Available online at www.sciencedirect.com
SCIENCE DIRECT•
An International Journal
computers & mathematics with applications
ELSEVIER Computers and Mathematics with Applications 51 (2006) 1453–1462
www.elsevier.com/locate/camwa

Integrability, Degenerate Centers, and Limit Cycles for a Class of Polynomial Differential Systems

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(Received May 2005; revised and accepted January 2006)

Abstract—We consider the class of polynomial differential equations $\dot{x} = P_n(x,y) + P_{n+1}(x,y) + P_{n+2}(x,y)$, $\dot{y} = Q_n(x,y) + Q_{n+1}(x,y) + Q_{n+2}(x,y)$, for $n \geq 1$ and where P_i and Q_i are homogeneous polynomials of degree i. These systems have a linearly zero singular point at the origin if $n \geq 2$. Inside this class, we identify a new subclass of Darboux integrable systems, and some of them having a degenerate center, i.e., a center with linear part identically zero. Moreover, under additional conditions such Darboux integrable systems can have at most one limit cycle. We provide the explicit expression of this limit cycle. © 2006 Elsevier Ltd. All rights reserved.

Keywords—Integrability, Algebraic limit cycle, Linearly zero singular point, Degenerate center, Polynomial vector field, Polynomial differential system.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Probably the three main open problems in the qualitative theory differential systems in $\mathbb{R}^2$ are the determination of the number of the limit cycles and their distribution in the plane (see, for instance, [1]); the distinction between a center and a focus, called the *center problem* (see, for instance, [2]); and the determination of their first integrals (see, for instance, [3]). This paper deals with these three problems for a class of polynomial differential systems. More explicitly,

The first author is partially supported by a DGICYT grant number MTM2005-06098-C02-02 and by a CICYT grant number 2005SGR 00550, and by DURSI of Government of Catalonia "Distinció de la Generalitat de Catalunya per a la promoció de la recerca universitària". The second author is partially supported by a DGICYT grant number MTM2005-06098-C02-01 and by a CICYT grant number 2005SGR 00550.

 Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
doi:10.1016/j.camwa.2006.01.005

we study the class of real planar polynomial differential systems of the form,

$$\begin{aligned} \dot{x} &= P_n(x,y) + P_{n+1}(x,y) + P_{n+2}(x,y), \\ \dot{y} &= Q_n(x,y) + Q_{n+1}(x,y) + Q_{n+2}(x,y), \end{aligned} \tag{1}$$

where P_i and Q_i are homogeneous polynomials of degree i.

Let $p \in \mathbb{R}^2$ be a singular point of a differential system in $\mathbb{R}^2$. We say that p is a *center* if there is a neighborhood U of p such that all the orbits of $U \setminus \{p\}$ are periodic, and we say that p is a *focus* if there is a neighborhood U of p such that all the orbits of $U \setminus \{p\}$ spiral either in forward or in backward time to p. A singular point p is called *linearly zero* if the singular point has zero linear part. A singular point p is a *monodromic* singular point of system (1) if there is no *characteristic orbit* associated to it, i.e., there is no orbit tending to the singular point with definite tangent at this point. When the vector field is analytic, a monodromic singular point p is either a center or a focus, see [4,5]. A singular point p is called *degenerate center* if the singular point is a center and it has zero linear part.

We say that an analytic differential system in the plane is *time-reversible* (with respect to an axis of symmetry through the origin) if after a rotation,

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

the system in the new variables (ξ,η) becomes invariant by a transformation of the form $(\xi,\eta,t) \mapsto (\xi,-\eta,-t)$. The phase portrait of this new system is symmetric with respect to the straight line $\xi = 0$.

The center problem for a nondegenerate singular point (i.e., distinguish when a singular point is either a center or a focus) has been partially solved, in the sense that there are several algorithms for deciding between a focus or a center, see [6,7]. Unfortunately, the implementation of this algorithm is very difficult due to the huge computations that it needs. In general, it does not exist an algorithm for the center problem of a linearly zero singular point, see, for instance, [2,7] and the references therein. The center problem for linearly zero singular points may be separated into two problems: the *monodromy problem*, to decide if the singular point is monodromic or not, and the *stability problem*, to decide when it is either a focus or a center. The monodromy conditions can be derived by an algorithmic method based in the blow-up technique, see, for instance, [2,6]. For the stability problem, some results are obtained in a series of papers, see [8] and the references inside. On the other hand, several authors have studied the center problem for particular subclasses of polynomial differential systems, see, for instance, [9,10]. In [11], sufficient conditions in order that the origin of system $\dot{x} = P_3(x,y) + P_4(x,y)$, $\dot{y} = Q_3(x,y) + Q_4(x,y)$ is a center are given.

For degenerate analytic centers, it is also known that, in general, they have no local analytic first integrals defined in its neighborhood, see, for instance, [3]. There are very few examples of degenerate analytic centers. Nemitskii and Stepanov in [12, page 122] give a real polynomial differential system which has a degenerate center, but the system has neither a local analytic first integral in its neighborhood, nor a formal one. In [13], Moussu gives another example of a real polynomial differential system having a degenerate center for which does not exist a local analytic first integral.

