The Gaussian Toeplitz Matrix

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ABSTRACT

An analytical expression for the LL^T decomposition for the Gaussian Toeplitz matrix with elements $T_{ij} = [1/(2\pi)^{1/2}\sigma]\exp[-(i-j)^2/2\sigma^2]$ is derived. An exact expression for the determinant and bounds on the eigenvalues follows. An analytical expression for the inverse T^{-1} is also derived.

INTRODUCTION

Toeplitz matrices arise in the study of a number of problems in engineering and mathematics [1]. The Gaussian Toeplitz matrix is of interest especially in the context of signal and image processing, the study of heat kernels in relation to the diffusion equation, etc. We consider Gaussian convolution

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defined as follows. Given an input function $f(x)$ defined for $x \in \mathbb{R}^n$, the output function $h(x)$ is obtained from

$$h(x) = \int_{\mathbb{R}^n} k(x-y,t) f(y)\, dy, \tag{1.1}$$

where

$$k(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t} \right). \tag{1.2}$$

In the context of the heat equation, the function $h(x)$ is the solution obtained by propagating $f(x)$ over a time period t. It is of great interest to ask the following: given the solution $h(x)$ at a later time t, is there a kernel k^{-1}, inverse to k, such that the solution at the earlier time $f(x)$ can be obtained? It is known [2] that unless suitable growth restrictions are placed on $f(x)$ for large x, an inverse kernel in general does not exist.

In the context of image processing $f(x)$ can be taken to be the original object and $h(x)$ the image. The kernel $k(x-y;t)$ describes the distortion or blur [3] produced by the instrument due to its finite resolution. In keeping with the experimental setup where the object function is discretized into pixels, we can discretize Equation (1.1) as follows (to keep the notation simple, we shall consider only one space dimension x):

$$h(i) = \sum_{j=1}^{N} T_{ij} f(j), \tag{1.3}$$

where

$$T_{ij} = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left[-\frac{(i-j)^2}{2\sigma^2} \right]. \tag{1.4}$$

We shall consider the solution of Equation (1.3), namely, finding the inverse of the matrix T for a finite but arbitrary integer N. We shall derive an analytical formula for the inverse of T. The question of the limiting behavior when N tends to infinity and the relation of the discrete problem (1.3) in the limit $N \to \infty$ to the continuum problem (1.1) are also of interest. As discussed in [2, 3], the continuum problem is ill conditioned. In general we do not expect the discrete solution $f(i)$ to converge to a continuum solution for arbitrary $h(i)$. We shall see in Section 3 that the condition number of the matrix T is large.

In the present paper we shall give the following results for T_{ij} as given by Equation (1.4):

(a) It is proved that the matrix T is positive definite for all values of σ and for arbitrary dimension N.

(b) It is shown that T admits a Cholesky decomposition

$$T = LL^T.$$

An analytic expression for L is written down.

(c) An analytic expression for $\det T$ is obtained, from which bounds on eigenvalues are derived.

(d) An explicit analytical expression for T^{-1} is derived.

2. THE CHOLESKY DECOMPOSITION

Let us define

$$a = \exp\left(\frac{-1}{2\sigma^2}\right). \tag{2.1}$$

Then clearly

$$T_{ij} = \frac{1}{\sqrt{2\pi}\,\sigma} a^{(1-j)^2}. \tag{2.2}$$

Ignoring the trivial overall normalization factor, we shall consider the $k \times k$ matrix

$$T(k) = \begin{pmatrix} 1 & a & a^4 & a^9 & \cdots & a^{(k-1)^2} \\ a & 1 & a & a^4 & \cdots & a^{(k-2)^2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a^{(k-1)^2} & a^{(k-2)^2} & a^{(k-3)^2} & a^{(k-4)^2} & \cdots & 1 \end{pmatrix}. \tag{2.3}$$

LEMMA.

$$T(k) = L(k)L(k)^T, \tag{2.4}$$

where the $(r+1)$*st row of the lower triangular matrix* $L(k)$ *is given by*

$$L_{r+1,1}(k) = a^{r^2},$$

$$L_{r+1,s+1}(k) = \frac{a^{(r-s)^2}(1-a^{2r})(1-a^{2(r-1)})\cdots(1-a^{2(r-s+1)})}{(1-a^2)^{1/2}(1-a^4)^{1/2}\cdots(1-a^{2s})^{1/2}},$$

$$r \geqslant s \geqslant 1. \tag{2.5}$$

It is a simple matter to check that $L(k)$ as given by Equation (2.5) satisfies Equation (2.4) for $s = 0, 1, 2, \ldots$ and for arbitrary r. The proof for the case when s is also arbitrary is given in the Appendix.

