Subgroups Isomorphic to $G_2(L)$ in Orthogonal Groups

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1. INTRODUCTION AND STATEMENT OF THE MAIN THEOREM

In this paper, we describe the embeddings of groups $G_2(L)$ in orthogonal groups such that the long root elements act as Siegel transvections. For finite orthogonal groups, this is contained in the results of Kantor [K] on subgroups of finite classical groups generated by a class of long root elements. The problem stated above is part of the determination of the subgroups $G$ generated by long root elements in algebraic groups $Y$ over arbitrary fields. For the case where $Y$ and $G$ are classical groups, see [S].

To state the Main Theorem, we introduce some notation.

1.1 Let $K$ be a commutative field, let $V$ be a finite-dimensional vector space over $K$, and let $Q: V \rightarrow K$ be a quadratic form (with associated bilinear form $b$). A subspace $U$ of $V$ is called singular, if $Q(u) = 0$ for all $u \in U$. We assume that $Q$ is nondegenerate (i.e., if $v \in \text{Rad}(V, b)$ with $Q(v) = 0$, then $v = 0$) and that $Q$ has Witt index at least 3 (i.e., $V$ contains three-dimensional singular subspaces).

Let $\mathcal{L}$ be a singular line of $V$ with basis $(x, y)$. For $c \in K$, the mapping

$$t_c: v \mapsto v - cb(v, x)y + cb(v, y)x \quad \text{for } v \in V$$

is called a Siegel transvection (see [T, Th. 5], [S, (1.1.3)]). The set $T_{\mathcal{L}} := \{t_c | c \in K\}$ is the Siegel transvection group corresponding to $\mathcal{L}$. Let $\Omega(V, Q)$

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\[ \Sigma := \langle \Sigma \rangle, \text{ where} \]
\[ \Sigma := \{ T_\tau \mid \tau \text{ a singular line in } V \} \]
is the class of Siegel transvection groups, be the associated orthogonal group.

Let \( G \) be a subgroup of \( Y := \Omega(V, Q) \) which is generated by Siegel transvections. For \( A \in \Sigma \), we set \( A^0 := A \cap G \), \( \Sigma^0 := \{ A^0 \mid A \in \Sigma, A^0 \neq 1 \} \) and further \( V_0 := [V, G] \). We assume that \( G \) and \( \Sigma^0 \) satisfy the following hypothesis:

\( (G_2) \) \( G \) is quasi-simple and there exists a commutative field \( L \) such that \( G := G/Z(G) = G_2(L) \) (resp. \( G_2(2') \)) and \( \Sigma^0 := \{ A^0 \mid A \in \Sigma^0 \} \) is the class of long root subgroups of \( G \).

We regard \( G_2(L) \) as the subgroup \( S \) of a seven-dimensional orthogonal group \( \Omega(W, B) \) which preserves the Dickson form as suggested by Aschbacher [A]; see Section 2. We say that \( W \) is the natural module for \( G_2(L) \).

The Main Theorem of this paper is:

1.2. MAIN THEOREM. Let \( Y = \Omega(V, Q) \), let \( G \) and \( G_2(L) \) be as in (1.1). Then the following hold:

(a) We have \( \dim V_0 = 7 \).

(b) There exists an embedding of fields \( \alpha : L \to K \), an injective semilinear (with respect to \( \alpha \)) mapping \( \varphi : W \to V_0 \) with \( V_0 = \langle W \varphi \rangle_K \), and an isomorphism \( \chi : G \to S \cong G_2(L) \) such that \( (w(g \chi))\varphi = (w\varphi)g \) for all \( w \in W, g \in G \).

(c) The quadratic form \( \bar{B} \) on \( V_0 \) defined by \( \bar{B}(w\varphi) := B(w)\alpha \) for \( w \in W \) is proportional to \( Q \), i.e., \( Q = d\bar{B} \) on \( V_0 \) for some \( d \in K \).

(d) If \( \text{Rad}(V_0) \subseteq \text{Rad}(V) \), then \( V = V_0 + C_V(G) \). If \( \text{Rad}(V_0) \nsubseteq \text{Rad}(V) \), then \( V + C_V(G) \) is a hyperplane of \( V \). In the latter case there exists an eight-dimensional subspace \( V_1 \) of \( V \) which contains \( V_0 \), such that \( \text{Rad}(V_1) = 0 \) and \( V = V_1 + C_V(G) \). The action of \( G \) on \( V_1 \) is uniquely determined by the action of \( G \) on \( V_0 \).

The Main Theorem shows that the embedding of \( G \) in \( Y \) is induced by a semilinear mapping. We can regard the commutator space \( V_0 \) as the natural module for \( G \) (tensored with the bigger field \( K \)).