A *limit cycle* of system (1) is a periodic orbit isolated in the set of periodic orbits of system (1). Let W be the domain of definition of a C^1 vector field (P,Q), and let U be an open subset of W. A function $V : U \to \mathbb{R}$ that satisfies the linear partial differential equation,

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V, \tag{2}$$

is called an *inverse integrating factor* of the vector field (P,Q) on U. We note that $\{V = 0\}$ is formed by orbits the vector field (P,Q). This function V is very important because $R = 1/V$

defines on $U \setminus \{V = 0\}$ an integrating factor of system (1) (which allows to compute a first integral of the system on $U \setminus \{V = 0\}$) and $\{V = 0\}$ contains the limit cycles of system (1) which are in U, see [14].

A differential system is completely integrable if all solutions to well-posed initial or boundary value problems can be presented beginning with elementary functions, using finitely many algebraic operations and compositions of functions, and evaluating limits. Thus, this definition holds for the cases in which solutions can be constructed explicitly and generalizes the notion of integrability by quadratures or Liouville integrability, see [15–17]. A particular case of Liouville integrability is the notion of Darboux integrability. A function of the form $f_1^{\lambda_1} \dots f_p^{\lambda_p} \exp(h/g)$, where f_i, g and h are polynomials in $\mathbb{C}[x,y]$ and the λ_i are complex numbers, is called a *Darboux function*. System (1) is called *Darboux integrable* if the system has a first integral or an integrating factor which is a Darboux function (for a definition of a first integral and of an integrating factor, see for instance [3,18]). The problem of determining when a polynomial differential system (1) is Darboux integrable is, in general, open.

Inside the class of the differential systems (1), we will characterize a new subclass of Darboux integrable systems, and under an additional assumption over the inverse integrating factor we shall show that they have at most 1 limit cycle and this upper bound is reached. Moreover, inside this family, we identify new examples of degenerate centers which, in general, are neither Hamiltonian nor time-reversible.

In order to present our results we need some preliminary notation and results. Thus, in polar coordinates (r, θ), defined by

$$x = r\cos\theta, \qquad y = r\sin\theta, \tag{3}$$

system (1) becomes

$$\begin{aligned} \dot{r} &= f_{n+1}(\theta)r^n + f_{n+2}(\theta)r^{n+1} + f_{n+3}(\theta)r^{n+2}, \\ \dot{\theta} &= g_{n+1}(\theta)r^{n-1} + g_{n+2}(\theta)r^n + g_{n+3}(\theta)r^{n+1}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} f_i(\theta) &= \cos\theta P_{i-1}(\cos\theta, \sin\theta) + \sin\theta Q_{i-1}(\cos\theta, \sin\theta), \\ g_i(\theta) &= \cos\theta Q_{i-1}(\cos\theta, \sin\theta) - \sin\theta P_{i-1}(\cos\theta, \sin\theta). \end{aligned}$$

We remark that f_i and g_i are homogeneous trigonometric polynomials in the variables $\cos\theta$ and $\sin\theta$ having degree in the set $\{i, i-2, i-4, \dots\} \cap \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers. This is due to the fact that $f_i(\theta)$ can be of the form $(\cos^2\theta + \sin^2\theta)^s \bar{f}_{i-2s}$ with $\bar{f}_{i-2s}$ a trigonometric polynomial of degree $i - 2s \geq 0$. A similar situation occurs for $g_i(\theta)$.

If we impose $g_{n+2}(\theta) = g_{n+3}(\theta) = 0$ and $g_{n+1}(\theta)$ either > 0 or < 0 for all θ, then system (4) becomes the following Abel differential equation,

$$\frac{dr}{d\theta} = \frac{1}{g_{n+1}(\theta)} \left[f_{n+1}(\theta)\ r + f_{n+2}(\theta)\ r^2 + f_{n+3}(\theta)\ r^3 \right]. \tag{5}$$

These kind of differential equations appeared in the studies of Abel on the theory of elliptic functions. For more details on Abel differential equations, see [19–21].

We say that all polynomial differential systems (1) forms the class F if

(i) $g_{n+2}(\theta) = g_{n+3}(\theta) = 0$;
(ii) either $g_{n+1}(\theta) > 0$, or $g_{n+1}(\theta) < 0$, for all θ; and
(iii) the following equality holds,

$$g_{n+1}(\theta) \left(f'_{n+3}(\theta) f_{n+2}(\theta) - f_{n+3}(\theta) f'_{n+2}(\theta) \right) = a f^3_{n+2}(\theta) - f_{n+1}(\theta) f_{n+2}(\theta) f_{n+3}(\theta), \tag{6}$$

for some $a \in \mathbb{R}$, with $' = \frac{d}{d\theta}$. Here, $f^n(\theta)$ means $[f(\theta)]^n$.

Since $g_{n+1}(\theta)$ either > 0 or < 0 for all θ, it follows that the polynomial differential systems (1) in the class F must satisfy that $n+1$ is even.

We shall prove that all polynomial differential systems (1) in the class F are Darboux integrable. We have found the subclass F thanks to cases (a), (b), (c), and (d) of Abel differential equations studied in [19, pp. 24–25]. Using similar techniques in [22,23] are found new Darboux integrable systems for polynomial systems with a center or a focus at the origin.

Our main result is the following one.