3. DETERMINANTS AND BOUNDS ON EIGENVALUES

We have

$$\begin{aligned} \det T(k) &= [\det L(k)]^2 \\ &= (1-a^2)^{k-1}(1-a^4)^{k-2}\cdots(1-a^{2(k-2)})^2(1-a^{2(k-1)}) \\ &> 0. \end{aligned} \tag{3.1}$$

Also, for every vector y the scalar product of y with $T(k)y$ is

$$\begin{aligned} (y^T, T(k)y) &= ((yL(k))^T, L(k)y) \\ &\geqslant 0. \end{aligned} \tag{3.2}$$

Since $\det T(k)$ is nonzero, $T(k)$ is positive definite.

Consider now the eigenvalues $\lambda_i^{(k)}$ of $T(k)$, and arrange them in sequence of decreasing values

$$\lambda_1^{(k)}, \lambda_2^{(k)}, \dots, \lambda_k^{(k)}.$$

From the interlacing property of eigenvalues

$$\lambda_1^{(k+1)} \geqslant \lambda_1^{(k)} \geqslant \lambda_2^{(k+1)} \geqslant \cdots \geqslant \lambda_k^{(k+1)} \geqslant \lambda_k^{(k)} \geqslant \lambda_{k+1}^{(k+1)} \tag{3.3}$$

and

$$\frac{\det T(k+1)}{\det T(k)} = \frac{\pi_{j=1}^{k+1} \lambda_j^{(k+1)}}{\pi_{j=1}^{k} \lambda_j^{(k)}}$$

$$= (1-a^2)(1-a^4) \cdots (1-a^{2k}),$$

we have an upper bound on the least eigenvalue:

$$\lambda_{k+1}^{(k+1)} \leqslant (1-a^2)(1-a^4) \cdots (1-a^{2k}). \tag{3.4}$$

Now for $T(k=2)$ the eigenvalues are $1-a$ and $1+a$, so trivially

$$\lambda_1^{(k+1)} \geqslant 1+a. \tag{3.5}$$

The condition number of a matrix is defined as the ratio of the largest to the smallest eigenvalues. We then have the bound

$$[\text{condition number of } T(k)] \geqslant \frac{1+a}{(1-a^2)(1-a^4) \cdots (1-a^{2(k-1)})}. \tag{3.6}$$

We note in passing that in most applications in signal processing a is very close to unity, so the condition number is very large.

A lower bound on the minimum eigenvalue can be found as follows. We have

$$\frac{\lambda_{k+1}^{(k+1)}}{\lambda_k^{(k)}} \cdots \frac{\lambda_i^{(k+1)}}{\lambda_{i-1}^{(k)}} \cdots \frac{\lambda_2^{(k+1)}}{\lambda_1^{(k)}} \lambda_1^{(k+1)} = (1-a^2)(1-a^4) \cdots (1-a^{2k}). \tag{3.7}$$

From the interlacing property, $\lambda_i^{(k+1)} \leqslant \lambda_{i-1}^{(k)}$. Further,

$$\operatorname{Trace} T(k+1) = \sum_{j=1}^{k+1} \lambda_j^{(k+1)}$$

$$= k+1, \tag{3.8}$$

so we have $\lambda_1^{(k+1)} \leqslant (k+1)$. Combining the above observations, we have

$$\lambda_{k+1}^{(k+1)} \geqslant \frac{(1-a^2)\cdots(1-a^{2k})}{k+1} \lambda_k^{(k)}, \tag{3.9}$$

which leads on iteration to

$$\lambda_{k+1}^{(k+1)} \geqslant \frac{(1-a^2)^k (1-a^4)^{k-1} \cdots (1-a^{2k})}{(k+1)!} \tag{3.10}$$

$$\lambda_{k+1}^{(k+1)} \geqslant \frac{\det T(k+1)}{(k+1)!}. \tag{3.11}$$

Finally we get an upper bound on the condition number:

$$[\text{condition number of } T(k)] \leqslant \frac{k \cdot k!}{(1-a^2)^{k-1} \cdots (1-a^{2(k-1)})}. \tag{3.12}$$

