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On dispersive effect of the Coriolis force for the stationary Navier–Stokes equations

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ABSTRACT

The dispersive effect of the Coriolis force for the stationary and non-stationary Navier–Stokes equations is investigated. Existence of a unique solution is shown for arbitrary large external force provided the Coriolis force is large enough. In addition to the stationary case, counterparts of several classical results for the non-stationary Navier–Stokes problem have been proven. The analysis is carried out in a new framework of the Fourier–Besov spaces.

1. Introduction

We consider the initial value problem for the 3D-Navier–Stokes equations with the Coriolis force:

$$u_t + (u \cdot \nabla)u + \Omega e_3 \times u - \Delta u + \nabla p = 0, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

$$u(0, x) = u_0(x), \tag{1.3}$$

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where $u = u(t,x) = (u^1(t,x), u^2(t,x), u^3(t,x))$ is the unknown velocity vector field and $p = p(t,x)$ is the unknown scalar pressure at the point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and time $t > 0$, and $u_0(x)$ is the initial velocity vector field. Here $\Omega \in \mathbb{R}$ is the Coriolis parameter, which is the doubled angular velocity of the rotation around the vertical unit vector $e_3 = (0,0,1)$. Moreover the kinematic viscosity coefficient is normalized by one. By "$\times$" we denote the exterior product, and hence, the Coriolis term is represented by $e_3 \times u = Ju$ with the corresponding skew-symmetric 3×3 matrix J.

Problems concerning large-scale atmospheric and oceanic flows are known to be dominated by rotational effects. By this reason, almost all of the models of oceanography and meteorology dealing with large-scale phenomena include a Coriolis force. For example, an oceanic circulation featuring a hurricane is caused by the large rotation. There is no doubt that other physical effects are of similar significance like salinity, natural boundary conditions and so on. However, the first step in the study of more complex model is to understand the behavior of rotating fluids. To this end, we treat in a standard manner the Navier–Stokes equations with the Coriolis force.

Let us look back on the history of the Coriolis force. In 1868, Kelvin observed that a sphere moving along the axis of uniformly rotating water takes with it a column of liquid as if this were a rigid mass (see [9] for references). After that, Hough [16], Taylor [23] and Proudmann [22] made important contributions. Mathematically, Poincaré [22], and more recently, Babin et al. [1,2] considered non-stationary Navier–Stokes equations with Coriolis force in periodic case. The periodicity is extended to the almost periodic case by several authors. In particular for the results on the local existence of solutions to the non-stationary rotating Navier–Stokes equations with spatially almost periodic data and its properties, we refer the reader to [10,13,14]. Moreover, for the results on the global in time existence of solutions and their long time behavior in the almost periodic setting, see [11,12,25] for example.

On the other hand, Chemin et al. [7] considered the decaying data case. They derived dispersion estimates on a linearized version of the 3D-Navier–Stokes equations with the Coriolis force to show the existence of a global in time solution to the non-stationary rotating Navier–Stokes system. To construct such estimate, they handled eigenvalues and eigenfunctions of the Coriolis operator.

The main result of this paper is to show existence of the solution to the stationary Navier–Stokes equations with the Coriolis force for arbitrary large external force provided that the Coriolis force is sufficiently large (compare it with results for the Navier–Stokes equations (2.8) with $\Omega = 0$, for example [18] for the case of exterior domain). To do so, we introduce a new type of function spaces, namely, Fourier–Besov spaces (FB), which allow us to describe, in a clear way, how the Coriolis force influences solutions to the considered system. A similar approach, based on function spaces which make the analysis of specific features of a system much easier, has been used by the first author in [17], where an asymptotic structure of solution to the stationary Navier–Stokes equations in $\mathbb{R}^2$ was investigated.

In FB spaces, we cannot expect energy type estimates and a structure of Hilbert spaces as in [7]. The main motivation to introduce those spaces is that, in this framework, we are able to present directly dispersive effect of the Coriolis force (see Proposition 2.5), which is in principle different from the dispersive effect from [7] (for the space-time estimates for different equations see for example [19,20]).

To emphasize the usefulness of introduced spaces we prove the existence of solutions to the non-stationary Navier–Stokes–Coriolis system in function spaces which are counterparts for well-known classical results in the Navier–Stokes theory (see [3,5,6]). Moreover, we can considerably simplify other results for the Navier–Stokes–Coriolis system, like recent results by Giga et al. [12].

2. Main results

In the paper we follow basic ideas from the Littlewood–Paley theory. We denote by $\varphi \in \mathcal{S}(\mathbb{R}^3)$ a radially symmetric function supported in $\{\xi \in \mathbb{R}^3\colon\ \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$ such that

$$\sum_{j \in \mathbb{Z}} \varphi\big(2^{-j}\xi\big) = 1 \quad \text{for all } \xi \neq 0.$$

We also introduce the following functions:

$$\varphi_j(\xi)=\varphi\bigl(2^{-j}\xi\bigr) \quad \text{and} \quad \psi_j(\xi)=\sum_{k\leqslant j-1}\varphi_k(\xi).$$

Now, we define the standard localization operators:

$$\Delta_j f=\varphi_j(D)f, \qquad S_j f=\sum_{k\leqslant j-1}\Delta_k f=\psi_j(D)f, \quad \text{for every } j\in\mathbb{Z}. \tag{2.1}$$

It is then easy to verify the following identities:

$$\Delta_j\Delta_k f=0 \quad \text{if } |j-k|\geqslant 2, \tag{2.2}$$

$$\Delta_j(S_{k-1}f\,\Delta_k f)=0 \quad \text{if } |j-k|\geqslant 5. \tag{2.3}$$

Moreover, we can follow Bony [4] and introduce the following decomposition:

$$fg=T_f g+T_g f+R(f,g), \tag{2.4}$$

where

$$T_f g=\sum_{j\in\mathbb{Z}}S_{j-1}f\,\Delta_j g, \qquad R(f,g)=\sum_{j\in\mathbb{Z}}\Delta_j f\,\tilde{\Delta}_j g, \qquad \tilde{\Delta}_j g=\sum_{|j'-j|\leqslant 1}\Delta_{j'}g. \tag{2.5}$$

The framework for our results is determined by the Fourier–Besov spaces defined as follows.

