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Distance distributions for graphs modeling computer networks

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Abstract

The Wiener polynomial of a graph G is a generating function for the distance distribution $dd(G) = (D_1, D_2, \ldots, D_t)$, where D_i is the number of unordered pairs of distinct vertices at distance i from one another and t is the diameter of G. We use the Wiener polynomial and several related generating functions to obtain generating functions for distance distributions of unweighted and weighted graphs that model certain large classes of computer networks. These provide a straightforward means of computing distance and timing statistics when designing new networks or enlarging existing networks.

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1. Introduction

Distance is an important concept in applications of graph theory to computer science, chemistry, and a variety of other fields. Indeed, the literature on the concept is so rich that Buckley and Harary [2] have an entire book dedicated to it. In this paper we explore generating functions for the distance distributions of graphs representing computer networks, although the methods and results may be translated to any setting requiring distance analysis in a network.

If a graph is being used to model a computer network, the distance (or weighted distance) between vertices represents the time required for the corresponding processors to communicate with each other. This concept is useful when estimating timing statistics associated with distributed computing. The statistics generated assume a uniform distribution of message length and frequency, but the results obtained can be useful even when these conditions are only approximately satisfied.

In Sections 2 and 3 of this article, we model several useful classes of computer networks with unweighted graphs and show how to use polynomials to both generate the distance distributions and compute corresponding distance/timing statistics; these network classes arise naturally in distributed computing settings. In Section 4, we introduce and discuss generating functions for weighted distance distributions in weighted graph models of computer networks; these have the form $\sum_{\lambda\in\Lambda} a_\lambda q^\lambda$ where the exponents in the set Λ need not be integers. In both situations, we show how to generate

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the appropriate generating functions for classes of networks with varying degrees of subnetwork homogeneity. We also show how to "update" the distance distribution generating functions when a new subnetwork is to be added to an existing network. Section 5 contains concluding remarks. The remainder of this introductory section is devoted to a discussion of the history, definitions and observations relevant to our work.

The use of polynomials to generate distance distributions for graphs was first suggested in Hosoya's paper [6] on various counting polynomials in chemistry. Sagan, Yeh and Zhang independently introduced similar polynomials in [8] and explored them in some detail. The inspiration for both papers was work done by Wiener [10] regarding thermodynamic properties of saturated hydrocarbon molecules.

We begin with an introduction and discussion of some terminology and notation, and refer the reader to [2,3] for terminology not defined herein. All graphs considered in this work are undirected and finite. For an arbitrary pair of vertices u and v in a connected graph G, the *distance* from u to v is the length of a shortest path (i.e., a *geodesic*) from u to v. We use $d(u, v)$ to denote the *distance* from u to v. For a connected graph G, Wiener [10] defined the parameter $W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u, v)$, since called the *Wiener index* of G; Wiener, who was using graphs to model various hydrocarbon molecules, used this parameter as a measure of "molecular compactness" and a predictor of boiling points in different saturated paraffins. More recent results concerning the Wiener index can be found in [1,4,7,9,11]. In [8], Sagan, Yeh and Zhang, defined the *Wiener polynomial* of a connected graph G, in terms of a parameter q, to be

$$W(G; q) = \sum_{\{u,v\}\subseteq V(G)} q^{d(u,v)},$$

where the sum is over all unordered pairs $\{u, v\}$ of distinct vertices in G. This is a slight variation of the Wiener polynomial introduced previously by Hosoya [6], and studied by Dobrynin [5]; Hosoya's version includes the constant term $|V(G)|$, which counts the number of "pairs of vertices at distance 0 from each other." Sagan et al. [8] pointed out that $W(G) = W'(G; 1)$, where $W'(G; 1)$ is the derivative of $W(G; q)$ with respect to q, evaluated at $q = 1$. Recalling that the *distance distribution* of a connected graph G of diameter t is the sequence $dd(G) = (D_1, D_2, \dots, D_t)$, where D_i is the number of pairs of vertices at distance i from one another, we see that the Wiener polynomial of [10] is the generating function for $dd(G)$. Taking a suggestion of Andreas Blass, Sagan et al. [8] found it useful to introduce the *ordered Wiener polynomial*