4. THE INVERSE $[T(k)]^{-1}$

We shall now show

$$[T(k)]^{-1} = \Lambda^T E^{-1} \Lambda, \tag{4.1}$$

where the lower triangular matrix Λ is given by

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ -a & 1 & 0 & 0 & 0 & \cdots \\ a^2 & \dfrac{-a(1-a^4)}{1-a^2} & 1 & 0 & 0 & \cdots \\ -a^3 & \dfrac{a^2(1-a^6)}{1-a^2} & \dfrac{-a(1-a^6)(1-a^4)}{(1-a^2)(1-a^4)} & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{4.2}$$

and E is the diagonal matrix

$$E = \begin{pmatrix} 1 & & & \mathbf{0} \\ & 1-a^2 & & \\ & & (1-a^2)(1-a^4) & \\ \mathbf{0} & & & \ddots \end{pmatrix}. \tag{4.3}$$

The general elements of Λ and E can be written as

$$\Lambda_{r+1,1} = (-1)^r a^r$$

$$\Lambda_{r+1,s+1} = (-1)^{r+s} a^{r-s} \frac{(1-a^{2r}) \cdots (1-a^{2(r-s+1)})}{(1-a^2) \cdots (1-a^{2s})}, \qquad 1 \leqslant s \leqslant r,$$

$$\Lambda_{r+1,s+1} = 0, \qquad s > r, \tag{4.4}$$

and

$$E_{11} = 1,$$

$$E_{ii} = (1-a^2)(1-a^4) \cdots (1-a^{2(i-1)}), \qquad i > 1. \tag{4.5}$$

To prove Equation (4.1) let us return to the Cholesky decomposition (2.4), (2.5) and write it in the form

$$T = FEF^T, \tag{4.6}$$

where E is the same matrix as in Equation (4.5) above, and F is given by

$$F_{r+1,1} = a^{r^2}, \tag{4.7}$$

$$F_{r+1,s+1} = a^{(r-s)^2} \frac{(1-a^{2r}) \cdots (1-a^{2(r-s+1)})}{(1-a^2) \cdots (1-a^{2s})}, \qquad r \geqslant s. \tag{4.8}$$

It is then sufficient to show that

$$\Lambda F = 1 \tag{4.9}$$

to prove that T^{-1} is as given by Equation (4.1). Since both Λ and F are left triangular matrices, it follows that

$$(\Lambda F)_{i+1,i+1} = \Lambda_{i+1,i+1} F_{i+1,i+1}$$
$$= 1.$$

Let us write

$$(\Lambda F)_{i+1,k+1} = \sum_{l=0}^{i-k} \Lambda_{i+1,l+k+1} F_{l+k+1,k+1}. \tag{4.10}$$

Consider the last two terms in the summation, corresponding to $l = 0$ and $l = 1$. We have

$$\Lambda_{i+1,k+1} + \Lambda_{i+1,k+2} F_{k+2,k+1}$$
$$= (-1)^{i-k} a^{i-k} R_{i,k} + (-1)^{i-k-1} a^{i-k-1} R_{i,k+1} a R_{k+1,k}, \tag{4.11}$$

where we have introduced the notation

$$R_{i,k} = \frac{(1-a^{2i})(1-a^{2(i-1)}) \cdots (1-a^{2(i-k+1)})}{(1-a^2)(1-a^4) \cdots (1-a^{2k})}. \tag{4.12}$$