Definition 2.1. We introduce the following homogeneous function spaces called Fourier–Besov spaces:

- for $1\leqslant p\leqslant\infty$, $1\leqslant q<\infty$,

$$\dot{FB}^s_{p,q}(\mathbb{R}^n)=\biggl\{f\in\mathcal{S}'\colon\ \hat{f}\in L^1_{\mathrm{loc}},\ \|f\|_{\dot{FB}^s_{p,q}(\mathbb{R}^n)}=\biggl(\sum_{k\in\mathbb{Z}}2^{ksq}\|\varphi_k\hat{f}\|^q_{L_p(\mathbb{R}^n)}\biggr)^{1/q}<\infty\biggr\},$$

- $$\dot{FB}^s_{p,\infty}(\mathbb{R}^n)=\Bigl\{f\in\mathcal{S}'\colon\ \|f\|_{\dot{FB}^s_{p,\infty}(\mathbb{R}^n)}=\sup_{k\in\mathbb{Z}}2^{ks}\|\varphi_k\hat{f}\|_{L_p(\mathbb{R}^n)}<\infty\Bigr\}.$$

Remark 1. Please note that in the case $p=q$ one may consider an equivalent norm on $\dot{FB}^s_{p,p}$, that is

$$\|f\|_{\dot{FB}^s_{p,p}}\sim\biggl(\int_{\mathbb{R}^n}|\xi|^{sp}\bigl|\hat{f}(\xi)\bigr|^p\,d\xi\biggr)^{1/p}.$$

Indeed, assuming for simplicity $s\geqslant 0$ (for $s<0$ the reasoning is analogous):

$$\begin{aligned}\|f\|^p_{\dot{FB}^s_{p,p}}&=\sum_{k\in\mathbb{Z}}2^{ksp}\|\varphi_k\hat{f}\|^p_{L_p(\mathbb{R}^n)}\\&\leqslant\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n}\varphi_k(\xi)|\xi|^{sp}\bigl|\hat{f}(\xi)\bigr|^p\,d\xi=\int_{\mathbb{R}^n}|\xi|^{sp}\bigl|\hat{f}(\xi)\bigr|^p\,d\xi,\end{aligned}$$

where we used the fact that $|\varphi_k(\xi)|^p\leqslant\varphi_k(\xi)$, $\operatorname{supp}\varphi_k\subset[2^k,2^{k+1}]$ and $\sum_k\varphi_k(\xi)\equiv 1$ for $\xi\in\mathbb{R}^n\setminus\{0\}$.

To obtain the second required inequality we first note that thanks to orthogonality of φ_k and that $0 \leqslant \varphi_k \leqslant 1$ one has

$$\left(\sum_k \varphi_k\right)^p = \left(\sum_k \varphi_k\right)^{[p]+1} \leqslant 3^{[p+1]} \sum_k \varphi_k^{[p+1]} \leqslant 3^{[p+1]} \sum_k \varphi_k^p. \tag{2.6}$$

Then

$$\int_{\mathbb{R}^n} |\xi|^{sp} \left|\hat{f}(\xi)\right|^p d\xi \leqslant 3^{[p+1]} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} 2^{ksp} \left|\varphi_k \hat{f}(\xi)\right|^p d\xi = 3^{[p+1]} \sum_{k\in\mathbb{Z}} 2^{ksp} \|\varphi_k \hat{f}\|^p_{L^p(\mathbb{R}^n)}, \tag{2.7}$$

which proves equivalence of those norms.[2]

Now, we are in a position to formulate our main results for the stationary and non-stationary Navier–Stokes equations with the Coriolis force. We would like to mention that it is not difficult to obtain also other results (like stability of solutions to the non-stationary case) in this framework. We refer readers to the paper by Cannone and Karch [5] as a reference for what can be expected. We do not prove those results to keep the paper more readable.

2.1. Non-stationary case

In the following theorem, we consider mild solutions to the non-stationary Navier–Stokes system with the Coriolis force (1.1)–(1.3).

Theorem 2.2. *Let $\Omega \in \mathbb{R}$ be an arbitrary constant. Let $u_0 \in X_0$ and $\|u_0\|_{X_0}$ be small enough (independently of Ω). Then there exists a unique global in time solution $u \in Y$ to problem (1.1)–(1.3), where X_0 and Y can be taken as follows:*

- $X_0 = \dot{FB}^{2-3/p}_{p,\infty}$, $Y = C^w([0,\infty); \dot{FB}^{2-3/p}_{p,\infty}) \cap L^\infty(0,\infty; \dot{FB}^{2-3/p}_{p,\infty})$, *where* $3 < p \leqslant \infty$,
- $X_0 = \dot{FB}^{2-3/p}_{p,p}$, $Y = C([0,\infty); \dot{FB}^{2-3/p}_{p,p}) \cap L^\infty(0,\infty; \dot{FB}^{2-3/p}_{p,p})$, *where* $3 < p < \infty$,
- $X_0 = \dot{FB}^{-1}_{1,1} \cap \dot{FB}^0_{1,1}$, $Y = C([0,\infty); \dot{FB}^0_{1,1}) \cap L^2(0,\infty; \dot{FB}^{-1}_{1,1} \cap \dot{FB}^0_{1,1})$,
- $X_0 = \dot{FB}^{2-3/p}_{p,\infty}$, $Y = L^\infty(0,\infty; \dot{FB}^{2-3/p}_{p,\infty}) \cap \tilde{L}^1(0,\infty; \dot{FB}^{4-3/p}_{p,\infty}) \cap C^w([0,\infty); \dot{FB}^{2-3/p}_{p,\infty})$, *for* $1 < p \leqslant \infty$,

where we used the standard notation for the norm $\tilde{L}^1$:

$$\|f\|_{\tilde{L}^1_T(\dot{FB}^{4-3/p}_{p,\infty})} = \sup_{k\in\mathbb{Z}} 2^{k(4-3/p)} \|\varphi_k \hat{f}\|_{L^1_T(L^p(\mathbb{R}^n))}.$$

Note. The mentioned cases have their counterparts in the current literature for the non-stationary Navier–Stokes equations. For example, the case $\dot{FB}^2_{\infty,\infty}$ was considered by Cannone and Karch [5], and the case $\dot{B}^{3/p-1}_{p,\infty}$ (which is a counterpart for $\dot{FB}^{2-3/p}_{p,\infty}$) by Cannone [6] and Planchon [21]. The Navier–Stokes system in the space $\dot{FB}^{2-3/p}_{p,p}$ was studied by Biswas and Swanson [3] in the periodic setting. Their result covers the whole range $1 < p \leqslant \infty$ due to the periodicity, more precisely, in their case they do not have problems with integrability (summability) close to 0 in the Fourier space. In our case, the analysis close to 0 in the Fourier space requires the assumption $p > 3$. A counterpart of the case $\dot{FB}^{-1}_{1,1} \cap \dot{FB}^0_{1,1}$, that is $FM_0^{-1} \cap FM_0$ spaces, has been published recently by Giga, Inui, Mahalov and Saal in [12].