1.3

The idea of the proof is as follows: We may write \( G_2(L) = \langle M, X \rangle \), where \( M = \text{SL}_2(L) \) is generated by long root subgroups and \( X = \text{SL}_2(L) \) is generated by two short root subgroups. By [S] the action of \( M \) on the
orthogonal space $V$ is known ($[V, M]$ is the direct sum of a natural and a dual module for $M$). We hence may determine the action of $X$ on $V$, using a subgroup $S_1 = \text{SL}_2(L)$ of $M$ with $[S_1, X] = 1$.

Because of the results of Kantor [K, Th. I, 12(B)], we may restrict to the case $|L| \geq 4$.

1.4

In the proof of the Main Theorem also the results of Borel and Tits [BT] on abstract homomorphisms of algebraic groups might be used (see also [St]). We give a short outline of this approach.

Let $L$ be an infinite field and $G$ an isotropic simple algebraic group defined over $L$. Assume that $G$ is split and simply connected. Let $V$ be a finite-dimensional vector space over an algebraically closed field $\overline{K}$ and denote by $\rho: G(L) \rightarrow \text{GL}(V)$ an irreducible representation of the group $G(L)$ of rational points. By [BT, (10.4)], $\rho$ is equivalent to a tensor product $\bigotimes_{i=1}^{r} \pi_i \circ \alpha_i$, where $\alpha_i: L \rightarrow \overline{K}$ is an embedding of fields, $\alpha G$ is the group obtained by transfer of base field, and $\pi_i$ is a nontrivial rational irreducible linear representation of $\alpha G$.

Let $K$ be any field with algebraic closure $\overline{K}$. Choose $G$ as in the above paragraph and of type $G_2$. Assume that $V$ is an absolutely irreducible $KG$-module of dimension at most 8 which is tensor indecomposable. (These properties may be verified for $[V, G]/C_{[V, G]}(G)$ under hypothesis ($G_2$).) We apply [BT, (10.4)] to $\overline{V} := \overline{K} \otimes_K V$ and use that the only irreducible modules over $\overline{K}$ of dimension at most 8 for an algebraic group of type $G_2$ are the seven-dimensional orthogonal module in characteristic $\neq 2$ and the six-dimensional symplectic module in characteristic 2 (see [KL, (5.4.12)], for example). Computing traces yields that the image of $L$ under the field embedding into $\overline{K}$ is contained in $K$ rather than in $\overline{K}$.

Hence in the Main Theorem $[V, G]/C_{[V, G]}(G)$ is a seven- or six-dimensional natural module for $G$ tensored with $K$. To finish the proof, we have to show that $[V, G]$ is the seven-dimensional orthogonal module for $G$ and that there is no cohomology for $[V, G]$ in characteristic $\neq 2$ and only one dimension of cohomology in characteristic 2.

In the case of characteristic not 2, we might also use the result of Premet and Suprunenko [PS] on quadratic modules for Chevalley groups. As a corollary of the Main Theorem (or from [BT]) we obtain that $G_2(L)$ does not occur as a subgroup of a linear group such that the long root elements act as transvections.

1.5. Corollary. Let $K$ be a commutative field, let $V$ be a finite-dimensional vector space over $K$, and let $\text{SL}(V) = \langle \Sigma \rangle$, where $\Sigma$ is the class of linear transvection groups. Then $\text{SL}(V)$ contains no subgroup $G$ generated by transvections satisfying hypothesis ($G_2$) of the Main Theorem.
2. G\textsubscript{2}(L) as a Group of Isometries of the Dickson Form

In this section, we describe how we can regard \(G_2(L)\) as a group of linear mappings preserving an alternating trilinear form (see [A]).

2.1

Let \(L\) be a field and let \(W = \langle x_1, x'_1 \rangle \perp \langle x_2, x'_2 \rangle \perp \langle x_3, x'_3 \rangle \perp \langle x_0 \rangle\) be a seven-dimensional vector space over \(L\) with associated quadratic form \(B\) such that \((x_i, x'_i)\) is a hyperbolic pair \((i = 1, 2, 3)\) and \(B(x_0) = -1\). Further, let \(f\) be the alternating trilinear form on \(W\) with monomials

\[
f = x_0 x_1 x'_1 + x_0 x_2 x'_2 + x_0 x_3 x'_3 + x_1 x_2 x_3 + x'_1 x'_2 x'_3.
\]

That is, \(f\) is the Dickson form (compare [A, p. 194]).

A singular line \(\zeta\) in \(W\) (singular with respect to the quadratic form \(B\)) is called doubly singular, if \(f(w, x, y) = 0\) for all \(w \in W\, x, y \in \zeta\) (compare [A, p. 194]). For example \(\langle x_1, x_2 \rangle\) is not doubly singular, since \(f(x_3, x_1, x_2) = 1\), and \(\langle x_1, x_2 \rangle\) is doubly singular.