The $R_{i,k}$ satisfy the recursion relations

$$R_{i+1,k} = \frac{1-a^{2(i+1)}}{1-a^{2(i-k+1)}} R_{i,k}, \tag{4.13}$$

$$R_{i,k+1} = \frac{1-a^{2(i-k)}}{1-a^{2(k+1)}} R_{i,k}. \tag{4.14}$$

It is then easy to see that Equation (4.11) can be written as

$$\begin{aligned} &\Lambda_{i+1,k+1} + \Lambda_{i+1,k+2} F_{k+2,i+1} \\ &= (-1)^{i-k} a^{i-k} R_{i,k}\left[1 - \frac{1-a^{2(i-k)}}{1-a^2}\right] \\ &= (-1)^{i-k+1} a^{(i-k)+2} R_{i,k} \frac{1-a^{2(i-k-1)}}{1-a^2}. \end{aligned} \tag{4.15}$$

Repeating the procedure with the $l = 2$ term in Equation (4.10), we get

$$\begin{aligned} &\Lambda_{i+1,k+1} + \Lambda_{i+1,k+2} F_{k+2,k+1} + \Lambda_{i+1,k+3} F_{k+3,i+1} \\ &= (-1)^{(i-k)+2} a^{(i-k)+2+4} R_{i,k} \frac{1-a^{2(i-k-1)}}{1-a^2} \frac{1-a^{2(i-k-2)}}{1-a^4}. \end{aligned} \tag{4.16}$$

Now a general term in the right hand side of Equation (4.10) is given by

$$\Lambda_{i+1,j+k+1} F_{j+k+1,k+1} = (-1)^{i-j-k} a^{i-j-k} R_{i,j+k} a^{j^2} R_{j+k,k}. \tag{4.17}$$

Using the equations (4.12)–(4.14) we can write

$$R_{i,j+k} R_{j+k,k} = R_{ik} \frac{(1-a^{2(i-k-j+1)}) \cdots (1-a^{2(i-k-1)})(1-a^{2(i-k)})}{(1-a^2)(1-a^4) \cdots (1-a^{2j})}. \tag{4.18}$$

Further,

$$(-1)^{i-j-k} a^{i-j-k} a^{j^2} = (-1)^{i-k+j} a^{i-k} a^{2+4+\cdots+2(j-1)}. \tag{4.19}$$

We can therefore write for the right hand side of Equation (4.10), summing up to $l = j$,

$$\sum_{l=0}^{j} \Lambda_{i+1,l+k+1} F_{l+k+1,k+1} = (-1)^{i-k+j} a^{(i-k)+2+4+\cdots+2j}$$

$$\times R_{i,k} \frac{1-a^{2(i-k-1)}}{1-a^2} \cdots \frac{1-a^{2(i-k-j)}}{1-a^{2j}}.$$

Setting $j = i - k - 1$, we find

$$\sum_{l=0}^{i-k-1} \Lambda_{i+1,l+k+1} F_{l+k+1,k+1} = -a^{(i-k)+2+4+\cdots+2(i-k-1)} R_{i,k}$$

$$= -a^{(i-k)^2} R_{i,k}$$

$$= -F_{i+1,k+1}.$$

5. CONCLUSION

The exact analytical expression that we have derived obviously has many applications. We have used it to study image processing in the case where the convolution kernel is Gaussian. For situations approximated by a Gaussian, we can use T^{-1} for preconditioning, which can be of considerable computational value. Finally, our solution will be useful for numerically studying problems of the heat equation type.

APPENDIX

Proof of the Cholesky decomposition (2.4). We need to prove (taking $r \geqslant s$)

$$\sum_{j=1}^{s+1} L_{r+1,j}(k) L_{s+1,j}(k) = a^{(r-s)^2}. \tag{A.1}$$

We have for $s = 0$ and 1 respectively

$$L_{r+1,1}(k)L_{1,1}(k) = a^{r^2},$$

$$L_{r+1,1}(k)L_{2,1}(k) + L_{r+1,2}(k)L_{2,2}(k) = a^{r^2}\cdot a + a^{(r-1)^2}(1-a^{2r})$$
$$= a^{(r-1)^2}.$$

