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Subgroups Isomorphic to $G_2(L)$ in Orthogonal Groups

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1. INTRODUCTION AND STATEMENT OF THE MAIN THEOREM

In this paper, we describe the embeddings of groups $G_2(L)$ in orthogonal groups such that the long root elements act as Siegel transvections. For finite orthogonal groups, this is contained in the results of Kantor [K] on subgroups of finite classical groups generated by a class of long root elements. The problem stated above is part of the determination of the subgroups G generated by long root elements in algebraic groups Y over arbitrary fields. For the case where Y and G are classical groups, see [S].

To state the Main Theorem, we introduce some notation.

1.1

Let K be a commutative field, let V be a finite-dimensional vector space over K, and let $Q: V \to K$ be a quadratic form (with associated bilinear form b). A subspace U of V is called singular, if $Q(u) = 0$ for all $u \in U$. We assume that Q is nondegenerate (i.e., if $v \in \mathrm{Rad}(V, b)$ with $Q(v) = 0$, then $v = 0$) and that Q has Witt index at least 3 (i.e., V contains three-dimensional singular subspaces).

Let ℓ be a singular line of V with basis $\{x, y\}$. For $c \in K$, the mapping

$$t_c: v \mapsto v - cb(v, x)y + cb(v, y)x \qquad \text{for } v \in V$$

is called a Siegel transvection (see [T, Th. 5], [S, (1.1.3)]). The set $T_\ell := \{t_c \mid c \in K\}$ is the Siegel transvection group corresponding to ℓ. Let $\Omega(V, Q)$

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$:= \langle \Sigma \rangle$, where

$$\Sigma := \{T_\ell \mid \ell \text{ a singular line in } V\}$$

is the class of Siegel transvection groups, be the associated orthogonal group.

Let G be a subgroup of $Y := \Omega(V, Q)$ which is generated by Siegel transvections. For $A \in \Sigma$, we set $A^0 := A \cap G$, $\Sigma^0 := \{A^0 \mid A \in \Sigma,\ A^0 \neq 1\}$ and further $V_0 := [V, G]$. We assume that G and Σ^0 satisfy the following hypothesis:

(G_2) G is quasi-simple and there exists a commutative field L such that $\overline{G} := G/Z(G) \simeq G_2(L)$ (resp. $G_2(2)'$) and $\overline{\Sigma^0} := \{\overline{A^0} \mid A^0 \in \Sigma^0\}$ is the class of long root subgroups of $\overline{G}$.

We regard $G_2(L)$ as the subgroup S of a seven-dimensional orthogonal group $\Omega(W, B)$ which preserves the Dickson form (as suggested by Aschbacher [A]; see Section 2). We say that W is the natural module for $G_2(L)$.

The Main Theorem of this paper is:

1.2. Main Theorem. *Let* $Y = \Omega(V, Q)$, *let* G *and* $G_2(L) \simeq S \leq \Omega(W, B)$ *be as in* (1.1). *Then the following hold*:

(a) *We have* $\dim V_0 = 7$.

(b) *There exists an embedding of fields* $\alpha: L \to K$, *an injective semilinear* (*with respect to* α) *mapping* $\varphi: W \to V_0$ *with* $V_0 = \langle W\varphi \rangle_K$, *and an isomorphism* $\chi: G \to S \simeq G_2(L)$ *such that* $(w(g\chi))\varphi = (w\varphi)g$ *for all* $w \in W$, $g \in G$.

(c) *The quadratic form* $\tilde{B}$ *on* V_0 *defined by* $\tilde{B}(w\varphi) := B(w)^\alpha$ *for* $w \in W$ *is proportional to* Q, *i.e.*, $Q = d\tilde{B}$ *on* V_0 *for some* $d \in K$.

(d) *If* $\mathrm{Rad}(V_0) \subseteq \mathrm{Rad}(V)$, *then* $V = V_0 + C_V(G)$. *If* $\mathrm{Rad}(V_0) \not\subseteq \mathrm{Rad}(V)$, *then* $V_0 + C_V(G)$ *is a hyperplane of* V. *In the latter case there exists an eight-dimensional subspace* V_1 *of* V, *which contains* V_0, *such that* $\mathrm{Rad}(V_1) = 0$ *and* $V = V_1 + C_V(G)$. *The action of* G *on* V_1 *is uniquely determined by the action of* G *on* V_0.

The Main Theorem shows that the embedding of G in Y is induced by a semilinear mapping. We can regard the commutator space V_0 as the natural module for G (tensored with the bigger field K).

1.3

The idea of the proof is as follows: We may write $G_2(L) = \langle M, X \rangle$, where $M \simeq \mathrm{SL}_3(L)$ is generated by long root subgroups and $X \simeq \mathrm{SL}_2(L)$ is generated by two short root subgroups. By [S] the action of M on the orthogonal space V is known ($[V, M]$ is the direct sum of a natural and a dual module for M). We hence may determine the action of X on V, using a subgroup $S_1 \simeq \mathrm{SL}_2(L)$ of M with $[S_1, X] = 1$.