[2] Authors are grateful to the referee for suggesting this clarification.

Our approach in this paper seems to be more suitable for the Navier–Stokes equations with the Coriolis force. Unfortunately, using our methods we were not able to include the case $p = 2$, $q = 2$ which has been recently proven by Hieber and Shibata [15].

2.2. Stationary case

In the following section we consider mild solutions to the stationary Navier–Stokes system with the Coriolis force:

$$-\nu\Delta u + \Omega e_3 \times u + u \cdot \nabla u + \nabla p = F, \qquad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3. \tag{2.8}$$

Definition 2.3. To study problem (2.8), we introduce the following function space for every $1 \leqslant p \leqslant \infty$,

$$X^p_{\mathcal{C},\Omega} = \left\{ f \in \mathcal{S}' \colon \|f\|_{X^p_{\mathcal{C},\Omega}} = \left\| w_1(\cdot)\hat{f}(\cdot)\right\|_{L^p} + \left\| w_2(\cdot)\hat{f}(\cdot)\right\|_{L^p} < \infty \right\}, \tag{2.9}$$

where

$$w_1(\xi) = \frac{|\xi|^{6-3/p}}{|\xi|^6 + \Omega^2|\xi_3|^2}, \qquad w_2(\xi) = \frac{\Omega|\xi_3||\xi|^{3-3/p}}{|\xi|^6 + \Omega^2|\xi_3|^2} R(\xi),$$

and $R(\xi)$ is the matrix (3.3).

The following theorem is the main result of our paper.

Theorem 2.4. *Let $3 < p \leqslant \infty$. Then for all $F \in X^p_{\mathcal{C},\Omega}$ such that $\|F\|_{X^p_{\mathcal{C},\Omega}}$ is small enough there exists a unique solution $u \in \dot{FB}^{2-3/p}_{p,p}$ to problem* (2.8)*. Moreover, the following estimate is valid*

$$\|u\|_{\dot{FB}^{2-3/p}_{p,p}} \leqslant C\|F\|_{X^p_{\mathcal{C},\Omega}}. \tag{2.10}$$

Analogous result holds also true for the spaces $\dot{FB}^{2-3/p}_{p,\infty}$.

Remark. It is important to emphasize here that $\dot{FB}^{-3/p}_{p,p} \subsetneq X^p_{\mathcal{C},\Omega}$ for $\Omega \neq 0$ (see the proof of Proposition 2.5 below). This means that the Coriolis force not only helps us to weaker the smallness assumptions on the force F (see [5] and Proposition 2.5 below) but also extends considerably the class of admissible external forces (for which we have the existence result). For example, on can choose as the external force function with the Fourier transform

$$\frac{|\xi|^6 + \Omega^2\xi_2^2}{|\xi|^6} \cdot \left(\frac{\xi_1\xi_3}{|\xi|^2}, \frac{\xi_2\xi_3}{|\xi|^2}, \frac{\xi_3^2}{|\xi|^2} \right) \tag{2.11}$$

which is an element of $X^\infty_{\mathcal{C},\Omega}$. This function, however, is not an element of the space of pseudo-measures $\mathcal{PM} = \dot{FB}^0_{\infty,\infty}$ from the paper [5].

In the case $F \in \dot{FB}^{-3/p}_{p,p}$, we can remove the smallness assumption provided that the Coriolis parameter Ω is large enough. This is precised in the following proposition (cf. the case $F \in \dot{FB}^{-3/p}_{p,\infty}$ in Remark 2 following the proof of Proposition 2.5).

Proposition 2.5. *Let $3 < p < \infty$. Then for any given function $F \in \dot{FB}^{-3/p}_{p,p}$ there exists Ω_0 such that for all $\Omega \in \mathbb{R}$ satisfying $|\Omega| \geqslant \Omega_0$ there exists the unique solution u to problem* (2.8) *such that $u \in \dot{FB}^{2-3/p}_{p,p}(\mathbb{R}^3)$.*

Remark 2. The counterpart of Proposition 2.5 for the case when $F \in \dot{FB}^{-3/p}_{p,\infty}$ requires additional assumptions on F. Method, which we presented in the previous proof requires smallness assumptions of the following form: there exists a number K such that

$$\sup_{|k|\geqslant K} 2^{-3k/p}\big(\|\varphi_k w_1 \hat{F}\|_{L^p} + \|\varphi_k w_2 \hat{F}\|_{L^p}\big) \quad \text{is small enough,} \tag{2.12}$$

where $w_1(\xi)$ and $w_2(\xi)$ are weights from the definition (2.9) of the space $X^p_{\mathcal{C},\Omega}$. In particular, this condition allows us to have $\|F\|_{\dot{FB}^{-3/p}_{p,\infty}}$ arbitrary large not only in frequencies within the region $[-K, K]$. The function can be large in frequencies with $|k| \geqslant K$ provided weights w_1 and w_2 make them small enough. The proof of this fact is analogous to the proof of Proposition 2.5.

3. Main estimates

In the formulation of lemmas in this section we are using the generator $\mathcal{G}$ of the Stokes–Coriolis semigroup. To be precise, for the following Stokes problem with the Coriolis force:

$$\begin{cases} u_t - \nu\Delta u + \Omega e_3 \times u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{3.1}$$

one may derive the following formula for the solution (see [10]):

$$\hat{u}(t,\xi) = \cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-\nu|\xi|^2 t} I\hat{u}_0(\xi) + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-\nu|\xi|^2 t} R(\xi)\hat{u}_0(\xi), \tag{3.2}$$

where $t \geqslant 0$, $\xi \in \mathbb{R}^3$, I is the identity matrix and

$$R(\xi) = \begin{pmatrix} 0 & \frac{\xi_3}{|\xi|} & -\frac{\xi_2}{|\xi|} \\ -\frac{\xi_3}{|\xi|} & 0 & \frac{\xi_1}{|\xi|} \\ \frac{\xi_2}{|\xi|} & -\frac{\xi_1}{|\xi|} & 0 \end{pmatrix}. \tag{3.3}$$

Thus by $\mathcal{G}$ we denote a function for which

$$\hat{\mathcal{G}}(t) = \cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-\nu|\xi|^2 t} I + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-\nu|\xi|^2 t} R(\xi). \tag{3.4}$$

We need the following auxiliary lemma, which we provide here without a proof (see [24]).

