatically satisfied on account of (7). Let $f(y) = |y|^\gamma \operatorname{sgn} y$, $0 < \gamma < 1$, then $c = (1-\gamma)/\gamma$ satisfies (6). Taking $\rho(t) = (t+k)^\alpha$ for any α, $0 \leqslant \alpha \leqslant \gamma$, it is easy to verify that condition (7) is also satisfied with $c = (1-\gamma)/\gamma$. Thus, for $k \neq 0$ condition (9) becomes

$$\lim_{T\to\infty} \int_0^T (t+k)^\alpha a(t)\, dt = +\infty, \qquad 0 \leqslant \alpha \leqslant \gamma,$$

which is a non-trivial extension of Belohorec's condition (4).

In a recent paper, Kwong and this writer gave another extension of Belohorec's result by proving the following analogue of Kiguradze's theorem in the superlinear case [6]:

THEOREM C (Kwong and Wong [7]). *If there exists a positive function $\varphi(t)$ such that $\varphi' \geqslant 0$, $\varphi'' \leqslant 0$ and $a(t)$ satisfies*

$$\lim_{T\to\infty} \int_0^T \varphi^\gamma(t)\, a(t)\, dt = +\infty, \tag{10}$$

then Eq. (3) is oscillatory.

We remark that Kigurzdze's result for Eq. (2) with $\gamma > 1$ has been extended to the more general Eq. (1) by Kamenev [4] subject to suitable superlinearity condition of f. The purpose of this note is to prove a further extension of Theorem C to Eq. (1) subject to condition (5), which also includes Theorem B as a special case.

THEOREM. *Let $f(y)$ satisfy (2), (5), and (6). If there exists a positive concave function $\varphi(t)$ such that $\varphi' \geqslant 0$, $\varphi'' \leqslant 0$ and $a(t)$ satisfies*

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \int_0^t \varphi^\lambda(s)\, a(s)\, ds\, dt = +\infty, \tag{11}$$

where $\lambda = 1/(1+c) < 1$, then Eq. (1) is oscillatory.

Note that for $c = (1-\gamma)/\gamma$, $\lambda = \gamma$, hence condition (4) implies condition (11) and so Theorem C is a corollary of this Theorem. On the other hand, taking $\rho(t) = \varphi^\lambda(t)$ one finds

$$\frac{\rho''(t)\,\rho(t)}{\rho'^2(t)} = \frac{\varphi''(t)\,\varphi(t)}{\gamma\varphi'^2(t)} + \frac{\lambda-1}{\lambda} \leqslant -c = \frac{\lambda-1}{\lambda}$$

which implies $\varphi'' \leqslant 0$. Thus, condition (7) with $\rho(t) = \varphi^{\lambda}(t)$ implies that $\varphi > 0$, $\varphi' \geqslant 0$, and $\varphi'' \leqslant 0$. Moreover condition (9) becomes

$$\lim_{T \to \infty} \int_0^T \varphi^{\lambda}(t)\, a(t)\, dt = +\infty$$

and is the same as condition (10) in Theorem C which clearly implies (11). This shows that Theorem B of Coles also follows from this more general result. Indeed, taking the simplest case when $\varphi(t) \equiv 1$ in condition (10), our Theorem reduces to an earlier result of Kamenev [5] for the sublinear equation.

Proof. The basic idea is to modify the proof of Theorem C. Let $y(t)$ be a nonoscillatory solution which may in view of (2) be assumed to be positive on $[t_0, \infty)$.

Define $z(t) = \varphi^{\lambda}(t)\, F(y(t))$, where $\lambda = 1/(1+c) < 1$. Differentiating twice and using (1), we find

$$z'' + \varphi^{\lambda} a = \lambda \varphi^{\lambda - 1} \varphi'' F(y) + \lambda(\lambda - 1)\, \varphi^{\lambda - 2} \varphi'^2 F(y) + 2\lambda \varphi^{\lambda - 1} \frac{y'\varphi'}{f(y)} - \frac{\varphi^{\lambda} f'(y)\, y'^2}{f^2(y)}. \tag{12}$$

(Notice here we used prime indiscriminately for both differentiation with respect to t and y.) Using $\varphi'' \leqslant 0$, $1 - \lambda = \lambda c$, and an alternative of (6), i.e.,

$$f'(y)\, F(y) \geqslant \frac{1}{c} \qquad \text{for all } y, \tag{13}$$

we can drop the first term on the right hand side of (12) and rewrite it as

$$\begin{aligned} z'' + \varphi^{\lambda} a &\leqslant -\frac{1}{c} \frac{\varphi^{\lambda - 2}}{F(y)} \left\{ (1-\lambda)^2 \varphi'^2 F^2(y) - 2(1-\lambda)\, \varphi \varphi' y' \frac{F(y)}{f(y)} + \frac{\varphi^2 y'^2}{f^2(y)} \right\} \\ &= -\frac{1}{c} \frac{\varphi^{\lambda - 2}}{F(y)} \left\{ (1-\lambda)\, \varphi' F(y) - \frac{\varphi y'}{f(y)} \right\}^2 \leqslant 0 \end{aligned}$$

which is the desired second order differential inequality $z'' + \varphi^{\lambda} a \leqslant 0$. Integrating this second order differential inequality twice and upon taking limit superior on both sides, one obtains the desired contradiction from condition (11). The proof is now complete.

To illustrate this result, we consider the example

$$y'' + t^{\alpha} \sin t\, |y|^{1/2} (1 + |y|) \operatorname{sgn} y = 0. \tag{14}$$

Here $a(t)$ is wildly oscillatory for $\alpha > 0$ and does not satisfy (8) or (10) for concave functions $\rho(t)$ or $\varphi(t)$, hence Theorems B and C are not applicable. Here the nonlinearity $f(y)$ is not in the simple form of a power of y as in Eq. (3), but satisfies (5). To apply our main result, we note first $f(y) = \sqrt{y}(1+y)$ for $y > 0$ and is odd, i.e., $f(-y) = -f(y)$. Also we observe

$$F(y) = \int_0^y \frac{dv}{\sqrt{v}(1+v)} = 2 \tan^{-1} \sqrt{y}$$

and that condition (13) gives

$$f'(y)\,F(y) = \frac{1}{\sqrt{y}}(1+3y) \tan^{-1} \sqrt{y} \geqslant 1.$$

Taking $c = 1$, $\lambda = \frac{1}{2}$, and $\varphi(t) = t$, we see that condition (11) will be satisfied when $a(t) = t^\alpha \sin t$ with $\alpha > \frac{1}{2}$. Hence Eq. (14) is oscillatory for $\alpha > \frac{1}{2}$, a result which cannot be concluded from any of the earlier results.

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