THEOREM 1. *For polynomial differential systems (1) in the class F the following statements hold.*

(a) *If $f_{n+1}(\theta)f_{n+2}(\theta)f_{n+3}(\theta)$ is not identically zero, then the system is Darboux integrable with the first integral $\tilde{H}(x,y) = H(r,\theta)$ obtained from*

$$\frac{r\exp\left(-\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)\exp\left[-\frac{1}{\sqrt{4a-1}}\arctan\left[\frac{(1+2rf_{n+3}(\theta)/f_{n+2}(\theta))}{\sqrt{4a-1}}\right]\right]}{\sqrt{r^2f_{n+3}^2(\theta)/f_{n+2}^2(\theta)+rf_{n+3}(\theta)/f_{n+2}(\theta)+a}}, \qquad \text{if } a>\frac{1}{4},$$

$$\frac{r\exp\left(-\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)\exp\left(\frac{1}{1+2rf_{n+3}(\theta)/f_{n+2}(\theta)}\right)}{1+2rf_{n+3}(\theta)/f_{n+2}(\theta)}, \qquad \text{if } a=\frac{1}{4},$$

$$\frac{r\exp\left(-\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)\left(\sqrt{1-4a}+1+\frac{2rf_{n+3}(\theta)}{f_{n+2}(\theta)}\right)^{\frac{1}{2}\left(-1+\frac{1}{\sqrt{1-4a}}\right)}}{\left(\sqrt{1-4a}-1-\frac{2rf_{n+3}(\theta)}{f_{n+2}(\theta)}\right)^{\frac{1}{2}\left(1+\frac{1}{\sqrt{1-4a}}\right)}}, \qquad \text{if } a\neq 0<\frac{1}{4},$$

$$\frac{r\exp\left(-\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)f_{n+3}(\theta)}{f_{n+2}(\theta)}, \qquad \text{if } a=0,$$

through the change of variables (3).

(b) *If $f_{n+1}(\theta)$ is not identically zero, $a=0$ and $f_{n+3}(\theta)f_{n+2}(\theta)$ is identically zero, then the system is Darboux integrable with the first integral $\tilde{H}(x,y) = H(r,\theta)$ obtained from*

$$\frac{\exp\left(\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)}{r}+\int \frac{\exp\left(\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)f_{n+2}(\theta)}{g_{n+1}(\theta)}d\theta, \qquad \text{if } f_{n+3}(\theta)\equiv 0,$$

$$\frac{\exp\left(2\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)}{r^2}+2\int \frac{\exp\left(2\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)f_{n+3}(\theta)}{g_{n+1}(\theta)}d\theta, \qquad \text{if } f_{n+2}(\theta)\equiv 0,$$

through the change of variables (2).

Theorem 1 will be proved in Section 2.

THEOREM 2. *For a polynomial differential system (1) in the class F the following statements hold.*

(a) *If $f_{n+1}(\theta)f_{n+2}(\theta)f_{n+3}(\theta)$ is not identically zero, then in the domain of definition of the inverse integrating factor,*

$$V(r,\theta) = r(r^2f_{n+3}^2(\theta)/f_{n+2}^2(\theta)+rf_{n+3}(\theta)/f_{n+2}(\theta)+a)\ , \tag{7}$$

system (1) has no limit cycles.

(b) *If $f_{n+1}(\theta)f_{n+3}(\theta)$ is not identically zero, $a=0$ and $f_{n+2}(\theta)$ is identically zero, then the maximum number of its limit cycles contained in the domain of definition of the inverse integrating factor,*

$$V(r,\theta) = \frac{r}{2}+r^3\exp\left(-2\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)\int \frac{\exp\left(2\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)f_{n+3}(\theta)}{g_{n+1}(\theta)}d\theta\ , \tag{8}$$

is one.

(c) *If $f_{n+1}(\theta)f_{n+2}(\theta)$ is not identically zero, $a=0$ and $f_{n+3}(\theta)$ is identically zero, then in the domain of definition of the inverse integrating factor,*

$$V(r,\theta)=r+r^2\exp\left(-\int\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)\int\frac{\exp\left(\int\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)f_{n+2}(\theta)}{g_{n+1}(\theta)}\,d\theta\,, \tag{9}$$

system (1) has no limit cycles.

Theorem 2 will be proved in Section 3.

Theorem 1 are related to the results obtained in [24] about the Darboux integrability of a system $\dot{x}=P_n(x,y)+P_m(x,y)$, $\dot{y}=Q_n(x,y)+Q_m(x,y)$ with $m>n$ and where P_i and Q_i are homogeneous polynomials of degree i verifying $P_m=xA(x,y)$ and $Q_m=yA(x,y)$ where A is a homogeneous polynomial of degree $m-1$.

It is easy to check that systems (1) with $n=1$ satisfying $g_3(\theta)=g_4(\theta)=0$ for all θ can be written into the form,

$$\begin{aligned}\dot{x}&=a_{10}x+a_{01}y+x\left(\alpha x+\beta y+Ax^2+Bxy+Cy^2\right),\\ \dot{y}&=b_{10}x+b_{01}y+y\left(\alpha x+\beta y+Ax^2+Bxy+Cy^2\right),\end{aligned}\tag{10}$$

where a_{ij}, b_{ij}, α,β, A,B and C are arbitrary constants. In the first corollary of the Appendix, we provide new classes of Darboux integrable systems (10) satisfying Statements (a) and (b) of Theorem 1. The case when the linear part of (10) is a focus, is studied in [23].

Systems (1) with $n=3$ satisfying $g_5(\theta)=g_6(\theta)=0$ for all θ can be written into the form,

$$\begin{aligned}\dot{x}&=a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3\\ &\quad+x\left(\alpha x^3+\beta x^2y+\gamma xy^2+\delta y^3+Ax^4+Bx^3y+Cx^2y^2+Dxy^3+Ey^4\right),\\ \dot{y}&=a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3\\ &\quad+y\left(\alpha x^3+\beta x^2y+\gamma xy^2+\delta y^3+Ax^4+Bx^3y+Cx^2y^2+Dxy^3+Ey^4\right),\end{aligned}\tag{11}$$

where a_{ij}, b_{ij}, $\alpha,\beta,\gamma,\delta$, A,B,C,D and E are arbitrary constants. In the second corollary of the Appendix, we provide new classes of Darboux integrable systems (11) satisfying Statements (a) and (b) of Theorem 1. The proofs of the first and second corollaries involve tedious computations using a computer-algebra program.