Lemma 3.1. *For $1 \leqslant q \leqslant p \leqslant \infty$ and any multiindex γ the following inequalities are valid*:

- $\operatorname{supp} \hat{f} \subset \{|\xi| \leqslant A2^j\} \Rightarrow \|(i\xi)^\gamma \hat{f}\|_{L^q(\mathbb{R}^n)} \leqslant C2^{j|\gamma| + nj(\frac{1}{q}-\frac{1}{p})}\|\hat{f}\|_{L^p(\mathcal{R}^n)}$.
- $\operatorname{supp} \hat{f} \subset \{B_1 2^j \leqslant |\xi| \leqslant B_2 2^j\} \Rightarrow \|\hat{f}\|_{L^q(\mathbb{R}^n)} \leqslant C2^{-j|\gamma|}\sup_{|\beta|=|\gamma|}\|(i\xi)^\beta \hat{f}\|_{L^p(\mathcal{R}^n)}$.

Lemma 3.2. *For $p \in [1,\infty]$ and $u_0 \in \dot{FB}^{2-3/p}_{p,\infty}$ one has*

$$\big\|\mathcal{G}(t)u_0\big\|_{L^\infty(0,T;\dot{FB}^{2-3/p}_{p,\infty}) \cap \tilde{L}^1(0,T;\dot{FB}^{4-3/p}_{p,\infty})} \leqslant \max\left(1, \frac{1}{\nu}\right)\|u_0\|_{\dot{FB}^{2-3/p}_{p,\infty}}. \tag{3.5}$$

Moreover one also has

$$\left\|\mathcal{G}(t)u_0\right\|_{L^\infty_T(\dot{FB}^s_{p,p})} \leqslant \|u_0\|_{\dot{FB}^s_{p,p}}. \tag{3.6}$$

Proof. While the second estimate is straightforward let us focus on the first inequality. We consider the case $p < \infty$. The case $p = \infty$ can be obtained analogously. Let us first estimate the norm $\|\mathcal{G}(t)u\|_{L^\infty_T(\dot{FB}^{2-3/p}_{p,\infty})}$:

$$\left\|\mathcal{G}(t)u\right\|_{L^\infty_T(\dot{FB}^{2-3/p}_{p,\infty})} \leqslant \sup_{0\leqslant t<T} \sup_k 2^{k(2-3/p)} \|\varphi_k \hat{u}_0\|_{L^p} \leqslant \|u_0\|_{\dot{FB}^{2-3/p}_{p,\infty}}.$$

The second part estimates as follows:

$$\begin{aligned}
\left\|\mathcal{G}(t)u\right\|_{\tilde{L}^1_T(\dot{FB}^{4-3/p}_{p,\infty})} &\leqslant \sup_k \int_0^T 2^{k(4-3/p)} e^{-\nu t 2^{2k}} \|\varphi_k \hat{u}_0\|_{L^p}\,dt \\
&\leqslant \sup_k \frac{1}{\nu} 2^{-2k} 2^{k(4-3/p)} \|\varphi_k \hat{u}_0\|_{L^p} \leqslant \frac{1}{\nu}\|u_0\|_{\dot{FB}^{2-3/p}_{p,\infty}}.
\end{aligned} \tag{3.7}$$

This finishes the proof of this lemma. □

Lemma 3.3. *Let $s \in \mathbb{R}$, $p, q \in [1, \infty]$ and $f \in \tilde{L}^r_T(\dot{FB}^s_{p,\infty})$. Then the following estimate is valid*:

$$\left\|\int_0^t \mathcal{G}(t-\tau) f(\tau)\,d\tau\right\|_{\tilde{L}^q_T(\dot{FB}^s_{p,\infty})} \leqslant \frac{1}{\nu}\|f\|_{\tilde{L}^r_T(\dot{FB}^{s-2-2/q+2/r}_{p,\infty})}. \tag{3.8}$$

Proof. Since

$$\left\|\int_0^t \mathcal{G}(t-\tau) f(\tau)\,d\tau\right\|_{\tilde{L}^q_T(\dot{FB}^s_{p,\infty})} = \sup_k 2^{sk} \left\|\int_0^t \left\|\hat{\mathcal{G}}(t-\tau)\hat{f}(\tau)\varphi_k\right\|_{L^p} d\tau\right\|_{L^q_T}$$

we may fix k and estimate the corresponding term:

$$2^{ks}\left\|\int_0^t \left\|\hat{\mathcal{G}}(t-\tau)\hat{f}(\tau)\varphi_k\right\|_{L^p} d\tau\right\|_{L^q_T} \leqslant 2\cdot 2^{ks}\left\|\int_0^t e^{(t-\tau)2^{2k}} \left\|\hat{f}(\tau)\varphi_k\right\|_{L^p} d\tau\right\|_{L^q_T}.$$

Using Young's inequality with $\tilde{q}$ such that $1 + \frac{1}{q} = \frac{1}{\tilde{q}} + \frac{1}{r}$, that is $\frac{1}{\tilde{q}} = 1 + \frac{1}{q} - \frac{1}{r}$ we get

$$\begin{aligned}
2^{ks}\left\|\int_0^t e^{(t-\tau)2^{2k}} \left\|\hat{f}(\tau)\varphi_k\right\|_{L^p} d\tau\right\|_{L^q_T} &\leqslant 2^{ks}\left\|e^{t2^{2k}}\right\|_{L^{\tilde{q}}_T} \left\|\hat{f}(t)\varphi_k\right\|_{L^r_T(L^p)} \\
&\leqslant 2^{k(s-2-\frac{2}{q}+\frac{2}{r})} \left\|\hat{f}(t)\varphi_k\right\|_{L^r_T(L^p)}.
\end{aligned}$$

Taking supremum over all $k \in \mathbb{Z}$ one obtains the desired estimate. □

Lemma 3.4. *The following estimates are valid*:

- *For* $1 < p \leqslant \infty$ *and* $V = L_T^\infty(0, T; \dot{FB}_{p,\infty}^{2-3/p}(\mathbb{R}^3)) \cap \tilde{L}_T^1(0, T; \dot{FB}_{p,\infty}^{4-3/p}))$ (V *introduced here for readability*):

$$\|uv\|_{\tilde{L}_T^1(\dot{FB}_{p,\infty}^{3-\frac{3}{p}})} \leqslant C\|u\|_V\|v\|_V. \tag{3.9}$$