2.2

Let \(O(W, f, B)\) be the subgroup of \(\text{GL}(W)\) consisting of all elements \(t \in \text{GL}(W)\) such that \(f(wt, xt, yt) = f(w, x, y)\) and \(B(wt) = B(w)\) for all \(w, x, y \in W\). By [A, (2.11), (3.4)] we have \(S := O(W, f, B) \cong G_2(L)\). Hence \(S \leq \Omega(W, B)\). Further, \(S\) is transitive on the doubly singular lines of \(W\) by [A, (7.3(2))] and \(T(x_1, x_2) \leq S\) by [A, (2.3)]. We denote by \(\Sigma^1\) the class of Siegel transvection groups of \(\Omega(W, B)\) corresponding to doubly singular lines of \(W\). Then the isomorphism mentioned above maps \(\Sigma^1\) to the class of long root subgroups of \(G_2(L)\).

2.3

Let \(W_3 = \langle x_3, x_2, x_1 \rangle, W_3' = \langle x'_3, x'_2, x'_1 \rangle, W_6 = W_3 \oplus W_3'\). We consider \(M := \text{SL}(W_3)\), where \(M\) acts naturally on \(W_3\), dually on \(W_3'\) (with \((x'_1, x'_2, x'_3)\) the dual basis of \((x_1, x_2, x_3)\)) and \(M\) fixes \(x_0\). Then \(M \leq S\) by [A, (2.3)].

2.4

By [A, (2.1)] we have \(X := \langle a(t), b(t) \mid t \in L \rangle \cong \text{SL}_2(L)\), where the matrices of \(a(t), b(t)\) with respect to the basis \((x_1, x'_1, x_2, x'_2, x_3, x'_3, x_0, x'_0)\) of
$W$ are as follows:

$$a(t) = \begin{pmatrix}
1 & t \\
t & 1 \\
1 & -t \\
-1 & 1 \\
1 & t \\
t^2 & 1 \\
1 & 2t \\
t & 1
\end{pmatrix},$$

$$b(t) = \begin{pmatrix}
1 & t \\
t & 1 \\
1 & -t \\
-1 & 1 \\
1 & 2t \\
t^2 & 1 \\
t & 1
\end{pmatrix} = a(-t)^\omega$$

with

$$\omega = \begin{pmatrix}
-1 & 1 \\
1 & -1 \\
1 & -1 \\
1 & 1
\end{pmatrix} = a(-1)b(1)a(-1).$$

Empty entries should be read as 0. We have $S = \langle M, X \rangle$ by [A, p. 205, (2.11)].

For $t \in L$, $t \neq 0$, we have $a(1)^{m(t)} = a(t)$ and $b(1)^{\omega^{-1}m(t)\omega} = b(t)$, where the matrix of $m(t)$ with respect to the basis $\{x_1, x_2, x_2', x_3, x_3, x_0, x_3'\}$ of $W$ is

$$\begin{pmatrix}
t & 1 \\
1 & t^{-1} \\
t^{-1} & 1 \\
t^{-1} & t
\end{pmatrix}.$$

Since $m(t), \omega^{-1}m(t)\omega \in M$, this yields $S = \langle M, a(1), b(1) \rangle$. 
3. THE ACTION $M = \text{SL}(W)$ ON THE ORTHOGONAL SPACE $V$

In this section, we describe the action of $M = \text{SL}(W)$ on the orthogonal space $V$, applying the results of [5]. We can regard $[V, M]$ as the direct sum of the natural and the dual module for $M$. In the following, we use the notation introduced so far.

3.1

Recall the definition of $S$ and $\Sigma$ in (2.2). The class $\Sigma$ of Siegel transvection groups of the orthogonal group $Y = \Omega(V, Q)$ is a class of abstract root subgroups in the sense of Timmesfeld [T]. The same holds for the class of long root subgroups of $G_2(L)$ and hence for the class $\Sigma^1$ of Siegel transvection groups in $S$. In particular, for $A, B \in \Sigma$, we have $[A, B] = 1$ or $\langle A, B \rangle = \text{SL}_2(K)$ or $[A, B] \in \Sigma$ (and similarly for $\Sigma^1$).

Two Siegel transvection groups $A, B \in \Sigma$ are commuting, if and only if $V, A \cap [V, B] \neq 0$ or $[V, A] \subseteq [V, B]^L$.

We have the following constellation:

$$
\begin{align*}
G & \xrightarrow{\chi_1} G/Z(G) \xrightarrow{\chi_2} G_2(L) \xrightarrow{\chi_3} S \\
\sigma & \quad \downarrow \quad Y
\end{align*}
$$

Here $\chi_1$ is the natural homomorphism, $\chi_2$ is the isomorphism occurring in hypothesis $(G_2)$, $\chi_3$ is the isomorphism of (2.2), and $\sigma$ is the inclusion mapping. We set $\chi = \chi_1 \chi_2 \chi_3$. Then $\Sigma^0 \chi = \Sigma^1$.

For each $A^1 = A^0 \chi \in \Sigma^1$, we have a corresponding element $A \in \Sigma$, defined by $A^0 \subseteq A$. The following relations between the Siegel transvection groups in $S = G_2(L)$ and the corresponding Siegel transvection groups on $V$ will be important throughout the whole paper.