A *characteristic direction* for the origin of system (1) is a root $(\lambda x,\lambda y)$ for all $\lambda\in\mathbb{R}$ of the homogeneous polynomial $xQ_n(x,y)-yP_n(x,y)$, which can be written by $\omega^*=[\cos\phi^*,\sin\phi^*]$ where ϕ^* is the argument of $x+iy$. It is obvious that, unless $xQ_n(x,y)-yP_n(x,y)\equiv 0$, the number of characteristic directions for the origin of system (1) is less or equal than $n+1$. It is well-known (see [25]) that if $\gamma(t)$ is a characteristic orbit for the origin of system (1) and $\omega^*=\lim_{t\to+\infty}\gamma(t)/\|\gamma(t)\|$, then ω^* is a characteristic direction for system (1). In particular, if all the roots of the polynomial $xQ_n(x,y)-yP_n(x,y)$ have nonzero imaginary part, then the origin is a monodromic singular point of system (1).

Systems (1) in the class F for $n\geq 2$ have a linearly zero singular point at the origin. It is obvious that n must be odd and greater or equal 3 in order that systems (1) in the class F have a linearly zero monodromic singular point at the origin. If a Darboux integrable system has a first integral defined in a neighborhood of the origin and the singular point is monodromic, then the system has a degenerate center at the origin. In the second corollary of the Appendix, we provide new classes of Darboux integrable systems (1) in the class F for $n=3$ satisfying Statements (a) and (b) of Theorem 1 which have a degenerate center at the origin.

2. PROOF OF THEOREM 1

PROOF OF THEOREM 1a. Following Case (d) of Abel differential equation studied in [19, p. 25], we do the change of variables $(r,\theta)\to(\eta,\xi)$ defined by $r=u(\theta)\eta(\xi)$, where

$$u(\theta)=\exp\left(\int\left[\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}\right]d\theta\right)$$

and $\xi = \int [u(\theta) f_{n+2}(\theta)/g_{n+1}(\theta)] d\theta$. This transformation writes the Abel differential equation (5) into the form,

$$\eta'(\xi) = g(\xi)\ [\eta(\xi)]^3 + [\eta(\xi)]^2, \tag{12}$$

where $g(\xi) = u(\theta) f_{n+3}(\theta)/f_{n+2}(\theta)$ and $' = \frac{d}{d\xi}$.

Doing the change $\xi \to t$ in the independent variable defined by $\xi' = -1/(t\eta(\xi))$, where now $' = \frac{d}{dt}$, equation (12) takes the form,

$$t^2 \xi''(t) + g(\xi(t)) = 0. \tag{13}$$

Note that $g(\xi) = a\xi$ means $u(\theta) f_{n+3}(\theta)/f_{n+2}(\theta) = a \int [u(\theta) f_{n+2}(\theta)/g_{n+1}(\theta)] d\theta$, or equivalently differentiating with respect to θ, we get

$$u(\theta) \frac{d}{d\theta} \left(\frac{f_{n+3}(\theta)}{f_{n+2}(\theta)} \right) = a\ u(\theta) \frac{f_{n+2}(\theta)}{g_{n+1}(\theta)} - u'(\theta) \frac{f_{n+3}(\theta)}{f_{n+2}(\theta)}.$$

Taking into account that $u'(\theta) = u(\theta) f_{n+1}(\theta)/g_{n+1}(\theta)$, we obtain

$$\frac{d}{d\theta} \left(\frac{f_{n+3}(\theta)}{f_{n+2}(\theta)} \right) = a\ \frac{f_{n+2}(\theta)}{g_{n+1}(\theta)} - \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)} \frac{f_{n+3}(\theta)}{f_{n+2}(\theta)}\ , \tag{14}$$

which is equivalent to condition (6). So, we have $g(\xi) = a\xi$. Now we note that equation (13) is an Euler differential equation. Therefore, doing the change $t = \exp(\tau)$ in the independent variable, equation (13) becomes the linear ordinary differential equation with constant coefficients,

$$\xi''(\tau) - \xi'(\tau) + a\xi(\tau) = 0, \tag{15}$$

where here $' = \frac{d}{d\tau}$. Equation (15) has the characteristic equation $k^2 - k + a = 0$, hence, its general solution is $\xi(\tau) = C_1 \exp(\tau/2) + C_2 \tau \exp(\tau/2)$ if $a = 1/4$, and $\xi(\tau) = C_1 \exp(k_1 \tau) + C_2 \exp(k_2 \tau)$ if $a \neq 1/4$, where k_1 and k_2 are the two roots of the characteristic equation. Going back to the independent variable $t = \exp(\tau)$ the solution of the Euler differential equation is $\xi(t) = C_1 \sqrt{t} + C_2 \sqrt{t} \ln t$ if $a = 1/4$, and $\xi(t) = C_1 t^{k_1} + C_2 t^{k_2}$ if $a \neq 1/4$.

Finally, going back to the variables (r, θ) and taking into account if the roots k_1 and k_2 are real or complex, after some tedious computations, we obtain the first integrals of Statement (a) according with the values of a.