- *For* $p > 3$:

$$\|uv\|_{L_T^\infty(\dot{FB}_{p,\infty}^{1-\frac{3}{p}})} \leqslant C\|u\|_{L_T^\infty(\dot{FB}_{p,\infty}^{2-3/p})}\|v\|_{L_T^\infty(\dot{FB}_{p,\infty}^{2-3/p})}. \tag{3.10}$$

Proof. In the following proof we follow in principle the reasoning from [8]. Let us focus on the first inequality. From the definition we have

$$\|uv\|_{\tilde{L}_T^1(\dot{FB}_{p,\infty}^{3-\frac{3}{p}})} = \sup_j \int_0^T 2^{j(3-\frac{3}{p})} \big\| \widehat{\Delta_j(uv)} \big\|_{L_p} \, dt. \tag{3.11}$$

For $\Delta_j(uv)$ we use decomposition (2.4), that is

$$\Delta_j(uv) = \sum_{|k-j|\leqslant 4} \Delta_j(S_{k-1}u\Delta_k v) + \sum_{|k-j|\leqslant 4} \Delta_j(S_{k-1}v\Delta_k u) + \sum_{k\geqslant j-2} \Delta_j(\Delta_k u\tilde{\Delta}_k v), \tag{3.12}$$

and denote each corresponding integral from (3.11) as I_j, II_j and III_j.

$$\begin{aligned} I_j &= 2^{j(3-3/p)} \int_0^T \Big\| \sum_{|k-j|\leqslant 4} \varphi_j(\psi_{k-1}\hat{u} * \varphi_k\hat{v}) \Big\|_{L^p} \, dt \\ &\leqslant 2^{j(3-3/p)} \int_0^T \sum_{|k-j|\leqslant 4} \|\psi_{k-1}\hat{u}\|_{L^1} \|\varphi_k\hat{v}\|_{L^p} \, dt. \end{aligned} \tag{3.13}$$

Now using Lemma 3.1 we have the following inequality:

$$\|\psi_{k-1}\hat{u}\|_{L^1} \leqslant \sum_{k'<k} \|\varphi_{k'}\hat{u}\|_{L^1} \leqslant \sum_{k'<k} 2^{k'3(1-1/p)} \|\varphi_{k'}\hat{u}\|_{L^p}, \tag{3.14}$$

which allows us to estimate I_j as follows:

$$\begin{aligned} I_j &\leqslant 2^{j(3-3/p)} \int_0^T \sum_{|k-j|\leqslant 4} \sum_{k'<k} 2^{k'} 2^{k'(2-3/p)} \|\varphi_{k'}\hat{u}\|_{L_p} \|\varphi_k\hat{v}\|_{L_p} \, dt \\ &\leqslant 2^{j(3-3/p)} \int_0^T \sum_{|k-j|\leqslant 4} 2^k \sup_{k'} 2^{k'(2-3/p)} \|\varphi_{k'}\hat{u}\|_{L_p} \|\varphi_k\hat{v}\|_{L_p} \, dt \end{aligned}$$

$$\leqslant 2^{j(4-3/p)} \int_0^T \|\varphi_k \hat{v}\|_{L_p}\, dt \sup_{k'} 2^{k'(2-3/p)} \|\varphi_{k'} \hat{u}\|_{L_p}$$
$$\leqslant \|v\|_{\tilde{L}^1_T(\dot{FB}^{4-3/p}_{p,\infty})} \|u\|_{L^\infty_T(\dot{FB}^{2-3/p}_{p,\infty})},$$

where we used the fact that since $|j-k|<4$ then $2^j \sim 2^k$.

Integral II_j is easily estimated in the same way as I_j. We will now focus on integral III_j

$$III_j = 2^{j(3-3/p)} \int_0^T \sum_{k \geqslant j-2} \big\|\varphi_j(\varphi_k u * \tilde{\varphi}_k v)\big\|_{L^p}\, dt$$
$$\leqslant 2^{j(3-3/p)} \int_0^T \sum_{k \geqslant j-2} \|\varphi_k \hat{u}\|_{L^1} \|\varphi_k \hat{v}\|_{L^p}\, dt$$
$$= \sum_{k \geqslant j-2} 2^{(j-k)(3-3/p)} \int_0^T \|\varphi_k \hat{u}\|_{L^p} 2^{k(2-3/p)} \|\tilde{\varphi}_k \hat{v}\|_{L^p} 2^{k(4-3/p)}\, dt$$
$$\leqslant \sup_k \|\varphi_k \hat{u}\|_{L^p} 2^{k(2-3/p)} \sup_k \int_0^T \|\tilde{\varphi}_k \hat{v}\|_{L^p} 2^{k(4-3/p)}\, dt$$
$$\leqslant \|u\|_{L^\infty_T(\dot{FB}^{2-3/p}_{p,\infty})} \|v\|_{\tilde{L}^1_T(\dot{FB}^{4-3/p}_{p,\infty})},$$

where we again used Lemma 3.1.

In order to obtain estimate (3.10) one proceeds in a similar way as for the case of (3.9), applying proper changes like $3-3/p$ is replaced by $1-3/p$. The requirement that $p>3$ comes from estimate of III_j, that is in the case of (3.9) one has the term $\sum_{k \geqslant j-2} 2^{(j-k)(3-3/p)}$, which is finite for $p>1$, while in case of estimate (3.10) one encounters the term $\sum_{k \geqslant j-2} 2^{(j-k)(1-3/p)}$, which is finite for $p>3$. □

In what follows we focus on estimates for the space $L^2_T(\dot{FB}^0_{1,1})$.