$$\overline{W}(G; q) = \sum_{(u,v)\in V(G)\times V(G)} q^{d(u,v)},$$

where, in contrast to the ordinary Wiener polynomial, the sum is over all ordered pairs (u, v) of not necessarily distinct vertices; the constant term in this polynomial is $|V(G)|$, the sum of all terms of the form $q^{d(u,u)}$. This polynomial is a generating function for what might be called an ordered distance frequency distribution; i.e., the coefficient of q^k in $\overline{W}(G; q)$ is the number the ordered pairs of vertices at distance k from each other. The Wiener polynomial and the ordered Wiener polynomial are related by the equation

$$\overline{W}(G; q) = 2W(G; q) + |V(G)|, \tag{1}$$

which allows one to compute either of the two polynomials if the other is known. For a variety of classes of graphs, Sagan et al. [8] computed these polynomials.

We will frequently use another variant on the Wiener polynomial that was briefly mentioned in [8]: if u is a vertex of a connected graph G, the *Wiener polynomial* of G *relative to* u is

$$W_u(G; q) = \sum_{v\in V(G)} q^{d(u,v)}.$$

We will refer to such a polynomial as a *relative Wiener polynomial*. Since the sum defining a relative Wiener polynomial includes one term with $v = u$, the constant term for every relative Wiener polynomial is 1. Recalling that the *distance degree sequence of a vertex* u is the sequence $dds(u) = (d_0(u), d_1(u), \dots, d_{e(u)}(u))$, where $d_i(u)$ is the number of vertices at distance i from u, and $e(u)$ (the *eccentricity* of u) is the maximum distance from u to any other vertex in G, we see that $W_u(G; q)$ is the generating function for $dds(u)$. Distance degree sequences, and therefore, relative

Wiener polynomials, can be easily found for specific graphs by employing breadth-first searches of the graphs. If we vary u over all vertices of G, we obtain the following relationship which is useful in determining the ordered Wiener polynomial, and thus the Wiener polynomial, of a specified graph.

$$\overline{W}(G;q)=\sum_{u\in V(G)} W_u(G;q).$$

If G is a distance degree regular graph (i.e., if all vertices have the same distance degree sequence), and r is any vertex of G, then this last equation reduces to

$$\overline{W}(G;q)=|V(G)|\cdot W_r(G;q).$$

For example, if Q_n is the n-cube, then $\overline{W}(Q_n;q)=2^n\sum_{k=0}^{n}\binom{n}{k}q^k=2^n(1+q)^n$.

Since the polynomials defined above are generating functions, we can use standard elementary probability arguments to obtain several facts about distance statistics for connected graphs. In the following, we use $W'(G;1)$, $W''(G;1)$, $\overline{W}'(G;1)$, $\overline{W''}(G;1)$, $W_u'(G;1)$, and $W_u''(G;1)$ to represent the first and second derivatives of W, $\overline{W}$, and W_u with respect to q, evaluated with $q=1$. Recall that $W'(G;1)=\sum_{\{u,v\}\subseteq V(G)}d(u,v)(=W(G))$, and observe that $W''(G;1)=\sum_{\{u,v\}\subseteq V(G)}(d(u,v))^2-\sum_{\{u,v\}\subseteq V(G)}d(u,v)$; similar expressions exist for $\overline{W'}(G;1)$ and $\overline{W''}(G;1)$. Using these observations and the standard definitions of mean and variance, we obtain the following:

Fact 1. *Let G be a connected graph having p vertices, and assume that the probability of selecting any unordered pair of distinct vertices in G is $1/\binom{p}{2}$. Then:*

(i) *the mean distance between pairs of distinct vertices in G is*

$$\mu=\frac{W'(G;1)}{\binom{p}{2}}=\frac{\overline{W'}(G;1)}{p(p-1)}\quad \textit{and}$$

(ii) *the variance of the distance distribution of G is*

$$\sigma^2=\frac{W''(G;1)}{\binom{p}{2}}+\mu-\mu^2=\frac{\overline{W''}(G;1)}{p(p-1)}+\mu-\mu^2.$$

Fact 2. *Let u be a vertex of a connected graph G having p vertices, and assume that $1/(p-1)$ is the probability of selecting any vertex of G other than u. Then:*