We shall work out the case $s = 2$ in a manner whose generalization to arbitrary s becomes evident. Let

$$\sum_{j=1}^{3} L_{r+1,1}(k)L_{3,j}(k) = A_1^{(0)} + A_2^{(0)} + A_3^{(0)} \tag{A.2}$$

with

$$A_1^{(0)} = a^{r^2}\cdot a^4, \tag{A.3}$$

$$A_2^{(0)} = a^{(r-1)^2}(1-a^{2r})a\frac{1-a^4}{1-a^2}, \tag{A.4}$$

and

$$A_3^{(0)} = a^{(r-2)^2}(1-a^{2r})(1-a^{2(r-1)}). \tag{A.5}$$

The sum in Equation (A.2) of course adds up to $a^{(r-2)^2}$. It is to be noted that the term in $a^{(r-2)^2}$ occurs only in $A_3^{(0)}$; it has coefficient unity. Further, all other terms in $A_3^{(0)}$ as well as in $A_2^{(0)}$ and $A_1^{(0)}$ have higher powers of a, and they mutually cancel on addition. Specifically open up the last bracket in (A.5) and write

$$A_3^{(0)} = a^{(r-2)^2}(1-a^{2r}) - a^{(r-2)^2}(1-a^{2r})a^{2(r-1)} \tag{A.6}$$

$$= A_3^{(1)} + \tilde{A}_3^{(0)}. \tag{A.7}$$

Combining $\tilde{A}_3^{(0)}$ with $A_2^{(0)}$ given by Equation (A.4), we have

$$A_2^{(0)} + \tilde{A}_3^{(0)} = a^{(r-1)^2+3} - a^{r^2+4} \tag{A.8}$$

$$= A_2^{(1)} + \tilde{A}_2^{(0)}. \tag{A.9}$$

Combining $\tilde{A}_2^{(0)}$ with the term $A_1^{(0)}$ given by Equation (A.3), we have

$$A_1^{(0)} + \tilde{A}_2^{(0)} = 0, \tag{A.10}$$

or

$$A_1^{(0)} + A_2^{(0)} + A_3^{(0)} = A_2^{(1)} + A_3^{(1)}. \tag{A.11}$$

Now

$$\begin{aligned} A_3^{(1)} &= a^{(r-2)^2} - a^{(r-2)^2+2r} \\ &= A_3^{(2)} + \tilde{A}_3^{(1)} \end{aligned} \tag{A.12}$$

and

$$A_2^{(1)} + \tilde{A}_3^{(1)} = 0, \tag{A.13}$$

so that

$$A_2^{(1)} + A_3^{(1)} = A_3^{(2)} = a^{(r-2)^2},$$

the desired result.

For arbitrary s we have

$$\begin{aligned} &\sum_{j=1}^{s+1} L_{r+1,j}(k) L_{s+1,j}(k) \\ &\quad = A_1^{(0)} + A_2^{(0)} + \cdots + A_{s+1}^{(0)} \end{aligned} \tag{A.14}$$

$$\begin{aligned} &\quad = a^{r^2} a^{s^2} + a^{(r-1)^2} a^{(s-1)^2} \frac{(1-a^{2r})(1-a^{2s})}{1-a^2} + \cdots \\ &\qquad + a^{(r-s+1)^2} a(1-a^{2r}) \cdots (1-a^{2(r-s+2)}) \frac{(1-a^{2s}) \cdots (1-a^4)}{(1-a^2) \cdots (1-a^{2(s-1)})} \\ &\qquad + a^{(r-s)^2}(1-a^{2r}) \cdots (1-a^{2(r-s+1)}). \end{aligned} \tag{A.15}$$