Because of the results of Kantor [K, Th. I, 12.(B)], we may restrict to the case $|L| \geq 4$.

1.4

In the proof of the Main Theorem also the results of Borel and Tits [BT] on abstract homomorphisms of algebraic groups might be used (see also [St]). We give a short outline of this approach.

Let L be an infinite field and G an isotropic simple algebraic group defined over L. Assume that G is split and simply connected. Let V be a finite-dimensional vector space over an algebraically closed field $\overline{K}$ and denote by ρ: $G(L) \to \mathrm{GL}(V)$ an irreducible representation of the group $G(L)$ of rational points. By [BT, (10.4)], ρ is equivalent to a tensor product $\otimes_{i=1}^{r} \pi_i \circ \alpha_i$, where α_i: $L \to \overline{K}$ is an embedding of fields, ${}^{\alpha_i}G$ is the group obtained by transfer of base field, and π_i is a nontrivial rational irreducible linear representation of ${}^{\alpha_i}G$.

Let K be any field with algebraic closure $\overline{K}$. Choose G as in the above paragraph and of type G_2. Assume that V is an absolutely irreducible KG-module of dimension at most 8 which is tensor indecomposable. (These properties may by verified for $[V, G]/C_{[V,G]}(G)$ under hypothesis (G_2).) We apply [BT, (10.4)] to $\overline{V} := \overline{K} \otimes_K V$ and use that the only irreducible modules over $\overline{K}$ of dimension at most 8 for an algebraic group of type G_2 are the seven-dimensional orthogonal module in characteristic $\neq 2$ and the six-dimensional symplectic module in characteristic 2 (see [KL, (5.4.12)], for example). Computing traces yields that the image of L under the field embedding into $\overline{K}$ is contained in K rather than in $\overline{K}$.

Hence in the Main Theorem $[V, G]/C_{[V,G]}(G)$ is a seven- or six-dimensional natural module for G tensored with K. To finish the proof, we have to show that $[V, G]$ is the seven-dimensional orthogonal module for G and that there is no cohomology for $[V, G]$ in characteristic $\neq 2$ and only one dimension of cohomology in characteristic 2.

In the case of characteristic not 2, we might also use the result of Premet and Suprunenko [PS] on quadratic modules for Chevalley groups. As a corollary of the Main Theorem (or from [BT]) we obtain that $G_2(L)$ does not occur as a subgroup of a linear group such that the long root elements act as transvections.

1.5. COROLLARY. *Let K be a commutative field, let V be a finite-dimensional vector space over K, and let* $\mathrm{SL}(V) = \langle \Sigma \rangle$, *where Σ is the class of linear transvection groups. Then* $\mathrm{SL}(V)$ *contains no subgroup G generated by transvections satisfying hypothesis* (G_2) *of the Main Theorem.*

2. $G_2(L)$ AS A GROUP OF ISOMETRIES OF THE DICKSON FORM

In this section, we describe how we can regard $G_2(L)$ as a group of linear mappings preserving an alternating trilinear form (see [A]).

2.1

Let L be a field and let $W = \langle x_1, x_1' \rangle \perp \langle x_2, x_2' \rangle \perp \langle x_3, x_3' \rangle \perp \langle x_0 \rangle$ be a seven-dimensional vector space over L with associated quadratic form B such that (x_i, x_i') is a hyperbolic pair $(i = 1, 2, 3)$ and $B(x_0) = -1$. Further, let f be the alternating trilinear form on W with monomials

$$f = x_0 x_1 x_1' + x_0 x_2 x_2' + x_0 x_3 x_3' + x_1 x_2 x_3 + x_1' x_2' x_3'.$$

That is, f is the Dickson form (compare [A, p. 194]).

A singular line ℓ in W (singular with respect to the quadratic form B) is called doubly singular, if $f(w, x, y) = 0$ for all $w \in W,\ x, y \in \ell$ (compare [A, p. 194]). For example $\langle x_1, x_2 \rangle$ is not doubly singular, since $f(x_3, x_1, x_2) = 1$, and $\langle x_1, x_2' \rangle$ is doubly singular.

2.2

Let $O(W, f, B)$ be the subgroup of $\mathrm{GL}(W)$ consisting of all elements $t \in \mathrm{GL}(W)$ such that $f(wt, xt, yt) = f(w, x, y)$ and $B(wt) = B(w)$ for all $w, x, y \in W$. By [A, (2.11), (3.4)] we have $S := O(W, f, B) \simeq G_2(L)$. Hence $S \leq \Omega(W, B)$. Further, S is transitive on the doubly singular lines of W by [A, (7.3)(2)] and $T_{\langle x_1, x_2' \rangle} \leq S$ by [A, (2.3)]. We denote by Σ^1 the class of Siegel transvection groups of $\Omega(W, B)$ corresponding to doubly singular lines of W. Then the isomorphism mentioned above maps Σ^1 to the class of long root subgroups of $G_2(L)$.