(a) $[A, B] = 1$, if $[A^1, B^1] = 1$.

(b) $\langle A, B \rangle = \text{SL}_2(K)$, if $\langle A^1, B^1 \rangle = \text{SL}_2(L)$.


(d) If $A^1 \in \Sigma^1$, $g \in G$, and $C^1 := (A^1)^g$, then $C = A^g$.

Let $M^0 := \langle T^0 \mid T^1 \in M \cap \Sigma^1 \rangle \subseteq G$. Then $M^0 \chi = M$.

3.2. We have $V = [V, M] \perp C_\Sigma(M)$ with $[V, M]$ a $6^0$-space (i.e., an orthogonal sum of three hyperbolic lines), which can be regarded as the direct sum of the natural module and the dual module for $M$. There exists an embedding of fields $\alpha: L \to K$ and a basis $\mathcal{B} := \{v_1, v_2, v_3, v'_1, v'_2, v'_3\}$ of
\[ [V, M] \] such that the following holds:

(a) We have \([V, M] = \langle v_1, v'_1 \rangle \perp \langle v_2, v'_2 \rangle \perp \langle v_3, v'_3 \rangle \) with \((v_i, v'_i)\) a hyperbolic pair \((i = 1, 2, 3)\).

(b) the matrix of \(m \in M^0\) with respect to \(\mathcal{B}\) is obtained by applying \(\alpha\) to the matrix of \(m\) with respect to the basis \(\{x_1, x_2, x_3, x'_1, x'_2, x'_3\}\) of \(W_6\).

Proof. Because of (3.1), we may apply [S, (6.2.1)] to describe the action of \(M\) on \(V\). (Condition (Z) of [S, (3.1.1)] holds for \(S\) and \(Y\) with \(G := G, \phi := \chi\), and \(\delta = \sigma\).)

Hence we obtain that \(V = [V, M] \perp C_y(M)\) with \([V, M]\) a 6*-space. Further, \([V, M] = U_1 \oplus U_2\) with \(U_1, U_2\) three-dimensional singular and invariant under \(M\) (i.e., \([U_i, T] \subseteq U_i\) for all \(T \in M \cap \Sigma^1\)). We can regard \(U_1\) as the natural module for \(M\) and \(U_2\) as the dual module for \(M\).

This means that there exists an embedding \(\alpha: L \to K\) and an injective semilinear (with respect to \(\alpha\)) mapping \(\varphi: W_3 \to U_1\) with \(\langle W_3 \varphi \rangle_K = U_1\) such that \((w(m))_\varphi = (w \varphi) m\) for all \(w \in W_3, m \in M^0\). Further \(\chi: M^0 \to M\) is an isomorphism.

We let \(v_i := x_i \varphi\) \((i = 1, 2, 3)\). Since we can regard \(U_2\) as the dual module for \(M\), there exists a basis \(\{v'_1, v'_2, v'_3\}\) of \(U_2\) such that the matrix of each \(m \in M^0\) with respect to this basis is the transpose inverse of the matrix of \(m\) with respect to \(\{v'_1, v'_2, v'_3\}\).

Since all matrices \((A^\alpha(A^\alpha)^{-1})\), where \(A \in \text{SL}_3(L)\), occur as matrices of elements \(m \in M\), we obtain that the fundamental matrix of \(b\) with respect to the basis \(\mathcal{B} := \{v_1, v_2, v_3, v'_1, v'_2, v'_3\}\) of \([V, M]\) is of the form \((\lambda^1 \lambda^2)\) for some \(\lambda \in K\). We replace \(v'_i\) by \(\lambda^{-1} v'_i\) \((i = 1, 2, 3)\). Now \(\mathcal{B}\) and \(\alpha\) satisfy the requirements of (3.2).

4. PROOF THAT THE COMMUTATOR SPACE \(V_0\) IS SEVEN-DIMENSIONAL

In this section, we show that \(\dim V_0 = 7\). For this we use that \([V, M]\) is a 6*-space.

Let \(B^1 := T(c_1x_1), A^1 := (B^1)^{a(1)} \in \Sigma^1\), and \(E := \langle M, A^1 \rangle\). As an intermediate step we show that \(E = S \approx G_2(L)\).

4.1. \(M\) is transitive on the singular points of \(W\) which have an \(x_0\)-component.