Lemma 3.5. *The following estimate is valid*:

$$\big\|e^{t\Delta} u_0\big\|_{L^2_T(\dot{FB}^s_{1,1})} \leqslant \|u_0\|_{\dot{FB}^{s-1}_{1,1}}. \tag{3.15}$$

Proof. This inequality is easily obtained

$$\big\|e^{t\Delta} u_0\big\|_{L^2_T(\dot{FB}^s_{1,1})} = \left\|\sum_k \int_{\mathbb{R}^3} \varphi_k e^{-t|\xi|^2} |\xi|^s \hat{u}_0(\xi)\, d\xi\right\|_{L^2_T}$$
$$\leqslant \sum_k \left(\int_0^T e^{-t2^{2k+1}} 2^{2sk} \big\|\varphi_k \hat{u}_0(\xi)\big\|^2_{L^1}\, dt\right)^{1/2}$$
$$\leqslant \sum_k 2^{(s-1)k} \big\|\varphi_k \hat{u}_0(\xi)\big\|_{L^1} = \|u_0\|_{\dot{FB}^{s-1}_{1,1}}. \qquad \square \tag{3.16}$$

Lemma 3.6. *The following estimate is valid*:

$$\|uv\|_{L^1_T(\dot{FB}^0_{1,1})} \leqslant \|u\|_{L^2_T(\dot{FB}^0_{1,1})} \|v\|_{L^2_T(\dot{FB}^0_{1,1})}. \tag{3.17}$$

Moreover if $u, v \in L^2_T(\dot{FB}^0_{1,1} \cap \dot{FB}^1_{1,1})$ *then the following estimate is valid*:

$$\|uv\|_{L^1_T(\dot{FB}^1_{1,1})} \leqslant \|u\|_{L^2_T(\dot{FB}^0_{1,1} \cap \dot{FB}^1_{1,1})} \|v\|_{L^2_T(\dot{FB}^0_{1,1} \cap \dot{FB}^1_{1,1})}. \tag{3.18}$$

Proof. First we note that $f \in \dot{FB}^0_{1,1} \Leftrightarrow \hat{f} \in L^1$. Then our inequality (3.17) is proven in the following way:

$$\begin{aligned}
\|uv\|_{L^1_T(\dot{FB}^0_{1,1})} &= \int_0^T \|\hat{u} * \hat{v}\|_{L^1} \leqslant \int_0^T \|\hat{u}\|_{L^1} \|\hat{v}\|_{L^1} \\
&\leqslant \|\hat{u}\|_{L^2_T(L^1)} \|\hat{v}\|_{L^2_T(L^1)} = \|u\|_{L^2_T(\dot{FB}^0_{1,1})} \|v\|_{L^2_T(\dot{FB}^0_{1,1})}.
\end{aligned}$$

To prove inequality (3.18) we proceed in a similar way:

$$\begin{aligned}
\|uv\|_{L^1_T(\dot{FB}^1_{1,1})} &= \int_0^T \int_{\mathbb{R}^n} |\xi| \int_{\mathbb{R}^n} \hat{u}(\xi - \eta, \tau)\hat{v}(\eta, \tau)\, d\eta\, d\xi\, d\tau \\
&\leqslant \int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \big(|\xi - \eta| + |\eta|\big)\hat{u}(\xi - \eta, \tau)\hat{v}(\eta, \tau)\, d\xi\, d\eta\, d\tau \\
&\leqslant \big\|\xi \hat{u}(\xi)\big\|_{L^2_T(L^1)} \|\hat{v}\|_{L^2_T(L^1)} + \big\|\hat{u}(\xi)\big\|_{L^2_T(L^1)} \big\|\eta \hat{v}(\eta)\big\|_{L^2_T(L^1)},
\end{aligned}$$

which finishes the proof of Lemma 3.6. □

Lemma 3.7. *The following inequality is valid*:

$$\left\| \int_0^t \mathcal{G}(t - \tau) f(\tau)\, d\tau \right\|_{L^2_T(\dot{FB}^s_{1,1})} \leqslant \frac{1}{\nu} \|f\|_{L^1_T(\dot{FB}^{s-1}_{1,1})}. \tag{3.19}$$

Proof. As previously we use triangle and Young's inequality to obtain

$$\begin{aligned}
&\left\| \int_0^t \mathcal{G}(t - \tau) f(\tau)\, d\tau \right\|_{L^2_T(\dot{FB}^s_{1,1})} \\
&\quad \leqslant \left\| \sum_k \int_0^t e^{(t-\tau)2^{2k}} 2^{sk} \big\|\varphi_k f(\tau)\big\|_{L^1}\, d\tau \right\|_{L^2_T} \leqslant \sum_k \big\|e^{t2^{2k}}\big\|_{L^2_T} 2^{sk} \big\|\varphi_k f(\tau)\big\|_{L^1_T(L^1)} \\
&\quad = \sum_k 2^{(s-1)k} \big\|\varphi_k f(\tau)\big\|_{L^1_T(L^1)} = \|f\|_{L^1_T(\dot{FB}^{s-1}_{1,1})}. \qquad \square
\end{aligned}$$

In what follows we focus on estimates for the space $L^\infty_T(\dot{FB}^{2-3/p}_{p,p})$, where $p>3$.

Lemma 3.8. *The following estimate is valid*:

$$\left\| \int_0^t \mathcal{G}(t-\tau)\nabla(u\otimes v)\,d\tau \right\|_{L^\infty_T(\dot{FB}^{2-3/p}_{p,p})} \leqslant \|u\|_{L^\infty_T(\dot{FB}^{2-3/p}_{p,p})}\|v\|_{L^\infty_T(\dot{FB}^{2-3/p}_{p,p})}. \tag{3.20}$$

Proof. First let us estimate the convolution $\hat{u}*\hat{v}$. We do this as follows:

$$\begin{aligned}\left|\hat{u}*\hat{v}(\xi)\right| &\leqslant \int_{\mathbb{R}^3} \frac{1}{|\eta|^{2-3/p}}\frac{1}{|\xi-\eta|^{2-3/p}}|\xi-\eta|^{2-3/p}\left|\hat{v}(\xi-\eta)\right|\left|\eta\right|^{2-3/p}\left|\hat{u}(\eta)\right|d\eta \\ &\leqslant \left(\int_{\mathbb{R}^3}\left(\frac{1}{|\eta|^{2-3/p}}\frac{1}{|\xi-\eta|^{2-3/p}}\right)^{p'}d\eta\right)^{1/p'} \\ &\quad\cdot\left(\int_{\mathbb{R}^3}\left(|\xi-\eta|^{2-3/p}\left|\hat{v}(\xi-\eta)\right|\left|\eta\right|^{2-3/p}\left|\hat{u}(\eta)\right|\right)^p d\eta\right)^{1/p}.\end{aligned}$$

Now in order to estimate the convolution $\frac{1}{|\xi|^{(2-3/p)\bar{p}}} * \frac{1}{|\xi|^{(2-3/p)p'}}$ we use the well-known fact

$$\mathcal{F}\left(|\xi|^{-\alpha}\right)(x) = C_{\alpha,n}|x|^{\alpha-n}, \tag{3.21}$$

for $0<\alpha<n$. Taking the Fourier transform of this convolution we get

$$\mathcal{F}\left(\frac{1}{|\xi|^{(2-3/p)p'}} * \frac{1}{|\xi|^{(2-3/p)p'}}\right)(x) = C_\alpha |x|^{2[(2-3/p)p'-3]}.$$