2.3

Let $W_3 = \langle x_1, x_2, x_3 \rangle$, $W_3' = \langle x_1', x_2', x_3' \rangle$, $W_6 = W_3 \oplus W_3'$. We consider $M := \mathrm{SL}(W_3)$, where M acts naturally on W_3, dually on W_3' (with $\{x_1', x_2', x_3'\}$ the dual basis of $\{x_1, x_2, x_3\}$) and M fixes x_0. Then $M \leq S$ by [A, (2.3)].

2.4

By [A, (2.1)] we have $X := \langle a(t), b(t) \mid t \in L \rangle \simeq \mathrm{SL}_2(L)$, where the matrices of $a(t), b(t)$ with respect to the basis $\{x_1, x_2', x_2, x_1', x_3, x_0, x_3'\}$ of

W are as follows:

$$a(t) = \left(\begin{array}{cc|cc|ccc} 1 & & & & & & \\ t & 1 & & & & & \\ \hline & & 1 & & & & \\ & & -t & 1 & & & \\ \hline & & & & 1 & t & t^2 \\ & & & & & 1 & 2t \\ & & & & & & 1 \end{array}\right),$$

$$b(t) = \left(\begin{array}{cc|cc|ccc} 1 & t & & & & & \\ & 1 & & & & & \\ \hline & & 1 & -t & & & \\ & & & 1 & & & \\ \hline & & & & 1 & & \\ & & & & 2t & 1 & \\ & & & & t^2 & t & 1 \end{array}\right) = a(-t)^{\omega}$$

with

$$\omega = \left(\begin{array}{cc|cc|ccc} & 1 & & & & & \\ -1 & & & & & & \\ \hline & & & -1 & & & \\ & & 1 & & & & \\ \hline & & & & & & 1 \\ & & & & & -1 & \\ & & & & 1 & & \end{array}\right) = a(-1)b(1)a(-1).$$

Empty entries should be read as 0. We have $S = \langle M, X \rangle$ by [A, p. 205, (2.11)].

For $t \in L$, $t \neq 0$, we have $a(1)^{m(t)} = a(t)$ and $b(1)^{\omega^{-1}m(t)\omega} = b(t)$, where the matrix of $m(t)$ with respect to the basis $\{x_1, x_2', x_2, x_1', x_3, x_0, x_3'\}$ of W is

$$\left(\begin{array}{cc|cc|ccc} t & & & & & & \\ & 1 & & & & & \\ \hline & & 1 & & & & \\ & & & t^{-1} & & & \\ \hline & & & & t^{-1} & & \\ & & & & & 1 & \\ & & & & & & t \end{array}\right).$$

Since $m(t)$, $\omega^{-1}m(t)\omega \in M$, this yields $S = \langle M, a(1), b(1) \rangle$.

3. THE ACTION $M = \mathrm{SL}(W_3)$ ON THE ORTHOGONAL SPACE V

In this section, we describe the action of $M = \mathrm{SL}(W_3)$ on the orthogonal space V, applying the results of [S]. We can regard $[V, M]$ as the direct sum of the natural and the dual module for M. In the following, we use the notation introduced so far.

3.1

Recall the definition of S and Σ^1 in (2.2). The class Σ of Siegel transvection groups of the orthogonal group $Y = \Omega(V, Q)$ is a class of abstract root subgroups in the sense of Timmesfeld [T]. The same holds for the class of long root subgroups of $G_2(L)$ and hence for the class Σ^1 of Siegel transvection groups in S. In particular, for $A, B \in \Sigma$, we have $[A, B] = 1$ or $\langle A, B\rangle \simeq \mathrm{SL}_2(K)$ or $[A, B] \in \Sigma$ (and similarly for Σ^1).

Two Siegel transvection groups $A, B \in \Sigma$ are commuting, if and only if $[V, A] \cap [V, B] \neq 0$ or $[V, A] \subseteq [V, B]^{\perp}$.

We have the following constellation:

$$\begin{array}{ccccccc} G & \xrightarrow{\chi_1} & G/Z(G) & \xrightarrow{\chi_2} & G_2(L) & \xrightarrow{\chi_3} & S \\ \sigma\big\downarrow & & & & & & \\ Y & & & & & & \end{array}$$

Here χ_1 is the natural homomorphism, χ_2 is the isomorphism occurring in hypothesis (G_2), χ_3 is the isomorphism of (2.2), and σ is the inclusion mapping. We set $\chi := \chi_1 \chi_2 \chi_3$. Then $\Sigma^0\chi = \Sigma^1$.