(i) *the mean distance from u to any other vertex of G is*

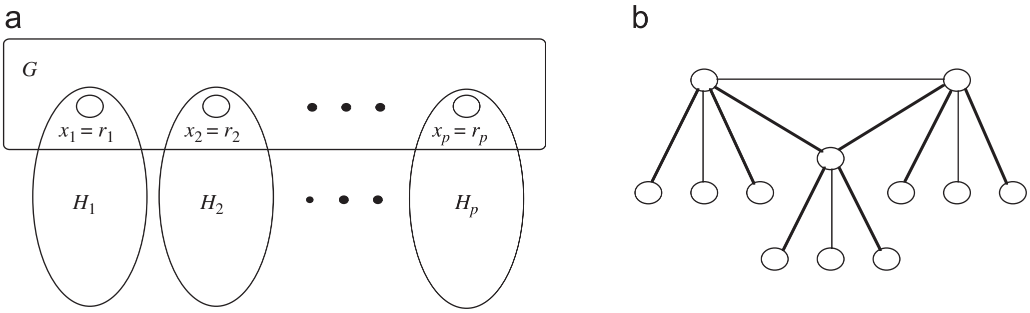
$$\mu_u=\frac{W_u'(G;1)}{p-1}\quad \textit{and}$$

(ii) *the variance of the distribution of distances from u to distinct vertices of G is*

$$\sigma_u^2=\frac{W_u''(G;1)}{p-1}+\mu_u-\mu_u^2.$$

2. Homogeneous articulated networks

By an *articulated network*, we mean a computer network consisting of a so-called *backbone* network and a collection of peripheral networks we call *perinets*. Each perinet has a designated *gateway* processor and either it shares its gateway processor (and no other processor) with the backbone network or it shares no processor with the backbone network but



Art[*G*, (*H*,*r*)] *Art*[C_3, (*K*(1,3),*r*)]

Fig. 1.

its gateway processor is linked to exactly one processor in the backbone. Many computer networks have this form, and many can be viewed as having been recursively constructed by making an initial network, taking that as the backbone of a new articulated network, taking the new articulated network as the backbone of yet another articulated network, etc. If the processors in the perinets of an articulated network are identical and connected in the same network topology, and either each processor of the backbone is the gateway of exactly one perinet or each processor of the backbone is linked to the gateway processor of exactly one perinet, then we say that the network is a *homogeneous* articulated network. In this section, we introduce two graph products that model homogeneous articulated networks and show how to calculate their Wiener polynomials.

If G and H are graphs and r is a specified root vertex of H, the *articulated product* $Art[G, (H, r)]$ is the graph having vertex set $V(G) \times V(H)$ and satisfying the property that two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either (i) $u_1v_1 \in E(G)$ and $u_2 = v_2 = r$, or (ii) $u_1 = v_1$ and $u_2v_2 \in E(H)$. This graph is a spanning subgraph of the cartesian product $G \times H$. While this definition provides a rationale for calling $Art[G, (H, r)]$ a graph product, there is an alternate definition that is easier to visualize and allows for simpler notation when writing proofs: Suppose $V(G)=\{x_1, x_2, \ldots, x_p\}$. For each $i=1, 2, \ldots, p$, let H_i be isomorphic to H, and let r_i be the vertex of H_i corresponding to r. Let $Art[G, (H, r)]$ be the graph obtained by identifying r_i and x_i, for $i = 1, 2, \ldots, p$; see Fig. 1a. In essence, $Art[G, (H, r)]$ is obtained by attaching $|V(G)|$ copies of H to a single copy of G by identifying each vertex of G with the root r of one of the copies of H. Fig. 1b shows the articulated product $Art[C_3, (K(1, 3), r)]$ of a 3-cycle with the star $K(1, 3)$; where r is the center of the star. One nice property of the articulated product $Art[G, (H, r)]$ is that it is automorphism group contains a subgroup isomorphic to that of G, so if G possesses a high degree of symmetry, then so does $Art[G, (H, r)]$. (If G has order p and automorphism group $aut(G)$, and if $stab(r)$ is the subgroup of $aut(H)$ consisting of those automorphisms of H that leave r fixed, then the automorphism group of $Art[G, (H, r)]$ is isomorphic to the direct product $aut(G) \times [stab(r)]^p$.)

Our first theorem provides a means for computing the Wiener polynomial of an articulated product.