Now using inverse Fourier transform we get

$$\frac{1}{|\xi|^{(2-3/p)p'}} * \frac{1}{|\xi|^{(2-3/p)p'}} \sim |\xi|^{-2[(2-3/p)p'-3]-3} = |\xi|^{-2p'(2-3/p)+3}.$$

This formula holds for $p>3$ (in dimension 3) in order to satisfy (two times) condition for validity of (3.21). We thus obtained the following formula:

$$\left(\int_{\mathbb{R}^3}\left(\frac{1}{|\eta|^{2-3/p}}\frac{1}{|\xi-\eta|^{2-3/p}}\right)^{p'}d\eta\right)^{1/p'} \sim |\xi|^{-2(2-3/p)+3/p'}. \tag{3.22}$$

Going back to our main estimate:

$$\begin{aligned}&\left\|\int_0^t \mathcal{G}(t-\tau)\nabla(uv)\,d\tau\right\|_{L^\infty_T(\dot{FB}^{2-3/p}_{p,p})} \\ &\quad\leqslant \sup_t\left(\int_{\mathbb{R}^3}|\xi|^{(2-3/p)p}\left(\int_0^t e^{\tau\xi^2}\,d\tau\right)^p |\xi|^p\left|\hat{u}*\hat{v}(\xi)\right|^p d\xi\right)^{1/p}\end{aligned}$$

$$\leqslant \sup_t \left(\int_{\mathbb{R}^3} |\xi|^A |\xi|^B \left(\int_{\mathbb{R}^3} \left(|\eta|^{2-3/p} \hat{v}(\eta) |\xi-\eta|^{2-3/p} \hat{u}(\xi-\eta)\right)^p d\eta \right)^{p/p} d\xi \right)^{1/p},$$

where $A = 2p - 3 - 2p + p$ and $B = [-2(2-3/p) + 3/p'] \cdot p$.

It is not hard to notice that $A + B = 0$ and thus the proof of the lemma follows easily from integration of the last term first with respect to ξ and then η. $\square$

4. Proofs of main results

4.1. Proof of Theorem 2.2

We use a rather standard approach to show existence, namely via the following Banach fixed point theorem [5]:

Lemma 4.1. *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space and $B : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ a bounded bilinear form satisfying $\|B(x_1, x_2)\|_{\mathcal{X}} \leqslant \eta \|x_1\|_{\mathcal{X}} \|x_2\|_{\mathcal{X}}$ for all $x_1, x_2 \in \mathcal{X}$ and a constant $\eta > 0$. Then if $0 < \epsilon < 1/(4\eta)$ and if $y \in \mathcal{X}$ such that $\|y\|_{\mathcal{X}} < \epsilon$, the equation $x = y + B(x, x)$ has a solution in $\mathcal{X}$ such that $\|x\|_{\mathcal{X}} \leqslant 2\epsilon$. This solution is the only one in the ball $\overline{B}(0, 2\epsilon)$. Moreover, the solution depends continuously on y in the following sense: if $\|\tilde{y}\|_{\mathcal{X}} \leqslant \epsilon$, $\tilde{x} = \tilde{y} + B(\tilde{x}, \tilde{x})$, and $\|\tilde{x}\|_{\mathcal{X}} \leqslant 2\epsilon$ then*

$$\|x - \tilde{x}\|_{\mathcal{X}} \leqslant \frac{1}{1 - 4\eta\epsilon} \|y - \tilde{y}\|_{\mathcal{X}}.$$

In our case the bilinear form B is defined as follows:

$$B(u, v)(t) = -\int_0^t \mathcal{G}(t-\tau) \mathbf{P} \operatorname{div}(u \otimes v)\, d\tau, \tag{4.1}$$

where $\mathcal{G}$ is the generator of the Stokes–Coriolis semigroup introduced earlier. It is now straightforward that, in order to prove the existence of solutions, we have to prove corresponding estimates in all spaces under considerations.

- In case $X_0 = \dot{FB}^{2-3/p}_{p,\infty}$, where $3 < p \leqslant \infty$ we use Lemma 3.3 with $r = \infty$ to get

$$\left\| \int_0^t \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{L^\infty_T(\dot{FB}^s_{p,\infty})} \leqslant \frac{1}{\nu} \|f\|_{L^\infty_T(\dot{FB}^{s-2}_{p,\infty})} \tag{4.2}$$

  and then for $f = \operatorname{div}(u \otimes v)$ we use inequality (3.10).
  An estimate for the convolution with initial data u_0 comes from Lemma 3.2.
- In the case $X_0 = \dot{FB}^{2-3/p}_{p,p}$, where $3 < p < \infty$, we use Lemma 3.8 to estimate the bilinear form. The initial datum u_0 is estimated trivially.
- In the case $X_0 = \dot{FB}^{-1}_{1,1} \cap \dot{FB}^0_{1,1}$, we make two steps. First we use Lemma 3.7 with $s = 0$ combined with Lemma 3.6 (inequality (3.17)) to estimate bilinear form $B(u, v)$ in the space $L^2([0,\infty); \dot{FB}^0_{1,1})$ and Lemma 3.5 with $s = 0$ to estimate initial data u_0. This gives us the unique solution in the space $L^2([0,\infty); \dot{FB}^0_{1,1})$. In the second step we notice that using inequality (3.18) and again Lemma 3.7 with $s = 1$ we obtain that the solution is in fact in the space $L^2([0,\infty); \dot{FB}^1_{1,1} \cap \dot{FB}^0_{1,1})$. This improved regularity is essential to show (in an elementary way) strong continuity of the solution, i.e. $u \in C([0,\infty); \dot{FB}^0_{1,1})$.

To prove the second part of Theorem 2.2, that is for $1 < p \leqslant \infty$ one uses the same results as in the case $3 < p \leqslant \infty$ but with estimate (3.9). Since these cases are of less interest to us (our paper focuses on the stationary case) we do not include more details in order to keep the paper more consistent.