For each $A^1 = A^0\chi \in \Sigma^1$, we have a corresponding element $A \in \Sigma$, defined by $A^0 \subseteq A$. The following relations between the Siegel transvection groups in $S \simeq G_2(L)$ and the corresponding Siegel transvection groups on V will be important throughout the whole paper.

(a) $[A, B] = 1$, if $[A^1, B^1] = 1$.

(b) $\langle A, B\rangle \simeq \mathrm{SL}_2(K)$, if $\langle A^1, B^1\rangle \simeq \mathrm{SL}_2(L)$.

(c) $[A, B] = C$, if $[A^1, B^1] = C^1$.

(d) If $A^1 \in \Sigma^1$, $g \in G$, and $C^1 := (A^1)^{g\chi}$, then $C = A^g$.

Let $M^0 := \langle T^0 \mid T^1 \in M \cap \Sigma^1\rangle \leq G$. Then $M^0\chi = M$.

3.2. *We have $V = [V, M] \perp C_V(M)$ with $[V, M]$ a 6^+-space (i.e., an orthogonal sum of three hyperbolic lines), which can be regarded as the direct sum of the natural module and the dual module for M. There exists an embedding of fields $\alpha\colon L \to K$ and a basis $\mathscr{B} := \{v_1, v_2, v_3, v_1', v_2', v_3'\}$ of*

$[V, M]$ *such that the following holds*:

(a) *We have* $[V, M] = \langle v_1, v_1' \rangle \perp \langle v_2, v_2' \rangle \perp \langle v_3, v_3' \rangle$ *with* (v_i, v_i') *a hyperbolic pair* $(i = 1, 2, 3)$.

(b) *the matrix of* $m \in M^0$ *with respect to* $\mathscr{B}$ *is obtained by applying* α *to the matrix of* $m\chi$ *with respect to the basis* $\{x_1, x_2, x_3, x_1', x_2', x_3'\}$ *of* W_6.

Proof. Because of (3.1), we may apply [S, (6.2.1)] to describe the action of M on V. (Condition (Z) of [S, (3.1.1)] holds for S and Y with $\hat{G} := G$, $\psi := \chi$, and $\delta = \sigma$.)

Hence we obtain that $V = [V, M] \perp C_V(M)$ with $[V, M]$ a 6^+-space. Further, $[V, M] = U_1 \oplus U_2$ with U_1, U_2 three-dimensional singular and invariant under M (i.e., $[U_i, T] \subseteq U_i$ for all $T^1 \in M \cap \Sigma^1$). We can regard U_1 as the natural module for M and U_2 as the dual module for M.

This means that there exists an embedding $\alpha\colon L \to K$ and an injective semilinear (with respect to α) mapping $\varphi\colon W_3 \to U_1$ with $\langle W_3\varphi \rangle_K = U_1$ such that $(w(m\chi))\varphi = (w\varphi)m$ for all $w \in W_3$, $m \in M^0$. Further $\chi\colon M^0 \to M$ is an isomorphism.

We let $v_i := x_i\varphi$ $(i = 1, 2, 3)$. Since we can regard U_2 as the dual module for M, there exists a basis $\{v_1', v_2', v_3'\}$ of U_2 such that the matrix of each $m \in M^0$ with respect to this basis is the transpose inverse of the matrix of m with respect to $\{v_1, v_2, v_3\}$.

Since all matrices $\left(\begin{smallmatrix} A^\alpha & \\ & (A^\alpha)^{-t} \end{smallmatrix}\right)$, where $A \in \mathrm{SL}_3(L)$, occur as matrices of elements $m \in M$, we obtain that the fundamental matrix of b with respect to the basis $\mathscr{B} := \{v_1, v_2, v_3, v_1', v_2', v_3'\}$ of $[V, M]$ is of the form $\left(\begin{smallmatrix} & \lambda I \\ \lambda I & \end{smallmatrix}\right)$ for some $\lambda \in K$. We replace v_i' by $\lambda^{-1} v_i'$ $(i = 1, 2, 3)$. Now $\mathscr{B}$ and α satisfy the requirements of (3.2). ∎

4. PROOF THAT THE COMMUTATOR SPACE V_0 IS SEVEN-DIMENSIONAL

In this section, we show that $\dim V_0 = 7$. For this we use that $[V, M]$ is a 6^+-space.

Let $B^1 := T_{\langle x_1', x_3 \rangle}$, $A^1 := (B^1)^{a(1)} \in \Sigma^1$, and $E := \langle M, A^1 \rangle$. As an intermediate step we show that $E = S \simeq G_2(L)$.