We anticipate that these terms add up to $a^{(r-s)^2}$, which occurs only in the last term $A_{s+1}^{(0)}$. We write

$$A_{s+1}^{(0)} = A_{s+1}^{(1)} + \tilde{A}_{s+1}^{(0)},$$

$$A_{s+1}^{(1)} = a^{(r-s)^2}(1-a^{2r}) \cdots (1-a^{2(r-s+2)}),$$

$$\tilde{A}_{s+1}^{(0)} = -a^{(r-s+1)^2+1}(1-a^{2r}) \cdots (1-a^{2(r-s+2)}),$$

Combining with $A_s^{(0)}$, we have

$$\tilde{A}_{s+1}^{(0)} + A_s^{(0)} = A_s^{(1)} + \tilde{A}_s^{(0)},$$

where

$$A_s^{(1)} = a^{(r-s+1)^2+3}(1-a^{2r}) \cdots (1-a^{2(r-s+3)}) \frac{1-a^{2(s-1)}}{1-a^2}$$

$$\tilde{A}_s^{(0)} = -a^{(r-s+2)^2+4}(1-a^{2r}) \cdots (1-a^{2(r-s+3)}) \frac{1-a^{2(s-1)}}{1-a^2}.$$

After j steps we obtain

$$A_{s+1-j}^{(1)} = a^{(r-s+j)^2+j^2+2j}(1-a^{2r}) \cdots (1-a^{2(r-s+j+2)}) \times \frac{1-a^{2(s-1)}}{1-a^2} \cdots \frac{1-a^{2(s-j)}}{1-a^{2j}} \tag{A.16}$$

and

$$\tilde{A}_{s+1-j}^{(0)} = -a^{(r-s+j+1)^2+(j+1)^2}(1-a^{2r}) \cdots (1-a^{2(r-s+j+2)}) \times \frac{1-a^{2(s-1)}}{1-a^2} \cdots \frac{1-a^{2(s-j)}}{1-a^{2j}} \tag{A.17}$$

so that finally

$$\begin{gathered} A_1^{(0)} + \tilde{A}_2^{(0)} = 0, \\ A_1^{(0)} + A_2^{(0)} + \cdots + A_s^{(0)} + A_{s+1}^{(0)} = A_2^{(1)} + \cdots + A_s^{(1)} + A_{s+1}^{(1)}. \end{gathered} \tag{A.18}$$

The sum on the right hand side of Equation (A.18) can again be reduced beginning with the last term. As in the previous step, at the intermediate stage we have

$$A^{(2)}_{s+1-j} = a^{(r-s+j)^2+j^2+4j}(1-a^{2r}) \cdots (1-a^{2(r-s+j+3)}) \times \frac{1-a^{2(s-2)}}{1-a^2} \cdots \frac{1-a^{2(s-1-j)}}{1-a^{2j}}, \tag{A.19}$$

$$\tilde{A}^{(1)}_{s+1-j} = -a^{(r-s+j+1)^2+(j+1)^2+2(j+1)}(1-a^{2r}) \cdots (1-a^{2(r-s+j+3)}) \times \frac{1-a^{2(s-2)}}{1-a^2} \cdots \frac{1-a^{2(s+1-j)}}{1-a^{2j}}. \tag{A.20}$$

Comparing Equation (A.16) and Equation (A.20), we get

$$A^{(1)}_{s-j} = -\tilde{A}^{(1)}_{s+1-j} \frac{1-a^{2(s-1)}}{1-a^{2(j+1)}}. \tag{A.21}$$

Therefore

$$A^{(1)}_2 + \tilde{A}^{(1)}_3 = 0.$$

This procedure can be iterated. At the mth iteration the analog of Equation (A.11) is

$$A^{(m-1)}_{s-j} = -\tilde{A}^{(m-1)}_{s+1-j} \frac{1-a^{2(s-m+1)}}{1-a^{2(j+1)}}, \tag{A.22}$$

Setting $j = s - m$, we find

$$A^{(m-1)}_m + \tilde{A}^{(m-1)}_{m+1} = 0. \tag{A.23}$$

After s iterations, it is easily seen that only the term $a^{(r-s)^2}$ from $A^{(0)}$ is left, so that

$$\begin{aligned} A_1^{(0)} + A_2^{(0)} + \cdots + A_{s+1}^{(0)} &= A_2^{(1)} + \cdots + A_{s+1}^{(1)} \\ &= A_3^{(2)} + \cdots + A_{s+1}^{(2)} \\ &= \cdots \\ &= a^{(r-s)^2}. \qquad \blacksquare \end{aligned} \tag{A.24}$$

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