4.2. Proof of Theorem 2.4

To prove existence results in the stationary case one may use the results from Theorem 2.2 in case $3 < p \leqslant \infty$ and $X = \dot{FB}^{2-3/p}_{p,\infty}$ or $X = \dot{FB}^{2-3/p}_{p,p}$ and repeat reasoning from the paper by Cannone and Karch [5]. The authors there use the following lemma which is essential to obtain this result:

Proposition 4.2. *The following two facts are equivalent*:

- $u = u(x)$ *is a stationary mild solution to the problem* (1.1)–(1.2), *that is*

$$u = \mathcal{G}(t)u - \int_0^t \mathcal{G}(t-\tau)\mathbf{P}\operatorname{div}(u \otimes u)\,d\tau + \int_0^t \mathcal{G}(\tau)\mathbf{P}F\,d\tau \tag{4.3}$$

  for every $t > 0$.
- $u = u(x)$ *satisfies the integral equation*

$$u = -\int_0^\infty \mathcal{G}(\tau)\mathbf{P}\operatorname{div}(u \otimes u)\,d\tau + \int_0^\infty \mathcal{G}(\tau)\mathbf{P}F\,d\tau, \tag{4.4}$$

  where $\mathbf{P}$ *is the Helmholtz projection.*

Using this proposition and results for non-stationary case we see that in order to obtain existence of solution using a fixed point argument we just need to obtain estimates for the term with the force F. We use the formula for the Stokes–Coriolis semigroup (3.4), that is

$$\hat{\mathcal{G}}(t) = \cos\left(\Omega \frac{\xi_3}{|\xi|} t\right) e^{-\nu|\xi|^2 t} I + \sin\left(\Omega \frac{\xi_3}{|\xi|} t\right) e^{-\nu|\xi|^2 t} R(\xi). \tag{4.5}$$

Integrating this formula with respect to t from 0 to ∞ we get

$$\int_0^\infty \hat{\mathcal{G}}(t)\,dt = \frac{|\xi|^4}{|\xi|^6 + \xi_3^2 \Omega^2} I + \frac{\xi_3 |\xi|}{|\xi|^6 + \xi_3^2 \Omega^2} R(\xi). \tag{4.6}$$

It is then straightforward (from the definition of $X^p_{\mathcal{C},\Omega}$) that

$$\left\| \int_0^\infty \mathcal{G}F\,dt \right\|_{\dot{FB}^{2-3/p}_{p,p}} \leqslant \|F\|_{X^p_{\mathcal{C},\Omega}}. \tag{4.7}$$

4.3. Proof of Proposition 2.5

First, we will show that for each $F \in \dot{FB}^{-3/p}_{p,p}$ and for all $\epsilon > 0$ there exists Ω_0 such that for all $|\Omega| \geqslant \Omega_0$,

$$\|F\|_{X^p_{C,\Omega}} \leqslant \epsilon \|F\|_{\dot{FB}^{-3/p}_{p,p}}. \tag{4.8}$$

This fact together with Theorem 2.4 proves the proposition.

First we have $\dot{FB}^{-3/p}_{p,p} \subset X^p_{C,\Omega}$. This is a simple observation since

$$\frac{|\xi|^4}{|\xi|^6 + \Omega^2|\xi_3|^2} = \int_0^\infty e^{-t|\xi|^2} \cos\left(\Omega \frac{\xi_3}{|\xi|} t\right) dt \leqslant \int_0^\infty e^{-t|\xi|^2}\, dt = |\xi|^{-2} \tag{4.9}$$

and

$$\frac{\Omega \xi_3 |\xi|}{|\xi|^6 + \Omega^2|\xi_3|^2} = \int_0^\infty e^{-t|\xi|^2} \sin\left(\Omega \frac{\xi_3}{|\xi|} t\right) dt \leqslant \int_0^\infty e^{-t|\xi|^2}\, dt = |\xi|^{-2}. \tag{4.10}$$

The proof of (4.8) is fairly simple. First, we decompose $\mathbb{R}^3$ into three regions: $\mathbb{R}^3 = A_\delta + B_\delta + C_\delta$, where $A_\delta = \{\xi\colon\ |\xi_3| > \delta \text{ and } \delta < |\xi| < \frac{1}{\delta}\}$, $B_\delta = \{\xi\colon\ |\xi_3| > \delta \text{ and } |\xi| > \frac{1}{\delta}\}$ and $C_\delta = \{\xi\colon\ |\xi_3| < \delta\}$.

For fixed F there exists a compact set $K \subset \mathbb{R}^3$ such that $\||\xi|^{-3/p} F\|_{L^p(\mathbb{R}^3 \setminus K)} \leqslant \epsilon/3$ and by (4.9) and (4.10) we have the following estimates, uniform with respect to Ω:

$$\left(\frac{|\xi|^{6-3/p}}{|\xi|^6 + \Omega^2|\xi_3|^2} |\hat{F}|\right)^p \leqslant \left(|\xi|^{-3/p} |\hat{F}|\right)^p \tag{4.11}$$

and

$$\left(\frac{\Omega \xi_3 |\xi|^{3-3/p}}{|\xi|^6 + \Omega^2|\xi_3|^2} \big|R(\xi)\big| |\hat{F}|\right)^p \leqslant \left(|\xi|^{-3/p} |\hat{F}|\right)^p. \tag{4.12}$$

From the definition of B_δ and C_δ we get that $|K \cap (B_\delta \cup C_\delta)| \to 0$ as $\delta \to 0$, hence for δ small enough we have $\|F\|_{X^p_{C,\Omega}(K \cap (B_\delta \cup C_\delta))} \leqslant \epsilon/2$. Once δ is fixed we get back to the integral over $K \cap A_\delta$:

$$\left(\int_{X^p_{C,\Omega}(K \cap A_\delta)} \left(\frac{|\xi|^{6-3/p}}{|\xi|^6 + \Omega^2|\xi_3|^2} |\hat{F}|\right)^p d\xi\right)^{1/p} \leqslant \frac{(1/\delta)^6}{\delta^6 + \Omega^2\delta^2} \|F\|_{\dot{FB}^{-3/p}_{p,p}} \leqslant \epsilon/4, \tag{4.13}$$

for Ω large enough (depending on ϵ, δ and $\|F\|$). Similarly, we have

$$\left(\int_{X^p_{C,\Omega}(K \cap A_\delta)} \left(\frac{\Omega \xi_3 |\xi|^{3-3/p}}{|\xi|^6 + \Omega^2|\xi_3|^2} |\hat{F}|\right)^p d\xi\right)^{1/p} \leqslant \frac{\Omega (1/\delta)^4}{\delta^6 + \Omega^2\delta^2} \|F\|_{\dot{FB}^{-3/p}_{p,p}} \leqslant \epsilon/4, \tag{4.14}$$

for Ω large enough. This completes the proof.

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