Cancellation of Azumaya Algebras

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1. Let \( R \) be a commutative ring with 1 and \( \text{Az}(R) \) the category of Azumaya algebras over \( R \). It was proved in [4] that if \( R \) is semilocal then, for \( X, Y, A \in \text{Obj } \text{Az}(R) \) the "cancellation law" \( X \otimes A \cong X \otimes Y \Rightarrow A \cong Y \) holds. In this note we prove a theorem which gives a set of equivalent conditions for cancellation over any ring. We derive, as corollaries, the above result, a theorem of Knus and the "cancellation law" for \( \text{Az}(k[x]) \) where \( k \) is a perfect field. We show (Proposition 2) that cancellation does not always hold over polynomial rings in two variables. To show this, we first prove that if \( D \) is any noncommutative division ring, there exist nonfree projective ideals in \( D[x, y] \). This result seems to be of some independent interest.

For unexplained terms we refer to Bass [2].

2. In this section, we denote by \( A, B, X, Y, Z, \ldots \) Azumaya algebras over a commutative ring \( R \). If \( A \subset Z \), \( A' \) will denote the commutant of \( A \) in \( Z \). All unadorned tensor products will be over \( R \). All homomorphisms will be \( R \)-homomorphisms.

**Theorem.** For any \( A \in \text{Obj } \text{Az}(R) \), the following conditions are equivalent.

1. For any integer \( n \geq 1 \) and any \( Y, M_n(Y) \subset M_n(A) \) implies \( Y \cong A \).
2. For any \( X, Y, X \otimes Y \cong X \otimes A \) implies \( Y \cong A \).
3. For any \( Z \supset A \) and \( Y \subset Z \), \( Y' \cong A' \) implies \( Y \cong A \).
4. For any integer \( n \geq 1 \) and any projective right \( A \)-module \( P \), \( P^n \cong A^n \) implies that \( P \) is an \( A \)-bimodule.

**Proof.** (1) \( \Rightarrow \) (2) Let \( X^0 \) denote the "opposite" of \( X \) and let \( X^0 \otimes X \cong \text{End}_R P \). We have \( X \otimes Y \cong X \otimes A \), hence \( X^0 \otimes X \otimes Y \cong X^0 \otimes X \otimes A \),

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i.e. $\text{End}(P) \otimes Y \cong (\text{End}(P) \otimes A)$. By [2, p. 476], there exists an $R$-projective module $Q$ such that $Q \otimes P \cong R^n$ for some $n \geq 1$. Tensoring with $\text{End}(Q)$, we get from the above isomorphism that $M_n(Y) \cong M_n(A)$. Now (1) implies $Y \cong A$.

(2) $\Rightarrow$ (3). We have $Z = A \otimes A' = Y \otimes Y'$. Since $Y' \cong A'$, (2) implies $Y \cong A$.

(3) $\Rightarrow$ (1). Let $\psi : M_n(A) \cong M_n(Y)$. The commutant of $A$ in $M_n(A)$ is $M_n(R)$ and hence the commutant of $\psi(A)$ in $M_n(Y)$ is isomorphic to $M_n(R)$. On the other hand the commutant of $Y$ in $M_n(Y)$ is $M_n(R)$. Now (3) implies $A \cong Y$.

(1) $\Rightarrow$ (4). Let $P^n \cong A^n$. We then have $\text{End}_A P^n \cong M_n(A)$. It is easy to see that $\text{End}_A P^n \cong M_n(\text{End}_A P)$. Thus, by (i) we have $\text{End}_A P \cong A$. It follows from this isomorphism that $P$ can be regarded as a left $A$-module and clearly this makes $P$ into an $A$-bimodule.

(4) $\Rightarrow$ (1). Let $\varphi : M_n(Y) \cong M_n(A)$ be an isomorphism. Let $e_{ij}$, $1 \leq i, j \leq n$ denote the canonical matrix units of $M_n(Y)$. Let $\varphi(e_{ij}) = f_{ij}$. Then $(f_{ij})$ is a system of matrix units of $M_n(A)$ and we have a direct-sum decomposition

$$ A^n = \prod_{1 \leq i \leq n} f_{ii}(A^n), $$

into projective right $A$-modules. We assert that all the summands are $A$-isomorphic. In fact multiplication by $f_{ij}$ gives an isomorphism of $f_{ii}(A^n)$ onto $f_{jj}(A^n)$, the inverse being multiplication by $f_{ij}$. Let $P = f_{11}(A^n)$. We have $P^n \cong A^n$. Now (4) implies that $P$ is an $A$-bimodule. Thus the map which sends any element of $A$ into left-multiplication in $P$ by that element gives a $R$-homomorphism of $A$ into $\text{End}_A P$. Since $A$ is an Azumaya algebra, this map is a monomorphism. On the other hand, from $P^n \cong A^n$, we have $M_n(\text{End}_A P) \cong M_n(A)$ and computing the ranks over $R$, we get $n^2 \text{rank } \text{End}_A P = n^2 \text{rank } A$ i.e. $\text{rank } \text{End}_A P = \text{rank } A$. Thus $A \cong \text{End}_A P$.