4.1. *M is transitive on the singular points of W which have an x_0-component.*

Proof. Let $P = \langle c_1x_1 + c_2x_2 + c_3x_3 + c_1'x_1' + c_2'x_2' + c_3'x_3' + x_0 \rangle$ be a singular point. Then $c_1c_1' + c_2c_2' + c_3c_3' = 1$. We show that there exists an

$m \in M$ with $\langle x_1 + x_1' + x_0 \rangle m = P$. Let $(c_4, c_5, c_6), (c_7, c_8, c_9) \in L^3$ be linearly independent with

$$(c_4, c_5, c_6)(c_1', c_2', c_3')^t = 0, \qquad (c_7, c_8, c_9)(c_1', c_2', c_3')^t = 0.$$

Replacing (c_4, c_5, c_6) by a scalar multiple, we may assume that the matrix A defined below has determinant 1. For $m \in \mathrm{GL}(W)$ whose matrix with respect to $\{x_1, x_2, x_3, x_1', x_2', x_3', x_0\}$ is

$$m = \begin{pmatrix} A & & \\ & A^{-t} & \\ & & 1 \end{pmatrix}, \qquad \text{where } A = \begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{pmatrix},$$

we have $m \in M$. Further $\langle x_1 + x_1' + x_0 \rangle m = P$, since the first row of A^{-t} is (c_1', c_2', c_3'). ∎

4.2. *Let W' be the space generated by the singular points y in W_6 with $y \subseteq [W, T^1]$ for some $T^1 \leq E$ with $[W, T^1] \not\subseteq W_6$. Then $W' = W_6$.*

Proof. We have $A^1 \leq E$ with $[W, A^1] = \langle x_1' - x_2, x_3 + x_0 + x_3' \rangle \not\subseteq W_6$. Hence $\langle x_1' - x_2 \rangle \subseteq W'$. We regard $W'' := \langle (x_1' - x_2)m \mid m \in M \rangle$. Then W'' is an LM-submodule of W_6. Since $W_6 = W_3 \oplus W_3'$ is the direct sum of two nonequivalent irreducible LM-modules, we obtain $W'' \in \{0, W_3, W_3', W_6\}$, hence $W'' = W_6$. This yields $W_6 = W'' \subseteq W' \subseteq W_6$, thus $W' = W_6$. ∎

4.3. *E is transitive on the singular points of W.*

Proof. Let $\langle x \rangle$ be a singular point in W_6. Then there exists a singular point $\langle y \rangle$ in W_6 with $y \in [W, T^1] =: L_1$, $T^1 \leq E$, $[W, T^1] \not\subseteq W_6$, such that $x \notin y^\perp$. Since if x is perpendicular to all these y, then (4.2) yields $x \in W_6 \cap W_6{}^\perp = 0$, a contradiction. For $1 \neq t \in T^1$, we have $xt = x + l$ with $l \in L_1$, $l \notin \langle y \rangle$, since $x \notin y^\perp$. Hence $\langle x \rangle t$ is a singular point with an x_0-component. By (4.1) the claim follows. ∎

4.4. *E is transitive on the doubly singular lines of W.*

Proof. Let $L_1 = P_1 \oplus P_2$ be a doubly singular line and $e \in E$ with $P_1^e = \langle x_1 \rangle$ by (4.3). Then L_1^e is a doubly singular line through x_1, hence $L_1^e = \langle x_1, \lambda x_2' + \mu x_3' \rangle$ with $\lambda, \mu \in L$ not both 0. We choose $m \in M$ with $x_1 m = x_1$ and $x_2' m = \lambda x_2' + \mu x_3'$. Then $L_1^{em^{-1}} = \langle x_1, x_2' \rangle$. ∎

4.5. *We have $E = S \simeq G_2(L)$.*

Proof. Let L_1 be a doubly singular line and $e \in E$ with $L_1 = \langle x_1, x_2' \rangle^e$ by (4.4). Then $T_{L_1} = T^e_{\langle x_1, x_2' \rangle} \leq M^e \leq E$. ∎

4.6. *We have* $\dim V_0 = 7$ *and* $C_{V_0}(M) = \langle v_0 \rangle$ *with* $v_0 \in V_0$ *not singular.*

Proof. Recall $A^1 = (T_{\langle x_1', x_3 \rangle})^{a(1)}$ and let $T^1 = T_{\langle x_1', x_2 \rangle} \in M \cap \Sigma^1$. Then $[W, T^1] + [W, A^1] = \langle x_1', x_2, x_3 + x_0 + x_3' \rangle$ is three-dimensional singular. Hence $[V, T] + [V, A]$ is also three-dimensional singular as in [S, (7.2.1)] and $\dim([V, M] \cap [V, A]) \geq 1$. Using (4.5), this yields $6 = \dim[V, M] \leq \dim V_0 \leq 6 + 2 - 1 = 7$.