Proof. Let \(P = \langle c_1 x_1 + c_2 x_2 + c_3 x_3 + c'_1 x'_1 + c'_2 x'_2 + c'_3 x'_3 + x_0 \rangle\) be a singular point. Then \(c_1 c'_1 + c_2 c'_2 + c_3 c'_3 = 1\). We show that there exists an
$m \in M$ with $\langle x_1 + x'_1 + x_0 \rangle = P$. Let $(c_4, c_5, c_6), (c_7, c_8, c_9) \in L^3$ be linearly independent with

$$(c_4, c_5, c_6)(c'_1, c'_2, c'_3)^t = 0, \quad (c_7, c_8, c_9)(c'_1, c'_2, c'_3)^t = 0.$$ 

Replacing $(c_4, c_5, c_6)$ by a scalar multiple, we may assume that the matrix $A$ defined below has determinant 1. For $m \in GL(W)$ whose matrix with respect to \{x_1, x_2, x_3, x'_1, x'_2, x'_3, x_0\} is

$$m = \begin{pmatrix} A & A^{-t} \\ & 1 \end{pmatrix}, \quad \text{where } A = \begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{pmatrix},$$

we have $m \in M$. Further $\langle x_1 + x'_1 + x_0 \rangle = P$, since the first row of $A^{-t}$ is $(c'_1, c'_2, c'_3)$.

4.2. Let $W'$ be the space generated by the singular points $y$ in $W_6$ with $y \subseteq [W, T^1]$ for some $T^1 \leq E$ with $[W, T^1] \not\subseteq W_6$. Then $W' = W_6$.

**Proof.** We have $A^1 \leq E$ with $[W, A^1] = \langle x'_1 - x_2, x_3 + x_0 + x'_3 \rangle \not\subseteq W_6$. Hence $\langle x'_1 - x_2 \rangle \subseteq W'$. We regard $W'' := \langle (x'_1 - x_2)m \mid m \in M \rangle$. Then $W''$ is an $LM$-submodule of $W_6$. Since $W_6 = W_3 \oplus W_3^+$ is the direct sum of two nonequivalent irreducible $LM$-modules, we obtain $W'' \subseteq \{0, W_3, W_3^+, W_6\}$, hence $W'' = W_6$. This yields $W_6 = W'' \subseteq W' \subseteq W_6$, thus $W' = W_6$.

4.3. $E$ is transitie on the singular points of $W$.

**Proof.** Let $\langle x \rangle$ be a singular point in $W_6$. Then there exists a singular point $\langle y \rangle$ in $W_6$ with $y \in [W, T^1] = L_1$, $T^1 \leq E$, $[W, T^1] \not\subseteq W_6$, such that $x \not\in y^\perp$. Since if $x$ is perpendicular to all these $y$, then (4.2) yields $x \in W_6 \cap W_6^+ = 0$, a contradiction. For $1 \neq t \in T^1$, we have $xt = x + l$ with $l \in L_1$, $l \not\in \langle y \rangle$, since $x \not\in y^\perp$. Hence $\langle x \rangle t$ is a singular point with an $x_0$-component. By (4.1) the claim follows.

4.4. $E$ is transitie on the doubly singular lines of $W$.

**Proof.** Let $L_1 = P_1 \oplus P_2$ be a doubly singular line and $e \in E$ with $P_1^e = \langle x_1 \rangle$ by (4.3). Then $L_1^e$ is a doubly singular line through $x_1$, hence $L_1^e = \langle x_1, \lambda x'_2 + \mu x'_3 \rangle$ with $\lambda, \mu \in L$ not both 0. We choose $m \in M$ with $x_1 m = x_1$ and $x'_2 m = \lambda x'_2 + \mu x'_3$. Then $L_1^{m^{-1}} = \langle x_1, x'_2 \rangle$.

4.5. We have $E = S = G_2(L)$.

**Proof.** Let $L_1$ be a doubly singular line and $e \in E$ with $L_1 = \langle x_1, x'_2 \rangle^e$ by (4.4). Then $T_{L_1}^e = T_{\langle x_1, x'_2 \rangle}^e \leq M^e \leq E$. 

4.6. We have \( \dim V_0 = 7 \) and \( C_{V_0}(M) = \langle v_0 \rangle \) with \( v_0 \in V_0 \) not singular.

Proof. Recall \( A^1 = (T_{(\xi, \eta, \zeta)})^{\text{nil}} \) and let \( T^1 = T_{(\xi, \eta, \zeta)} \in M \cap \Sigma^1 \). Then 
\[ [W, T] + [W, A^1] = \langle x_1, x_2, x_3 + x_0 + x'_3 \rangle \] 
is three-dimensional singular. Hence \( [V, T] + [V, A] \) is also three-dimensional singular as in [S, (7.2.1)] and \( \dim([V, M] \cap [V, A]) \geq 1 \). Using (4.5), this yields \( 6 = \dim[V, M] \leq \dim V_0 = 6 + 2 - 1 = 7 \).

We assume \( V_0 = [V, M] \). Let \( \langle \nu_1, \nu_2, \nu_3, \nu'_1, \nu'_2, \nu'_3 \rangle \) be the basis of \( [V, M] \) occurring in (3.2). Then \( [V, T] = \langle \nu'_1, \nu'_2 \rangle \). Let \( [V, A] \cap [V, T] = P = \langle \alpha \nu'_1 + \beta \nu'_2 \rangle \). Then \( P \neq \langle \nu'_1 \rangle \). Since otherwise \( [V, T] \cap [V, A] \neq 0 \) for \( D^1 = T_{(\xi, \eta, 0)} \). Hence \( [D, A] = 1 \), and also \([D, A^1] = 1\), a contradiction. Similarly, \( P \neq \langle \nu'_2 \rangle \).