Theorem 1. *If G and H are connected nontrivial graphs, and r is a vertex of H, then*

$$W(Art[G, (H, r)]; q) = W(G; q)[W_r(H; q)]^2 + |V(G)|W(H; q).$$

Equivalently,

$$\overline{W}(Art[G, (H, r)]; q) = (\overline{W}(G; q) - |V(G)|)[W_r(H; q)]^2 + |V(G)|\overline{W}(H; q).$$

Proof. Suppose $V(G) = \{x_1, x_2, \ldots, x_p\}$. For each $i = 1, 2, \ldots, p$, let H_i be isomorphic to H, and let r_i be the vertex of H_i corresponding to r. Following our alternate definition of the articulated product, let $Art[G, (H, r)]$ be the graph obtained by identifying r_i and x_i, for $i = 1, 2, \ldots, p$. In the derivation that follows, we will take $r_i = x_i$ and let V_i be the vertex set of H_i, for each i. We will use the fact that $\{V_i | i = 1, 2, \ldots, p\}$ is a partition of $V(Art[G, (H, r)])$. We will also break the sum defining $W(Art[G, (H, r)]; q)$ into two sums depending on whether the vertices in a 2-set $\{u, v\}$ of vertices belong to the same copy of H in $Art[G, (H, r)]$ or to different copies. It is important to observe that

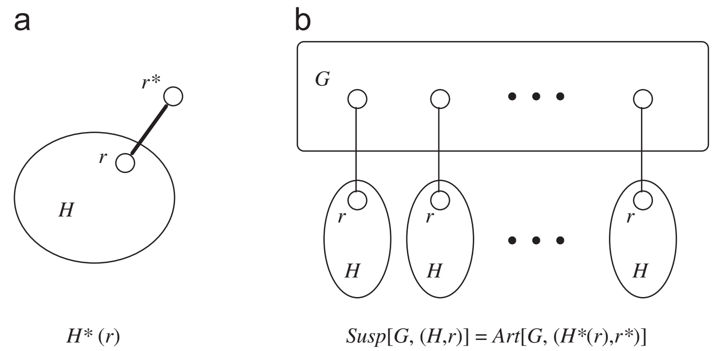


Fig. 2.

if $u \in V_i$ and $v \in V_j$, with $i \neq j$, then each u–v geodesic in $Art[G, (H, r)]$ consists of a u–r_i geodesic in H_i, followed by an x_i–x_j geodesic in G, followed by an r_j–v geodesic in H_j. Furthermore, we note that $W(H_i; q) = W(H; q)$ and $W_{r_i}(H_i; q) = W_r(H; q)$, for each i. With these observations, we see that

$$\begin{aligned}
W(Art[G, (H, r)]; q) &= \sum_{\{u,v\}\subseteq V(Art[G,(H,r)])} q^{d(u,v)} \\
&= \sum_{i=1}^{p} \sum_{\{u,v\}\subseteq V_i} q^{d(u,v)} + \sum_{1\leqslant i<j\leqslant p} \sum_{u\in V_i} \sum_{v\in V_j} q^{d(u,v)} \\
&= \sum_{i=1}^{p} W(H_i; q) + \sum_{1\leqslant i<j\leqslant p} \sum_{u\in V_i} \sum_{v\in V_j} q^{d(u,r_i)+d(x_i,x_j)+d(r_j,v)} \\
&= pW(H; q) + \sum_{1\leqslant i<j\leqslant p} q^{d(x_i,x_j)} \sum_{u\in V_i} q^{d(u,r_i)} \sum_{v\in V_j} q^{d(r_j,v)} \\
&= pW(H; q) + W(G; q)[W_r(H; q)]^2,
\end{aligned}$$

as claimed. Using this result and identity (1) given in the introduction, we obtain the equation $\overline{W}(Art[G, (H, r)]; q) = (\overline{W}(G; q) - |V(G)|)[W_r(H; q)]^2 + |V(G)| \cdot \overline{W}(H; q)$. $\square$