Now, we are going to prove that systems of Statement (a) are Darboux integrable. For systems (1) in the class F with $f_{n+1}(\theta) f_{n+2}(\theta) f_{n+3}(\theta)$ not identically zero, it is easy to check that an inverse integrating factor for its associated Abel differential equation (5) is given by (7). As this inverse integrating factor $V(r, \theta)$ is an elementary function in cartesian coordinates (see [16,17] for more details and a definition of elementary function), then systems (1) in the class with $f_{n+1}(\theta) f_{n+2}(\theta) f_{n+3}(\theta)$ not identically zero have a Liouvillian first integral according with the results of Singer, see [17], and this completes the proof of Statement (a). ∎

PROOF OF THEOREM 1b. If $f_{n+3}(\theta)$ is identically zero or $f_{n+2}(\theta)$ is identically zero, the Abel differential equation (5) is the Bernoulli differential equation $\frac{dr}{d\theta} = r^2 f_{n+2}(\theta)/g_{n+1}(\theta) + r f_{n+1}(\theta)/g_{n+1}(\theta)$, or $\frac{dr}{d\theta} = r^3 f_{n+3}(\theta)/g_{n+1}(\theta) + r f_{n+1}(\theta)/g_{n+1}(\theta)$; respectively. Solving these Bernoulli equations we obtain the first integrals of Statement (b).

Systems of Statement (b) are Darboux integrable because their first integrals are obtained by integrating elementary functions, see, for more details, [17]. ∎

Now, we study if it is possible to find other integrable subclasses from the well-known integrable cases of the Abel differential equation. Following the Case (a) of Abel differential equation studied in [19, p. 24], first we do the change of variables $(r, \theta) \to (\eta, \xi)$ defined by $r = w(\theta)\eta(\xi) - f_{n+2}(\theta)/(3 f_{n+3}(\theta))$, where

$$w(\theta) = \exp \left(\int \left[\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)} - \frac{f_{n+2}^2(\theta)}{3 f_{n+3}(\theta) g_{n+1}(\theta)} \right] d\theta \right)$$

and $\xi = \int [f_{n+3}(\theta)w^2(\theta)/g_{n+1}(\theta)]\, d\theta$. This transformation writes the Abel equation (5) into the normal form,

$$\eta'(\xi) = [\eta(\xi)]^3 + I(\theta), \tag{16}$$

where

$$I(\theta) = \frac{g_{n+1}(\theta)}{f_{n+3}(\theta)w^3(\theta)} \left[\frac{d}{d\theta}\left(\frac{f_{n+2}(\theta)}{3f_{n+3}(\theta)}\right) - \frac{f_{n+1}(\theta)f_{n+2}(\theta)}{3f_{n+3}(\theta)g_{n+1}(\theta)} + \frac{2f_{n+2}^3(\theta)}{27f_{n+3}^2(\theta)g_{n+1}(\theta)}\right].$$

From the definition of $w(\theta)$, we have

$$\begin{aligned} \ln|w(\theta)| &= \int \left[\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)} - \frac{f_{n+2}^2(\theta)}{3f_{n+3}(\theta)g_{n+1}(\theta)}\right] d\theta \\ &= \int \frac{f_{n+2}(\theta)}{f_{n+3}(\theta)} \left[\frac{f_{n+1}(\theta)f_{n+3}(\theta)}{f_{n+2}(\theta)g_{n+1}(\theta)} - \frac{f_{n+2}(\theta)}{3g_{n+1}(\theta)}\right] d\theta. \end{aligned} \tag{17}$$

In the case $a \neq 0$, using (6) or equivalently (14) in (17), we obtain

$$\begin{aligned} -\frac{1}{3a}\int \frac{\frac{d}{d\theta}\left(\frac{f_{n+3}(\theta)}{f_{n+2}(\theta)}\right)}{\frac{f_{n+3}(\theta)}{f_{n+2}(\theta)}} d\theta + \left(1 - \frac{1}{3a}\right)\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)} d\theta \\ = -\frac{1}{3a}\ln\left|\frac{f_{n+3}(\theta)}{f_{n+2}(\theta)}\right| + \left(1 - \frac{1}{3a}\right)\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)} d\theta. \end{aligned}$$

Using this result, we get that

$$w(\theta) = \left|\frac{f_{n+3}(\theta)}{f_{n+2}(\theta)}\right|^{-1/3a} \exp\left[(1 - 1/(3a)) \int \left[\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}\right] d\theta\right]$$

and therefore $I(\theta)$ becomes

$$I(\theta) = \left[\frac{2-9a}{27}\right] \left(\frac{f_{n+3}(\theta)}{f_{n+2}(\theta)}\right)^{(1-3a)/a} \exp\left[\frac{1-3a}{a}\int \frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}\, d\theta\right].$$

It is easy to see that for $a = 2/9$ and for $a = 1/3$ we have $I(\theta) = 0$ and $I(\theta) = -1/27$, respectively. For these two cases, the differential equation (16) is of separable variables and we can obtain the associated first integrals. But $I(\theta) = 0$ and $I(\theta) = -1/27$ implies that equality (6) holds with $a = 2/9$ and for $a = 1/3$, respectively. So, we obtain cases already studied. New cases of integrability would be able to appear for $I(\theta) \neq 0, -1/27$.

We must mention that Cases (b) and (c) of Abel differential equation studied in [19, p. 25] provide again the case studied for $a = 2/9$.