Let \([V, A] = P \oplus \langle c_1 \nu_1 + c_2 \nu_2 + c_3 \nu_3 + \nu'_1 + \nu'_2 + \nu'_3 \rangle \). Then \( c_1 \nu'_1 + c_2 \nu'_2 + c_3 \nu'_3 = 0 \). Since \([V, A] \subseteq [V, T]^{\perp} \), we obtain \( c_1 = 0, c_2 = 0 \). Hence \( c_3 = 0 \). We first consider the case \( c'_3 = 0 \). Let \([V, A] \cap \nu^\perp = \langle y \rangle \).

Then \( y = \lambda \nu_1 + \mu \nu_3 \), with \( \lambda, \mu \in K \). We have \( \mu \neq 0 \), since otherwise \( \nu_2 \in [V, A] \cap [V, T] = P \), a contradiction. Hence \([V, A] + [V, T] = \langle \nu'_1, \nu_2, \nu'_3 \rangle \). Thus \([V, B] \subseteq [V, A] + [V, T] \subseteq [V, A]^{\perp} \), where \( B^1 = T_{(\xi, \eta, 0)} \). Hence \([B^1, A^1] = 1 \), a contradiction. Similarly, the case \( c_3 = 0 \) leads to a contradiction.

Hence \( \dim V_0 = 7 \). Because of \( V = [V, M] \perp C_{V_0}(M) \), we have \( V_0 = [V, M] + C_{V_0}(M) \). This shows \( C_{V_0}(M) = \langle v_0 \rangle \) with \( v_0 \in V_0 \). If \( v_0 \) is singular, then \( V_0/\langle v_0 \rangle \) is a 6-dimensional space on which \( S = G_2(L) \) acts by Siegel transformations. This is not possible by the previous part of the proof.

5. THE ACTION OF \( a(1), b(1) \) ON \( V_0 \)

In this section, we construct a basis of \( V_0 \) such that for \( a(1), b(1) \), and all \( m \in M \) the matrix with respect to this basis is obtained by applying the embedding of fields \( \alpha \) to the matrix with respect to the basis \( \{x_1, x'_2, x_2, x'_3, x_3, x_0, x'_0\} \) of \( W \). Our starting point is the basis \( \langle \nu_1, \nu_2, \nu_3, \nu'_1, \nu'_2, \nu'_3, \nu_0 \rangle \) constructed in the proof of (3.2) and (4.6).

5.1. The subspaces \( \langle \nu_1, \nu'_1 \rangle, \langle \nu'_1, \nu_2 \rangle, \) and \( \langle \nu_3, \nu_3, \nu'_3 \rangle \) are invariant under \( X \) of (2.4).

Proof. Let \( SL_2(L) = S_1 \leq M \), where \( S_1 \) acts naturally on \( \langle x_1, x_2 \rangle \), dually on \( \langle x'_1, x'_2 \rangle \) (with \( \{x'_1, x'_2\} \) the dual basis of \( \{x_1, x_2\} \)) and \( S_1 \) fixes \( x_3, x'_3, x_0 \).

The action of \( S_1 \) on \( V_0 \) is known by (3.2). Since \( [S_1, X] = 1 \), we obtain that \( [V, S_1] = \langle \nu_1, \nu_2, \nu'_1, \nu'_2, \nu'_3 \rangle \) and \( C_{V_0}(S_1) = \langle \nu_3, \nu_3, \nu'_3 \rangle \) are invariant under \( X \). Further, a matrix calculation shows that with respect to the basis
\( \{v_1, v_2, v'_1, v'_2, v_3, v_0, v'_3\} \) the matrix of \( x \in X \) is of the form
\[
\begin{pmatrix}
  a & -b \\
  c & d \\
  -c & d \\
\end{pmatrix}
\]
with coefficients \( a, b, c, d \in K \) and a \( 3 \times 3 \) matrix \( C \). This yields the claim.

5.2. We have \( v_1 a(\lambda) \in \langle v_1 \rangle \) and \( v'_2 b(\lambda) \in \langle v'_2 \rangle \) for \( \lambda \in L \).

Proof. For \( \lambda \in L \) and \( T^{-1} = T_{(x_1, x_2)} \), we have \( T^{-1} = (T^1)^{a(\lambda)} \). Hence
\[
\langle v_1, v'_2 \rangle = [V, T] = [V, T]a(\lambda) = \langle v_1 a(\lambda), v'_2 a(\lambda) \rangle.
\]
This yields \( v_1 a(\lambda) \in \langle v_1 \rangle \) for \( \lambda \in L \). Similarly, \( v'_2 b(\lambda) \in \langle v'_2 \rangle \) for \( \lambda \in L \), using \( T_{(v'_2, v_2)} \).