We now consider a restricted class of articulated products. Let H be a graph and let r be a specified vertex of H. Let $H^*(r)$ denote the graph obtained by adding a single vertex r^* to H together with an edge joining r^* to r; see Fig. 2a. We define the *suspended product* $Susp[G, (H, r)]$ to be the articulated product $Art[G, (H^*(r), r^*)]$; see Fig. 2b. The suspended product corresponds to a network having a backbone modeled by the graph G, and $|V(G)|$ identical perinets each modeled by the graph H and having a gateway processor corresponding to the vertex r. Each processor of the backbone network is connected by a single link to the gateway processor of exactly one perinet and each perinet's gateway processor is linked to exactly one processor in the backbone network. To find the ordered Wiener polynomial of $Susp[G, (H, r)](=Art[G, (H^*(r), r^*)])$, we may find the ordered Wiener polynomial of the graph $H^*(r)$ and the Wiener polynomial of $H^*(r)$ relative to the vertex r^* and apply Theorem 1.

Lemma 1. *If r is a specified vertex of a connected graph H, and if r^* and $H^*(r)$ are as defined in the preceding paragraph, then*

(i) $W_{r^*}(H^*(r); q) = 1 + qW_r(H; q)$, *and*
(ii) $W(H^*(r); q) = W(H; q) + qW_r(H; q)$, *or equivalently*, $\overline{W}(H^*(r); q) = \overline{W}(H; q) + 2qW_r(H; q) + 1$.

Proof. We first establish Eq. (i). For each vertex v different from r^*, an r^*–v geodesic consists of the path of length one from r^* to r followed by an r–v geodesic in H, so $d(r^*, v) = 1 + d(r, v)$. Thus,

$$\begin{aligned} W_{r^*}(H^*(r); q) &= \sum_{v \in V(H^*(r))} q^{d(r^*,v)} \\ &= q^{d(r^*,r^*)} + \sum_{v \in V(H)} q^{d(r^*,v)} \\ &= 1 + \sum_{v \in V(H)} q^{1+d(r,v)} \\ &= 1 + q \sum_{v \in V(H)} q^{d(r,v)} \\ &= 1 + qW_r(H; q). \end{aligned}$$

The key idea in the following derivation of the first statement of (ii) is that the set of 2-element subsets of $V(H^*(r))$ can be partitioned into those 2-sets containing r^* and those that do not contain r^*. We also use the statement (i) established above:

$$\begin{aligned} W(H^*(r); q) &= \sum_{\{u,v\} \subseteq V(H^*(r))} q^{d(u,v)} \\ &= \sum_{\{u,v\} \subseteq V(H)} q^{d(u,v)} + \sum_{v \in V(H)} q^{d(r^*,v)} \\ &= W(H; q) + [W_{r^*}(H^*(r); q) - q^{d(r^*,r^*)}] \\ &= W(H; q) + qW_r(H; q). \end{aligned}$$

This proves the first equation of (ii). By applying Eq. (1) from the introduction to this equation, one may derive $\overline{W}(H^*(r); q) = \overline{W}(H; q) + 2qW_r(H; q) + 1$ in a straightforward manner. $\square$

Using Theorem 1, Lemma 1, and the fact that $Susp[G, (H, r)] = Art[G, (H^*(r), r^*)]$, we get the following result.

Theorem 2. *If G is a connected graph and r is a specified vertex of a connected graph H, then*

$$W(Susp[G, (H, r)]; q) = W(G; q)(1 + qW_r(H; q))^2 + |V(G)|(W(H; q) + qW_r(H; q)),$$

and

$$\overline{W}(Susp[G, (H, r)]; q) = \overline{W}(G; q)(1 + qW_r(H; q))^2 + |V(G)|(\overline{W}(H; q) - q^2(W_r(H; q))^2).$$

To illustrate the utility of these theorems, we will compute the mean distances between vertices in some examples of suspended products formed using several graphs often encountered as models of local area computer networks. Wiener polynomials of the graphs used to form these suspended products are given in the following lemma which we state without proof. Some parts of this lemma are results that are either stated in [8] or follow immediately from results in [8]; the remaining statements are not difficult to verify. In Lemma 2 and Corollary 1, we use P_n, Q_n and $K_{1,n}$ to represent, respectively, the path with n vertices, the n-dimensional hypercube, and the star with n leaves; $P_m \times P_n$ (denoted by some authors as $P_m \Box P_n$) is the cartesian product of the paths P_m and P_n and will be referred to as the $m \times n$ grid. Also, in statement 1 of Lemma 2, we use the standard q-analog of n which is $[n] = 1 + q + \cdots + q^{n-1}$.