We now show that $\text{End}_A P \cong Y$. For $\alpha \in \text{End}_A P$, the element

$$ \bar{\alpha} = \sum_{1 \leq i, j \leq n} f_{1i} f_{ij}, $$

clearly commutes with $f_{ij}$, $1 \leq i, j \leq n$ and $\varphi^{-1}(\bar{\alpha})$ is therefore an element of the commutant of $M_n(R)$ in $M_n(Y)$, i.e. $\varphi^{-1}(\bar{\alpha}) \in Y$. We thus have a map $\theta : \text{End}_A P \to Y$ which is easily seen to be an $R$-algebra homomorphism. It is in fact an isomorphism, the inverse map $Y \to \text{End}_A P$ being given by $y \mapsto \varphi(y) | P$. This proves (4) $\Rightarrow$ (1).
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COROLLARY 1 ([4, Proposition 3.2]). If $R$ is a semilocal ring, any Azumaya algebra $A$ over $R$ satisfies the conditions (1)-(4) of the Theorem.

Proof. We check (4). Let $P^n \cong A^n$. Since $R$ is semilocal, $A/\text{rad } A$ is an Artinian ring and the isomorphism $P^n \cong A^n$ implies $(P/P \text{rad } A)^n \cong (A/\text{rad } A)^n$. By Krull–Schmidt theorem, it follows that $P/P \text{rad } A \cong A/\text{rad } A$. Hence $P \cong A$.

COROLLARY 2. Let $A = \text{End}_B Q$, where $B$ is an Azumaya algebra and $Q$ a projective $B$-module such that $f\text{-rank}_B Q > \text{dim Max}(R)$. If $K^0(B)$ has no torsion, then $A$ satisfies the conditions (1)-(4) of the theorem.

Proof. We check (4). Let $P^n \cong A^n$. Since $A = \text{End}_B Q$, the algebras $A$ and $B$ are Morita-equivalent [2, p. 67]. Under this equivalence, $A$ corresponds to the $B$-module $A \otimes_A Q = Q$ and $P$ corresponds to $P' = P \otimes_A Q$. We thus have a $B$-isomorphism $P^n \cong Q^n$. Since $K^0(B)$ is without torsion, this implies that there exists a finitely generated projective $B$-module $Q'$ such that $P' \oplus Q' \cong Q \oplus Q'$. Since $f\text{-rank}_B Q > \text{dim Max}(R)$, the “cancellation theorem” of Bass shows that $P' \cong Q$, i.e. $P \cong A$, by Morita-equivalence.

COROLLARY 3. Let $R = K[x]$, with $K$ a perfect field. Then any Azumaya algebra $A$ satisfies the conditions (1)-(4) of the theorem.

Proof. By a theorem of Auslander–Goldman [1], the algebra $A$ is Brauer-equivalent to $D \otimes_K K[x] = D[x]$ for some central division algebra $D$ over $K$. Hence, by [3, p. 109], $A$ is Morita-equivalent to $D[x]$. The equation $S^n \cong T^n$, where $S$ and $T$ are projective (and therefore free) modules over $D[x]$ clearly implies $S \cong T$. Hence the same is true for projective modules over $A$. Thus, the projective module $P$ of condition (4) is isomorphic to $A$.

Remark. If $R = K[x, y]$, where $K$ is a field of characteristic zero and if $\text{Br}(K) = (0)$, then, by a theorem of Auslander–Goldman [1], $\text{Br}(R) = (0)$. Hence, any Azumaya algebra $A$ over $R$ is Brauer- and therefore Morita-equivalent to $K[x, y]$. Since projective $R$-modules are free by a theorem of Seshadri [5], it follows that $A$ satisfies the conditions (1)-(4) of the theorem. However, if $\text{Br}(K) \neq (0)$, there always exist Azumaya algebras over $R$ which do not satisfy these conditions. To show this, we prove first the following Proposition which is of some independent interest.

PROPOSITION 1. Let $D$ be any noncommutative division ring. Then $A = D[x, y]$ contains a non-free projective ideal $P$ such that $A \oplus P \cong A$.