5.3. With respect to \( \{v_1, v'_2\} \) we have
\[
a(1) = \begin{pmatrix} 1 \\ x \\ 1 \end{pmatrix}, \quad b(1) = \begin{pmatrix} 1 \\ x^{-1} \\ 1 \end{pmatrix}
\]
for some \( 0 \neq x \in K \).

Proof. For \( 0 \neq t \in L \), we denote by \( h(t) \) the element of \( M \) with matrix
\[
h(t) :=
\begin{pmatrix}
  t & t^{-1} \\
  t & t^{-1} \\
  t^{-2} & 1 \\
\end{pmatrix}
\]
with respect to the basis \( \{x_1, x_2, x'_1, x'_2, x_3, x_0, x'_3\} \).

Then \( a(c) h(t) = a(t^2 c) \) for \( c \in L \). Since \( |L| \geq 4 \), there exists a \( 0 \neq t \in L \) with \( t^2 \neq 1 \). Let \( \lambda \in L \) with \( \lambda(t^2 - 1) = 1 \). Then \( [a(\lambda), h(t)] = a(\lambda(t^2 - 1)) = a(1) \). Hence by (5.1), (5.2), and (3.2) the matrix of \( a(1) \) is of the form
\[
a(1) \sim \begin{pmatrix} a \\ c \\ d \end{pmatrix}^{-1} \begin{pmatrix} t^a & t^{-a} \\ t^{-a} & t^a \end{pmatrix}^{-1} \begin{pmatrix} a \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ 1 \end{pmatrix}
\]
for some \( x \in K \). Similarly, \( b(1) = (1 \; \frac{1}{x}) \).
It remains to show, that \( xy = 1 \). For \( \omega = a(-1)b(1)a(-1) \), we have \( T_\omega^{(x, y)} = T_{(x, y)} \). Hence \( \langle v'_2, v'_3 \rangle = \langle v_1 \omega, v'_2 \omega \rangle \) and \( v_1 \omega \in \langle v'_2 \rangle \). On the other hand, a matrix calculation yields that the first row of the matrix of \( \omega \) with respect to \( \{ v_1, v'_2 \} \) is \( (1 - xy, y) \). Hence \( xy = 1 \).

5.4

Next, we make some suitable replacements.

1. Replacing \( v'_i \) by \( x^{-1}v'_i \) \( (i = 1, 2, 3) \), we may assume that the matrices of \( a(1), b(1) \) with respect to \( \{ v_1, v'_2 \} \) are

\[
a(1) \sim \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad b(1) \sim \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

Further, this replacement does not affect the elements \( m \in M \).

2. The fundamental matrix of \( b \) with respect to the basis \( \{ v_1, v_2, v_3, v'_1, v'_2, v'_3 \} \) is of the form \( (\lambda I, \lambda') \) for some \( \lambda \in K \) as in the proof of (3.2). Replacing \( Q \) by the proportional quadratic form \( \lambda^{-1}Q \), we may assume that \( (v_j, v'_i) \) is a hyperbolic pair \( (i, 1, 2, 3) \).

3. Let \( Q(v_0) = r \), where \( C_{Tr}(M) = \langle v_0 \rangle \) as in (4.6), and let \( \overline{K} \) be the algebraic closure of \( K \). We replace \( V \) by \( \overline{K} \otimes V \) with \( Q \) extended to \( \overline{K} \otimes V \) and choose \( c \in \overline{K} \) with \( c^2 = -r^{-1} \). Replacing \( v_0 \) by \( cv_0 \), we may assume \( Q(v_0) = -1 \). Later, we will show that \( c \in K \), i.e., it was not necessary to pass to the algebraic closure.

5.5

The action of \( a(1), b(1) \) on \( \langle v_1, v'_2, v_2, v'_3 \rangle \) is already determined. Next, we describe the action on \( \langle v_3, v_0, v'_3 \rangle \).

As in (5.2), \( \langle v'_3 \rangle \) is invariant under \( a(\lambda) \). Therefore, \( C_{Tr}(S_v) \cap v'_3 = \langle v_0, v'_3 \rangle \) is also invariant under \( a(\lambda) \). Hence the matrix of \( a(\lambda) \) with respect to the basis \( \{ v_3, v_0, v'_3 \} \) is of the form

\[
a(\lambda) \sim \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = A.
\]