3. EXISTENCE OF LIMIT CYCLES IN THE CLASS F

In order to study the existence and nonexistence of the limit cycles of system (1), we shall use the following result.

THEOREM 3. *Let (P, Q) be a C^1 vector field defined in the open subset U of $\mathbb{R}^2$. Let $V = V(x, y)$ be a C^1 solution of the linear partial differential equation (2) defined in U. If γ is a limit cycle of (P, Q) in the domain of definition U, then γ is contained in $\{(x, y) \in U : V(x, y) = 0\}$.*

PROOF. See Theorem 9 of [1,14]. ∎

We recall that under the assumptions of Theorem 3, the function $1/V$ is an integrating factor in $U \setminus \{V(x, y) = 0\}$. Again, for more details, see [3,18]. As we have seen, the function V is called an inverse integrating factor. In fact, using this notion, recently it is proved that any topological finite configuration of limit cycles is realizable by algebraic limit cycles of a Darboux integrable polynomial differential systems, see [1].

PROOF OF THEOREM 2A. For systems (1) in the class F with $f_{n+1}(\theta)f_{n+2}(\theta)f_{n+3}(\theta)$ not identically zero, it is easy to check that an inverse integrating factor of its associated Abel differential equation (5) is given by (7). By Theorem 3, if system (1) and consequently its associated Abel equation (5) have limit cycles, those of the Abel equation must be contained into the set $\{V(r,\theta)=0\}$. From the expression of the inverse integrating factor, the unique possible limit cycles must be given by

$$r(\theta)=\begin{cases}\dfrac{\left(-1\pm\sqrt{1-4a}\right)f_{n+2}(\theta)}{2f_{n+3}(\theta)}, & \text{if } a<\dfrac{1}{4},\\[2ex] -\dfrac{f_{n+2}(\theta)}{2f_{n+3}(\theta)}, & \text{if } a=\dfrac{1}{4}.\end{cases}$$

Since $n+2$ is odd, the function $f_{n+2}(\theta)$ has zeroes, therefore the above expressions of $r(\theta)$ cannot be positive for all θ. Consequently, there are no limit cycles in the domain of definition of V. ∎

PROOF OF THEOREM 2B. For systems (1) in the class F with $f_{n+1}(\theta)f_{n+3}(\theta)$ not identically zero, $a=0$ and $f_{n+2}(\theta)$ is identically zero, it is easy to check that an inverse integrating factor of its associated Abel differential equation (5) is given by (8). By Theorem 3, if system (1) and consequently its associated Abel equation (5) have limit cycles, those of the Abel equation must be contained into the set $\{V(r,\theta)=0\}$. From the expression of the inverse integrating factor, the unique possible limit cycles must be given by

$$r(\theta)=\pm\frac{\exp\left(\int\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}\,d\theta\right)}{\sqrt{-2\int\frac{\exp\left(2\int\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}\,d\theta\right)f_{n+3}(\theta)}{g_{n+1}(\theta)}\,d\theta}}.$$

In order that this expression of $r(\theta)$ define limit cycles, we must have $r(\theta)>0$, for all θ. Consequently, the maximum number of possible limit cycles in the domain of definition of $V(r,\theta)$ is at most 1.

Now it only remains to prove that this upper bound for the number of the limit cycles is reached. Systems (1) in the class F with $f_{n+1}(\theta)f_{n+3}(\theta)$ not identically zero and $f_{n+2}(\theta)$ identically zero, which first integrals are given in Theorem 1b, can have limit cycles as the following examples show. For $n=1$, the system $\dot{x}=-y-x(x^2+y^2-1)$, $\dot{y}=x-y(x^2+y^2-1)$ has exactly one limit cycle given by the circle $x^2+y^2-1=0$. For $n=3$, the system $\dot{x}=-x^3+x^2y-y^3+x^3(x^2+y^2-2xy)$, $\dot{y}=x^3-x^2y+xy^2+x^2y(x^2+y^2-2xy)$, has the limit cycle given by the circle $x^2+y^2-1=0$. This is due to the fact that this circle is an invariant algebraic curve and the system has a monodromic point at the origin because it has not characteristic directions since $xQ_n(x,y)-yP_n(x,y)=x^4+y^4$. Both systems have a focus at the origin. These systems for $n=1$ have been studied in [23]. ∎

PROOF OF THEOREM 2C. For systems (1) in the class F with $f_{n+1}(\theta)f_{n+2}(\theta)$ not identically zero, $a=0$ and $f_{n+3}(\theta)$ is identically zero, it is easy to check that an inverse integrating factor of its associated Abel differential equation (5) is given by (9). By Theorem 3, if system (1) and consequently its associated Abel equation (5) have limit cycles, those of the Abel equation must be contained into the set $\{V(r,\theta)=0\}$. From the expression of the inverse integrating factor, the unique possible limit cycles must be given by

$$r(\theta)=-\frac{\exp\left(\int\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)}{\int\frac{\exp\left(\int\frac{f_{n+1}(\theta)}{g_{n+1}(\theta)}d\theta\right)f_{n+2}(\theta)}{g_{n+1}(\theta)}\,d\theta}.$$

Since $n+2$ is odd, the function $f_{n+2}(\theta)$ has zeroes, therefore the above expressions of $r(\theta)$ cannot be positive for all θ. Consequently, there are no limit cycles in the domain of definition of V. ∎

4. THE APPENDIX

Systems (1) with $n = 1$ satisfying $g_3(\theta) = g_4(\theta) = 0$ inside the class F (i.e., the cubic systems (10)) having a focus or a center at the origin were studied in [23]. The following corollary provides the cubic polynomial systems (10) which belong to the class F without a focus or a center at the origin.