We assume $V_0 = [V, M]$. Let $\{v_1, v_2, v_3, v_1', v_2', v_3'\}$ be the basis of $[V, M]$ occurring in (3.2). Then $[V, T] = \langle v_1', v_2 \rangle$. Let $[V, A] \cap [V, T] = P = \langle \alpha v_1' + \beta v_2 \rangle$. Then $P \neq \langle v_2 \rangle$. Since otherwise $[V, D] \cap [V, A] \neq 0$ for $D^1 = T_{\langle x_2, x_3' \rangle}$. Hence $[D, A] = 1$, and also $[D^1, A^1] = 1$, a contradiction. Similarly, $P \neq \langle v_1' \rangle$.

Let $[V, A] = P \oplus \langle c_1 v_1 + c_2 v_2 + c_3 v_3 + c_1' v_1' + c_2' v_2' + c_3' v_3' \rangle$. Then $c_1 c_1' + c_2 c_2' + c_3 c_3' = 0$. Since $[V, A] \subseteq [V, T]^{\perp}$, we obtain $c_1 = 0$, $c_2' = 0$. Hence $c_3 c_3' = 0$. We first consider the case $c_3' = 0$. Let $[V, A] \cap v_1^{\perp} = \langle y \rangle$. Then $y = \lambda v_2 + \mu v_3$ with $\lambda, \mu \in K$. We have $\mu \neq 0$, since otherwise $v_2 \in [V, A] \cap [V, T] = P$, a contradiction. Hence $[V, A] + [V, T] = \langle v_1', v_2, v_3 \rangle$. Thus $[V, B] \subseteq [V, A] + [V, T] \subseteq [V, A]^{\perp}$, where $B^1 = T_{\langle x_1', x_3 \rangle}$. Hence $[B^1, A^1] = 1$, a contradiction. Similarly, the case $c_3 = 0$ leads to a contradiction.

Hence $\dim V_0 = 7$. Because of $V = [V, M] \perp C_V(M)$, we have $V_0 = [V, M] + C_{V_0}(M)$. This shows $C_{V_0}(M) = \langle v_0 \rangle$ with $v_0 \in V_0$. If v_0 is singular, then $V_0 / \langle v_0 \rangle$ is a 6^+-space on which $S \simeq G_2(L)$ acts by Siegel transvections. This is not possible by the previous part of the proof. ∎

5. THE ACTION OF $a(1), b(1)$ ON V_0

In this section, we construct a basis of V_0 such that for $a(1)$, $b(1)$, and all $m \in M$ the matrix with respect to this basis is obtained by applying the embedding of fields α to the matrix with respect to the basis $\{x_1, x_2', x_2, x_1', x_3, x_0, x_3'\}$ of W. Our starting point is the basis $\{v_1, v_2, v_3, v_1', v_2', v_3', v_0\}$ constructed in the proof of (3.2) and (4.6).

5.1. *The subspaces* $\langle v_1, v_2' \rangle$, $\langle v_1', v_2 \rangle$, *and* $\langle v_3, v_0, v_3' \rangle$ *are invariant under* X *of* (2.4).

Proof. Let $\mathrm{SL}_2(L) \simeq S_1 \leq M$, where S_1 acts naturally on $\langle x_1, x_2 \rangle$, dually on $\langle x_1', x_2' \rangle$ (with $\{x_1', x_2'\}$ the dual basis of $\{x_1, x_2\}$) and S_1 fixes x_3, x_3', x_0.

The action of S_1 on V_0 is known by (3.2). Since $[S_1 X] = 1$, we obtain that $[V, S_1] = \langle v_1, v_2, v_1', v_2' \rangle$ and $C_{V_0}(S_1) = \langle v_3, v_0, v_3' \rangle$ are invariant under X. Further, a matrix calculation shows that with respect to the basis

$\{v_1, v_2, v_1', v_2', v_3, v_0, v_3'\}$ the matrix of $x \in X$ is of the form

$$\left(\begin{array}{cc|cc|c} a & & & b & \\ & a & -b & & \\ \hline & c & d & & \\ -c & & & d & \\ \hline & & & & C \end{array}\right)$$

with coefficients $a, b, c, d \in K$ and a 3×3 matrix C. This yields the claim. ∎

5.2. *We have* $v_1 a(\lambda) \in \langle v_1 \rangle$ *and* $v_2' b(\lambda) \in \langle v_2' \rangle$ *for* $\lambda \in L$.

Proof. For $\lambda \in L$ and $T^1 = T_{\langle x_1, x_3' \rangle}$, we have $T^1 = (T^1)^{a(\lambda)}$. Hence $\langle v_1, v_3' \rangle = [V, T] = [V, T]a(\lambda) = \langle v_1 a(\lambda), v_3' a(\lambda) \rangle$. This yields $v_1 a(\lambda) \in \langle v_1 \rangle$ for $\lambda \in L$. Similarly, $v_2' b(\lambda) \in \langle v_2' \rangle$ for $\lambda \in L$, using $T_{\langle x_2', x_3 \rangle}$.