1 This result was kindly communicated to us by Knus.
Proof. Let \( a, b \in D \) with \( c = ab - ba \neq 0 \). Consider the exact sequence of right \( A \)-modules

\[
0 \to P \to A^2 \to A \to 0,
\]

where \( \varphi(\lambda, \mu) = (X + a)\lambda - (Y + b)\mu \).

The homomorphism \( \varphi \) is surjective since \( \varphi(Y + b, X + a) = c \) is invertible in \( A \). The module \( P \) is projective, since the sequence splits. We show that \( P \) is not free. For this, it is enough to show that \( P \) cannot be generated by a single element. It can be verified that the element \((\lambda_1, \mu_1)\) where

\[
\lambda_1 = b(b^{-1}ab - a)^{-1}a^{-1}x + a^{-1}ba(ba - ab)^{-1}y \\
+ (b^{-1}ab - a)^{-1}a^{-1}xy + (ba - ab)^{-1}y^2,
\]

\[
\mu_1 = b^{-1}ab(b^{-1}ab - a)^{-1}a^{-1}x + a(ba - ab)^{-1}y \\
+ (b^{-1}ab - a)^{-1}a^{-1}x^2 + (ba - ab)^{-1}xy
\]

and the element \((\lambda_2, \mu_2)\), where

\[
\lambda_2 = (ab^{-1} - b^{-1}a)^{-1} + (a^{-1}b^{-1}a(ab^{-1} - b^{-1}a)^{-1} \\
+ a^{-1}ba(ba - ab)^{-1}y + (ba - ab)^{-1}y^2,
\]

\[
\mu_2 = b^{-1}a(ab^{-1} - b^{-1}a)^{-1} + b^{-1}(ab^{-1} - b^{-1}a)^{-1}x \\
+ a(ba - ab)^{-1}y + (ba - ab)^{-1}xy
\]

are both in \( P \). If \( P \) were to be generated by a single element \((\lambda_0, \mu_0)\), then the above elements would be multiples of \((\lambda_0, \mu_0)\) and it is clear by comparing degrees that \( \lambda_0 \) must be a quadratic polynomial in \( x, y \). On the other hand, it is easily seen that \( P \) cannot contain any element \((\lambda, \mu)\) with \( \lambda \) linear in \( x, y \). Thus \( \lambda_0 \) must be of degree 2. Then, if \((\lambda_i, \mu_i) = s_i(\lambda_0, \mu_0), i = 1, 2\), the elements \( s_i \) must be non-zero elements of \( D \). Now \( \lambda_1 = s_1\lambda_0 \), implies that \( \lambda_0 \) has no constant term. On the other hand \( \lambda_2 = s_2\lambda_0 \) implies that \( \lambda_0 \) has a non-zero constant term. This is a contradiction, which proves that \( P \) is not free.

Since, for any \((\lambda, \mu) \in P\), we have \( \lambda = 0 \Rightarrow (\lambda, \mu) = 0 \), it follows that the projection mapping \( P \to A \) given by \((\lambda, \mu) \mapsto \lambda\) is an \( A \)-isomorphism of \( P \) on to a right ideal of \( A \).

We use Proposition 1 to prove the following

**Proposition 2.** Let \( D \) be a noncommutative central division algebra over a field \( K \). Then the Azumaya algebra \( A = D[x, y] \) does not satisfy the condition (1) of the theorem.
Proof. Let $P$ be the projective $A$-module of Proposition 1. From $P \oplus A \cong A^3$, we get $P^3 \oplus A^3 \cong A^3 \oplus A^3$. Since $f$-rank $A^3 > 2 = \dim \operatorname{Max}(K[x,y])$, it follows from the cancellation theorem of Bass [2, p. 184], that $P^3$ is isomorphic to $A^3$. Hence $M_3(\operatorname{End}_A P) \cong M_3(A)$. We assert that $B = \operatorname{End}_A P \cong A$. Since $P^3 \cong A^3$, $P$ is faithfully projective and we have that the mappings

$$f : P \otimes_A \operatorname{Hom}_A(P, A) \to B$$

and

$$g : \operatorname{Hom}_A(P, A) \otimes_B P \to A$$

defined respectively by $f(p \otimes p')(q) = p \cdot p'(q)$ and $g(p' \otimes p) = p'(p)$ are isomorphisms of $B$- and $A$-bimodules respectively [2, p. 68]. Suppose now $B \cong A$. This means, in view of the above isomorphisms, that $P$ is an invertible $A$-bimodule. Since by [3, p. 108], $\operatorname{Pic} A \cong \operatorname{Pic}(K[x,y]) = (0)$, it follows that $P$ is free, which is a contradiction. This proves the Proposition.

References