Since \( a(\lambda) \) preserves \( Q \), we have \( Q(v_3) = 0 = Q(v_3 a(1)) \), hence \( ac - b^2 = 0 \). We abbreviate the fundamental matrix of the bilinear form associated to \( Q \) with \( J \). The equation \( A^2 = J \) shows that the matrix of \( a(\lambda) \) is of the form

\[
a(\lambda) \sim \begin{pmatrix} a & b & a^{-1}b^2 \\ d & e & 2a^{-1}bd \\ a^{-1} \end{pmatrix}.
\]
Let $\lambda, \iota \in L$ with $a(\lambda) = [a(\lambda), b(\iota)]$ as in the proof of (5.3). Then a matrix calculation shows that the matrix of $a(1)$ with respect to $\{v_3, v_0, v'_3\}$ is of the form

$$
a(1) \sim \begin{pmatrix} 1 & y & y^2 \\ 1 & 2y & 1 \end{pmatrix}.
$$

Similarly, the matrix of $b(1)$ is of the form

$$
b(1) \sim \begin{pmatrix} 1 \\ 2x & 1 \\ x^2 & x & 1 \end{pmatrix}.
$$

For $T^0 = T_{(x_1 + x_2, x_3)}$, we have $[V, T'] = \langle v'_1 + v'_3, v_2 \rangle$. Let $A^0 := (T^0)_w = T_{(x_1 + x_2, x_3)}$. Then $\langle v_2 + v'_3, -v'_1 \rangle = [V, A] = [V, T]_w = \langle v_2 + v'_3, -v'_1 \rangle$, since the action of $X$ on $\langle v_1, v'_2, v_2, v'_1 \rangle$ is already determined. This yields $v'_2 \in K$.

We have $\omega = a(-1)b(1)\alpha(-1)$. A matrix calculation shows that the last row of the matrix of $\omega$ is $(x^2, x(1-xy), (xy-1)^2)$. Hence $x^2 = 1, x(1-xy) = 0$, and $(xy-1)^2 = 0$, i.e., $x^2 = 1$ and $xy = 1$. If $x = 1$, then $x = y = 0$. If $x = -1$, then we replace $v_0$ by $-v_0$ and may thus assume that $x = y = 1$.

The present $v_0$ is of the form $\pm cv_0$, where $v_0$ is the original $v_0$ and $c$ is an element of the algebraic closure of $K$. Hence with respect to the original $v_0$ the matrix of $a(1)$ is a matrix over $K$ of the form

$$
\begin{pmatrix}
1 & \pm c^{-1} \\
1 & 1 & 2 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & \pm c \\
1 & 1 \\
1 & 1
\end{pmatrix}.
$$

Thus $c \in K$.

6. THE PROOF OF THE MAIN THEOREM

6.1

Since $S = \langle M, a(1), b(1) \rangle$, we have shown that $\{v_1, v_2, v_3, v'_1, v'_2, v'_3, v_0\}$ is a basis of $V_0$ over $K$ such that the matrix of $g \in G$ with respect to this basis is obtained by applying the embedding of fields $\alpha$ to the matrix of $g X$ with respect to the basis $\{x_1, x_2, x_3, x'_1, x'_2, x'_3, x_0\}$ of $W$.

The semilinear (with respect to $\alpha$) mapping $\varphi: W \to V_1$ with $x_i \mapsto v_i, x'_i \mapsto v'_i, x_0 \mapsto v_0$ $(i = 1, 2, 3)$ satisfies $(w(g X))\varphi = (w \varphi)g$ for all $w \in W$. 
$g \in G$. Hence the embedding of $G$ in $Y$ is induced by a semilinear mapping and $\chi: G \to S$ is an isomorphism. Further, $\varphi$ is injective and $\langle W\varphi \rangle_K = V_0$. The quadratic form defined by $B(w\varphi) := B(w)^k$ for $w \in W$ is proportional to $Q$.

6.2

We are left with the last part of the Main Theorem. Calculating dimensions shows that $V = V_0 + C_V(G)$ if and only if $\text{Rad}(V_0) \subseteq \text{Rad}(V)$. If $\text{Rad}(V_0) \not\subseteq \text{Rad}(V)$, then $V_0 + C_V(G)$ is a hyperplane of $V$. We choose an eight-dimensional subspace $V_1$ of $V$, which contains $V_0$, such that $\text{Rad}(V_1) = 0$. Then $V = V_1 + C_V(G)$. The action of $G$ on $V_1$ is uniquely determined by the action of $G$ on $V_0$, since there is only one possibility to extend Siegel transvections on $V_0$ to Siegel transvections on $V_1$ by [S, (4.3.1)].

6.3. Proof of Corollary 1.5. We assume that $\text{SL}(V)$ has a subgroup $G$ generated by transvections satisfying hypothesis $(G_2)$ of the Main Theorem. By (4.5) we may write $G_2(L) = \langle M, T \rangle$, where $T$ is a long root subgroup and $M = \text{SL}_2(L)$ is generated by long root subgroups. Hence the commutator space $[V, G]$ is at most four-dimensional and $G$ is a subgroup of $\text{SL}_4(K)$ such that long root elements act as transvections. Because of $\text{PSL}_4(K) = P\Omega^+(L)K$, we may apply the Klein correspondence and obtain $G$ as subgroup of a six-dimensional orthogonal group such that long root elements act as Siegel transvections. This is a contradiction to the Main Theorem, (1.2(a)).

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