5.3. *With respect to* $\{v_1, v_2'\}$ *we have*

$$a(1) = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}, \qquad b(1) = \begin{pmatrix} 1 & x^{-1} \\ & 1 \end{pmatrix}$$

for some $0 \neq x \in K$.

Proof. For $0 \neq t \in L$, we denote by $h(t)$ the element of M with matrix

$$h(t) := \left(\begin{array}{cc|cc|ccc} t & & & & & & \\ & t^{-1} & & & & & \\ \hline & & t & & & & \\ & & & t^{-1} & & & \\ \hline & & & & t^{-2} & & \\ & & & & & 1 & \\ & & & & & & t^2 \end{array}\right)$$

with respect to the basis $\{x_1, x_2', x_2, x_1', x_3, x_0, x_3'\}$.

Then $a(c)^{h(t)} = a(t^2 c)$ for $c \in L$. Since $|L| \geq 4$, there exists a $0 \neq t \in L$ with $t^2 \neq 1$. Let $\lambda \in L$ with $\lambda(t^2 - 1) = 1$. Then $[a(\lambda), h(t)] = a(\lambda(t^2 - 1)) = a(1)$. Hence by (5.1), (5.2), and (3.2) the matrix of $a(1)$ is of the form

$$a(1) \sim \begin{pmatrix} a & \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} t^{\alpha} & \\ & t^{-\alpha} \end{pmatrix}^{-1} \begin{pmatrix} a & \\ c & d \end{pmatrix} \begin{pmatrix} t^{\alpha} & \\ & t^{-\alpha} \end{pmatrix} = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}$$

for some $x \in K$. Similarly, $b(1) = \left(\begin{smallmatrix} 1 & y \\ & 1 \end{smallmatrix}\right)$.

It remains to show, that $xy = 1$. For $\omega = a(-1)b(1)a(-1)$, we have $T^{\omega}_{\langle x_1, x_3' \rangle} = T_{\langle x_2', x_3 \rangle}$. Hence $\langle v_2', v_3 \rangle = \langle v_1 \omega, v_3' \omega \rangle$ and $v_1 \omega \in \langle v_2' \rangle$. On the other hand, a matrix calculation yields that the first row of the matrix of ω with the respect to $\{v_1, v_2'\}$ is $(1 - xy, y)$. Hence $xy = 1$.

5.4

Next, we make some suitable replacements.

(1) Replacing v_i' by $x^{-1}v_i'$ $(i = 1, 2, 3)$, we may assume that the matrices of $a(1), b(1)$ with respect to $\{v_1, v_2'\}$ are

$$a(1) \sim \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}, \qquad b(1) \sim \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}.$$

Further, this replacement does not affect the elements $m \in M$.

(2) The fundamental matrix of b with respect to the basis $\{v_1, v_2, v_3, v_1', v_2', v_3'\}$ is of the form $\begin{pmatrix} & \lambda I \\ \lambda I & \end{pmatrix}$ for some $\lambda \in K$ as in the proof of (3.2). Replacing Q by the proportional quadratic form $\lambda^{-1}Q$, we may assume that (v_i, v_i') is a hyperbolic pair $(i = 1, 2, 3)$.

(3) Let $Q(v_0) = r$, where $C_{V_0}(M) = \langle v_0 \rangle$ as in (4.6), and let $\overline{K}$ be the algebraic closure of K. We replace V by $\overline{K} \otimes V$ with Q extended to $\overline{K} \otimes V$ and choose $c \in \overline{K}$ with $c^2 = -r^{-1}$. Replacing v_0 by cv_0, we may assume $Q(v_0) = -1$. Later, we will show that $c \in K$, i.e., it was not necessary to pass to the algebraic closure.

5.5

The action of $a(1), b(1)$ on $\langle v_1, v_2', v_2, v_1' \rangle$ is already determined. Next, we describe the action on $\langle v_3, v_0, v_3' \rangle$.

As in (5.2), $\langle v_3' \rangle$ is invariant under $a(\lambda)$. Therefore, $C_{V_0}(S_1) \cap v_3'^{\perp} = \langle v_0, v_3' \rangle$ is also invariant under $a(\lambda)$. Hence the matrix of $a(\lambda)$ with respect to the basis $\{v_3, v_0, v_3'\}$ is of the form

$$a(\lambda) \sim \begin{pmatrix} a & b & c \\ & d & e \\ & & f \end{pmatrix} =: A.$$

Since $a(\lambda)$ preserves Q, we have $Q(v_3) = 0 = Q(v_3 a(1))$, hence $ac - b^2 = 0$. We abbreviate the fundamental matrix of the bilinear form associated to Q with J. The equation $AJA^t = J$ shows that the matrix of $a(\lambda)$ is of the form

$$a(\lambda) \sim \begin{pmatrix} a & b & a^{-1}b^2 \\ & d & 2a^{-1}bd \\ & & a^{-1} \end{pmatrix}.$$