Lemma 2. *For positive integers m and n*:

1. $W(P_n; q) = (q/(1-q))(n-[n])$, $\overline{W}(P_n; q) = n + (2q/(1-q))(n-[n])$, *and if r is an end-vertex of* P_n, *then* $W_r(P_n; q) = [n]$;
2. $\overline{W}(P_m \times P_n; q) = \overline{W}(P_m; q)\overline{W}(P_n; q)$;
3. $W(Q_n; q) = 2^{n-1}((1+q)^n - 1)$, $\overline{W}(Q_n; q) = 2^n(1+q)^n$, *and if r is any vertex of* Q_n, *then* $W_r(Q_n; q) = (1+q)^n$;
4. $W(K_{1,n}; q) = nq + (n(n-1)/2)q^2$ *and, if r is the center of the star* $K_{1,n}$, *then* $W_r(K_{1,n}; q) = 1 + nq$.

Using Theorems 1, 2, Fact 1, Lemma 2, and basic features of the *Mathematica* software package in a straightforward way, we obtained the Wiener polynomials and the mean distances between vertices for several representative examples of suspended products that model homogeneous articulated networks. The means are given in the following corollary.

Corollary 1. *In each of the following statements,* μ *represents the mean distance between pairs of distinct vertices in the specified graph*:

(a) *For the graph* $Susp[P_m, (Q_n, r)]$, *where r is any specified vertex of* Q_n,

$$\mu = \frac{-2(1+2^{1+n}+7*4^n)+2(1+2^n)^2*m^2-3n4^n+6m(2^n+4^n)(2+n)}{6(1+2^n)(-1+m+m*2^n)}.$$

(b) *For* $Susp[P_m, (P_n, r)]$, *where r is an end-vertex of* P_n,

$$\mu = \frac{-1-2n-2n^2+m^2(1+n)+3mn(1+n)}{3(-1+m+mn)}.$$

(c) *For* $Susp[P_m, (K_{1,n}, r)]$, *where r is the center of* $K_{1,n}$,

$$\mu = \frac{-10-22n-7n^2+m^2(2+n)^2+6m(2+5n+2n^2)}{3(2+n)(-1+m(2+n))}.$$

(d) *For* $Susp[P_m \times P_n, (P_s, r)]$, *where r is an end-vertex of* P_s,

$$\mu = -\frac{n+s+ns+2s^2-m^2n(1+s)-m(1+s)(-1+n^2+3ns)}{3(-1+mn(1+s))}.$$

(e) *For* $Susp[P_m \times P_n, (Q_s, r)]$, *where r is an arbitrary specified vertex of* Q_s,

$$\mu = \frac{-2(1+2^s)^2n+2(1+2^s)^2m^2n-3\cdot 4^s(4+s)+2(1+2^s)m(-1-2^s+(1+2^s)n^2+3\cdot 2^s n(2+s))}{6(1+2^s)(-1+(1+2^s)mn)}.$$

(f) *For* $Susp[P_m \times P_n, (K_{1,n}, r)]$, *where r is the center of* $K_{1,n}$,

$$\mu = \frac{-n(2+s)^2+m^2n(2+s)^2-6(1+3s+s^2)+m(2+s)(-2-s+n^2(2+s)+6n(1+2s))}{3(2+s)(-1+mn(2+s))}.$$

(g) *For* $Susp[Q_m, (Q_n, r)]$, *where r is an arbitrary specified vertex of* Q_n,

$$\mu = \frac{2^m(1+2^n)^2m+2(2(-1+2^{m+n}+2^{m+2n})+(-6+2^{m+n}+2^{m+2n})n-2n^2}{2(1+2^n)(-1+2^m+2^{m+n})}.$$

3. Nonhomogeneous articulated networks

As mentioned earlier, networks modeled by articulated products or suspended products are termed homogeneous articulated networks. In these networks the perinets are mutually identical, and either each processor of the backbone is the gateway processor of exactly one perinet or each processor of the backbone is linked to the gateway processor of exactly one perinet. In practice, of course, not all articulated networks are homogeneous. An articulated network need not have a perinet dedicated to each processor in the backbone network and/or different perinets might have different topologies. In this section, we consider these nonhomogeneous articulated networks.