COROLLARY 4. *Cubic systems (10) without a focus or a center at the origin belong to the class F if and only if one of the following statements holds.*

(a) $\alpha = \beta = 0$.
(b) $a_{10} = b_{10} = \alpha = 0$, $A = 0$, *and* $a = (a_{01}B + b_{01}C)/\beta^2$.
(c) $b_{10} = \alpha = 0$, $A = B = 0$, *and* $a = (b_{01}C)/\beta^2$.
(d) $\alpha = 0$, $A = 0$, $C = -(a_{10}B)/b_{10}$, *and* $a = (a_{01}b_{10} - a_{10}b_{01})B/(b_{10}\beta^2)$.
(e) $b_{10} = \alpha = 0$, $b_{01} = 2a_{10}$, $B = -(2a_{01}A)/a_{10}$ *,and* $a = 2(a_{10}^2B - a_{01}^2A)/(a_{10}\beta^2)$.
(f) $B = A(a_{10}\alpha\beta - a_{01}\alpha^2 + b_{01}\alpha\beta - b_{10}\beta^2)/(\alpha(b_{01}\alpha - b_{10}\beta))$, $C = A\beta(a_{10}\beta - a_{01}\alpha)/(\alpha(b_{01}\alpha - b_{10}\beta))$, *and* $a = A(a_{10}b_{01} - a_{01}b_{10})/(\alpha(\alpha b_{01} - \beta b_{10}))$.
(g) $b_{01} = (b_{10}\beta)/\alpha$, $A = 0$, $C = (\beta B)/\alpha$, *and* $a = b_{10}B/\alpha^2$.
(h) $a_{01} = (a_{10}\beta)/\alpha$, $b_{01} = (b_{10}\beta)/\alpha$, $C = \beta(\alpha B - \beta A)/\alpha^2$, *and* $a = (a_{10}\alpha A + b_{10}\alpha B - b_{10}\beta A)/\alpha^3$.
(i) $a_{01} = \beta = B = C = 0$, *and* $a = a_{10}A/\alpha^2$.
(j) $a_{01} = b_{01} = \beta = C = 0$, *and* $a = (a_{10}A + b_{10}B)/\alpha^2$.
(k) $a_{10} = 2b_{01}$, $b_{10} = -(b_{01}B)/(2C)$, $a_{01} = \beta = 0$, *and* $a = b_{01}(4AC - B^2)/(2C\alpha^2)$.
(m) $a_{01} = b_{10} = b_{01} = 0$, $A = 0$, $C = (\beta B)/\alpha$, *and* $a = 0$.
(n) $a_{01} = 0$, $b_{10} = (\alpha a_{10})/\beta$, $b_{01} = 2a_{10}$, $C = (\beta B)/(2\alpha)$, *and* $a = (a_{10}B)/(\alpha\beta)$.
(o) $b_{10} = \alpha(a_{10}\beta - 2a_{01}\alpha)/\beta^2$, $b_{01} = (2a_{10}\beta - 3a_{01}\alpha)/\beta$, $A = (2a_{10}\alpha\beta B - 4a_{01}\alpha^2 B - a_{10}\beta^2 C + 3a_{01}\alpha\beta C)/(2a_{01}\beta^2)$, *and* $a = (2a_{10}\beta C - 4a_{01}\alpha C + a_{01}\beta B)/\beta^3$.

System (a) is Darboux integrable with the first integral given by Theorem 1(b) with $n = 1$ and where $f_3(\theta) = 0$. The other systems are Darboux integrable with the first integral given by Theorem 1(a) with $n = 1$.

Systems (1) with $n = 3$ satisfying $g_5(\theta) = g_6(\theta) = 0$ inside the class F has a linearly zero singular point at the origin. The following corollary provides some quintic polynomial systems of the form (11) which belong to the class F.

COROLLARY 5. *Systems (1) with $n = 3$ satisfying $g_5(\theta) = g_6(\theta) = 0$ belong to the class F if one of the following statements holds.*

(a) $\alpha = \beta = \gamma = \delta = 0$.
(b) $b_{30} = b_{12} = b_{03} = 0$, $b_{21} = a\beta^2/C$, $\alpha = \gamma = \delta = 0$, *and* $A = B = D = E = 0$.
(c) $A = B = C = D = E = 0$ *and* $a = 0$.

Systems (a) and (c) are Darboux integrable with the first integral given by Theorem 1b with $n = 3$ and where $f_5(\theta) = 0$ and $f_6(\theta) = 0$, respectively. System (b) is Darboux integrable with the first integral given by Theorem 1a with $n = 3$. Consequently, these quintic systems with a linearly zero singular point at the origin are Darboux integrable.