Let $\lambda, t \in L$ with $a(1) = [a(\lambda), h(t)]$ as in the proof of (5.3). Then a matrix calculation shows that the matrix of $a(1)$ with respect to $\{v_3, v_0, v_3'\}$ is of the form

$$a(1) \sim \begin{pmatrix} 1 & y & y^2 \\ & 1 & 2y \\ & & 1 \end{pmatrix}.$$

Similarly, the matrix of $b(1)$ is of the form

$$b(1) \sim \begin{pmatrix} 1 & & \\ 2x & 1 & \\ x^2 & x & 1 \end{pmatrix}.$$

For $T^0 = T_{\langle x_1' + x_3', x_2 \rangle}$, we have $[V, T] = \langle v_1' + v_3', v_2 \rangle$. Let $A^0 := (T^0)^\omega = T_{\langle x_2 + x_3, -x_1' \rangle}$. Then $\langle v_2 + v_3, -v_1' \rangle = [V, A] = [V, T]\omega = \langle v_2 + v_3'\omega, -v_1' \rangle$, since the action of X on $\langle v_1, v_2', v_2, v_1' \rangle$ is already determined. This yields $v_3'\omega = v_3$.

We have $\omega = a(-1)b(1)a(-1)$. A matrix calculation shows that the last row of the matrix of ω is $(x^2, x(1 - xy), (xy - 1)^2)$. Hence $x^2 = 1$, $x(1 - xy) = 0$, and $(xy - 1)^2 = 0$, i.e., $x^2 = 1$ and $xy = 1$. If $x = 1$, then $x = y = 1$. If $x = -1$, then we replace v_0 by $-v_0$ and may thus assume that $x = y = 1$.

The present v_0 is of the form $\pm c v_0$, where v_0 is the original v_0 and c is an element of the algebraic closure of K. Hence with respect to the original v_0 the matrix of $a(1)$ is a matrix over K of the form

$$\begin{pmatrix} 1 & & \\ & \pm c^{-1} & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ & 1 & 2 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & \pm c & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & \pm c & 1 \\ & 1 & \pm 2c^{-1} \\ & & 1 \end{pmatrix}.$$

Thus $c \in K$.

6. THE PROOF OF THE MAIN THEOREM

6.1

Since $S = \langle M, a(1), b(1) \rangle$, we have shown that $\{v_1, v_2, v_3, v_1', v_2', v_3', v_0\}$ is a basis of V_0 over K such that the matrix of $g \in G$ with respect to this basis is obtained by applying the embedding of fields α to the matrix of $g\chi$ with respect to the basis $\{x_1, x_2, x_3, x_1', x_2', x_3', x_0\}$ of W.

The semilinear (with respect to α) mapping $\varphi\colon W \to V_0$ with $x_i \mapsto v_i$, $x_i' \mapsto v_i'$, $x_0 \mapsto v_0$ $(i = 1, 2, 3)$ satisfies $(w(g\chi))\varphi = (w\varphi)g$ for all $w \in W$,

$g \in G$. Hence the embedding of G in Y is induced by a semilinear mapping and $\chi: G \to S$ is an isomorphism. Further, φ is injective and $\langle W\varphi \rangle_K = V_0$. The quadratic form defined by $\tilde{B}(w\varphi) := B(w)^{\alpha}$ for $w \in W$ is proportional to Q.

6.2

We are left with the last part of the Main Theorem. Calculating dimensions shows that $V = V_0 + C_V(G)$ if and only if $\mathrm{Rad}(V_0) \subseteq \mathrm{Rad}(V)$. If $\mathrm{Rad}(V_0) \nsubseteq \mathrm{Rad}(V)$, then $V_0 + C_V(G)$ is a hyperplane of V. We choose an eight-dimensional subspace V_1 of V, which contains V_0, such that $\mathrm{Rad}(V_1) = 0$. Then $V = V_1 + C_V(G)$. The action of G on V_1 is uniquely determined by the action of G on V_0, since there is only one possibility to extend Siegel transvections on V_0 to Siegel transvections on V_1 by [S, (4.3.1)].

6.3. *Proof of Corollary* 1.5. We assume that $\mathrm{SL}(V)$ has a subgroup G generated by transvections satisfying hypothesis (G_2) of the Main Theorem. By (4.5) we may write $G_2(L) = \langle M, T \rangle$, where T is a long root subgroup and $M \simeq \mathrm{SL}_3(L)$ is generated by long root subgroups. Hence the commutator space [V, G] is at most four-dimensional and G is a subgroup of $\mathrm{SL}_4(K)$ such that long root elements act as transvections. Because of $\mathrm{PSL}_4(K) \simeq \mathrm{P}\Omega_6^+(+)K$, we may apply the Klein correspondence and obtain G as subgroup of a six-dimensional orthogonal group such that long root elements act as Siegel transvections. This is a contradiction to the Main Theorem, (1.2)(a).

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