Inside Family (a) of Corollary 5 we have examples with a degenerate center. For instance, the system $\dot{x} = y(x^2 - y^2) - 2x^4y$, $\dot{y} = x(x^2 + y^2) - 2x^3y^2$ has a monodromic singular point at the origin because it has not characteristic directions. Therefore, the system has a center or a focus at the origin. Moreover, this system has a degenerate center at the origin because it is a time-reversible system (i.e., it is invariant under the change $(x, y, t) \to (x, -y, -t)$. Its first integral is given by $H = (x^4 + y^4)\exp[2\arctan(x^2 - y^2/x^2 + y^2)]/(x^2 + y^2 - 1)^2$. Another example is the system,

$$\begin{aligned} \dot{x} &= -y(x^2 + y^2) - x((3a + b)x^4 - 3cx^3y - 3bx^2y^2 - 3dxy^3 - 3ay^4)/3, \\ \dot{y} &= x(x^2 + y^2) - y((3a + b)x^4 - 3cx^3y - 3bx^2y^2 - 3dxy^3 - 3ay^4)/3. \end{aligned} \tag{18}$$

System (18) has a monodromic singular point at the origin because it has not characteristic directions. Moreover, this system has a degenerate center at the origin because it has the first integral $H = (x^2 + y^2)^2/f(x, y)$ where $f(x, y) = -48(x^2 + y^2) + 3(3d + 5c)x^4 + 32(3a + b)x^3y + 18(d - c)x^2y^2 + 96axy^3 - 3(5d + 3c)y^4$ which is well defined at the origin. We note that this degenerate center is neither time-reversible nor Hamiltonian.

Therefore, an exhaustive study of the family F will give a lot of examples of degenerate centers which are Darboux integrable.

The proof of Corollaries 4 and 5 follows doing tedious computations and using a computer-algebra program and Statements (a) and (b) of Theorem 1 when $n = 1$ and $n = 3$, respectively.

REFERENCES

1. J. Llibre and G. Rodriguez, Finite limit cycles configurations and polynomial vector fields, *J. Differential Equations* **198**, 374–380, (2004).
2. V. Mañosa, On the center problem for degenerate singular points of planar vector fields, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **12**, 687–707, (2002).
3. J. Chavarriga, H. Giacomini, J. Giné and J. Llibre, On the integrability of two-dimensional flows, *J. Differential Equations* **157**, 163–182, (1999).
4. J. Écalle, Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac, In *Actualités Mathématiques*, Hermann, Paris, (1992).
5. Yu. S. Il'yashenko, Finiteness theorems for limit cycles, In *Translations of Mathematical Monographs, Volume 94*, (Translated from the Russian by H.H. McFaden), American Mathematical Society, Providence, RI, (1991).
6. V.I. Arnold and Yu. S. Il'yashenko, Ordinary differential equations, In *Encyclopaedia of Math. Sci., Volume 1*, Springer-Verlag, Berlin, (1988).
7. H. Giacomini, J. Giné and J. Llibre, The problem of distinguishing between a center and a focus for nilpotent and degenerate analytic systems, (2005).
8. N.B. Medvedeva and E. Batcheva, The second term of the asymptotics of the monodromy map in case of two even edges of Newton diagram, *Electron. J. Qual. Th. Diff. Eqs.* **19**, 1–15, (2000).
9. A. Gasull, J. Llibre, V. Mañosa and F. Mañosas, The focus-center problem for a type of degenerate systems, *Nonlinearity* **13**, 699–730, (2000).
10. A. Gasull, V. Mañosa and F. Mañosas, Monodromy and stability of a generic class of degenerate planar critical points, *J. Differential Equations* **182**, 169–190, (2002).
11. J. Giné, Sufficient conditions for a center at completely degenerate critical point, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **12**, 1659–1666, (2002).
12. V.V. Nemytskii and V.V. Stepanov, *Qualitative theory of differential equations*, Dover Publ., New York, (1989).
13. R. Moussu, Une démonstration d'un théorème de Lyapunov-Poincaré, *Astérisque* **98-99**, 216–223, (1982).
14. H. Giacomini, J. Llibre and M. Viano, On the nonexistence, existence, and uniquennes of limit cycles, *Nonlinearity* **9**, 501–516, (1996).
15. W.-X. Ma, Integrability, In *Encyclopedia of Nonlinear Science*, (Edited by A. Scott), Taylor and Francis Books, Inc., New York, (2005).
16. M.J. Prelle and M.F. Singer, Elementary first integrals of differential equations, *Trans. Amer. Math. Soc.* **279**, 215–229, (1983).
17. M.F. Singer, Liouvillian first integrals of differential equations, *Trans. Amer. Math. Soc.* **333**, 673–688, (1992).
18. C.J. Christopher and J. Llibre, Integrability via invariant algebraic curves for planar polynomials differential systems, *Annals of Differential Equations* **16**, 5–19, (2000).
19. E. Kamke, Differentialgleichungen "losungsmethoden und losungen", In *Col. Mathematik und Ihre Anwendungen*, Akademische Verlagsgesellschaft Becker und Erler Kom-Ges., Leipzig, (1943).
20. E.S. Cheb-Terrab and A.D. Roche, An Abel ODE class generalizing known integrable classes, *Eur. J. Appl. Math.* **14**, 217–229, (2003).
21. A. Gasull and J. Llibre, Limit cycles for a class of Abel equations, *SIAM J. Math. Anal.* **21**, 1235–1244, (1990).
22. J. Giné and J. Llibre, Integrability and algebraic limit cycles for polynomial differential systems with homogeneous nonlinearities, *J. Differential Equations* **197**, 147–161, (2004).
23. J. Giné and J. Llibre, A family of isochronous foci with Darbouxian first integral, *Pacific J. Math.* **218**, 343–356, (2005).
24. J. Chavarriga, I.A. García, and J. Giné, On integrability of differential equations defined by the sum of homogeneous vector fields with degenerate infinity, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **11**, 711–722, (2001).
25. A.A. Andronov, E.A. Leontovich, I.I. Gordon and A.G. Maier, *Qualitative Theory of Second-Order Dynamic Systems*, John Wiley and Sons, New York; Israel Program for Scientific Translations, Jerusalem, (1973).