Let G and H be connected graphs, and let r be a specified vertex of H. Furthermore, let $X=\{x_1, x_2, \ldots, x_n\}$ be a nonempty subset of $V(G)$, let $H_1, H_2, \ldots, H_n$ be graphs isomorphic to H, and, for each $i=1,2,\ldots,n$, let r_i be a vertex of H_i that corresponds to r. We define $Paste[(G, X), (H, r)]$ to be the graph obtained by identifying x_i with r_i (as a single vertex) for each $i=1,2,\ldots,n$. Also, we define $Link[(G, X), (H, r)]$ to be the graph obtained by joining x_i to r_i with an edge for each $i=1,2,\ldots,n$. These graphs are subgraphs of $Art[G, (H, r)]$ and $Susp[G, (H, r)]$, respectively, and $Art[G, (H, r)] = Paste[(G, V(G)), (H, r)]$, while $Susp[G, (H, r)] = Link[(G, V(G)), (H, r)]$. Defining r^* and $H^*(r)$ as in the paragraph preceding Lemma 1, we note that $Link[(G, X), (H, r)] = Paste[(G, X), (H^*(r), r^*)]$. To help find the distance distribution for these graphs, we define the *Wiener polynomial of G restricted to X* as

$$W(G|X;q)=\sum_{\{u,v\}\subseteq X} q^{d(u,v)},$$

and the *Wiener polynomial of G relative to X* as

$$W_X(G;q)=\sum_{x\in X} W_x(G;q).$$

In the Wiener polynomial of G restricted to X, the constant term is 0 and the coefficient of q is the number of edges in the subgraph of G induced by X; the exponents $d(u,v)$ are distances in G. The constant term of the Wiener polynomial of G relative to X is $|X|$.

Theorem 3. *Let G and H be connected graphs, let* $X=\{x_1, x_2, \ldots, x_n\}$ *be a nonempty subset of* $V(G)$, *and let r be a specified vertex of H. Then*

$$\begin{aligned} W(Paste[(G, X), (H, r)]) &= W(G;q)+W_X(G;q)[W_r(H;q)-1] \\ &\quad + W(G|X;q)[W_r(H;q)-1]^2 + n[W(H;q)-W_r(H;q)+1] \end{aligned}$$

and

$$W(Link[(G, X), (H, r)];q) = W(G;q)+qW_X(G;q)W_r(H;q)+q^2W(G|X;q)[W_r(H;q)]^2+nW(H;q).$$

Proof. We first obtain the Wiener polynomial of $Paste[(G, X), (H, r)]$. In the following derivation, we use several facts. The first is that for each pair of distinct vertices in $Paste[(G, X), (H, r)]$, exactly one of the following is true: (i) both vertices belong to G; (ii) exactly one vertex of the pair is in G; (iii) for some distinct indices i and j, one vertex is in H_i, the other is in H_j, and neither is in G; (iv) for some index i, both vertices are in H_i and neither is in G. Using this fact, the sum that defines $W(Paste[(G, X), (H, r)]; q)$ will be broken into four sums. In order to transform these four sums into more meaningful expressions, we will use the facts that for each $i=1,2,\ldots,n$, we have $x_i=r_i$, $W(H_i;q)=W(H;q)$ and $W_{r_i}(H_i;q)=W_r(H;q)$. We now proceed with the derivation.

$$\begin{aligned} W(Paste[(G, X), (H, r)];q) &= \sum_{\{u,v\}\subseteq V(Paste[(G,X),(H,r)])} q^{d(u,v)} \\ &= \sum_{\{u,v\}\subseteq V(G)} q^{d(u,v)} + \sum_{i=1}^{n}\sum_{u\in V(G)}\sum_{\substack{v\in V(H_i)\\ v\neq r_i}} q^{d(u,v)} \\ &\quad + \sum_{1\leqslant i<j\leqslant n}\sum_{\substack{u\in V(H_i)\\ u\neq r_i}}\sum_{\substack{v\in V(H_j)\\ v\neq r_j}} q^{d(u,v)} + \sum_{i=1}^{n}\sum_{\{u,v\}\subseteq V(H_i)\setminus\{r_i\}} q^{d(u,v)} \end{aligned}$$

$$
\begin{aligned}
&= W(G;q) + \sum_{i=1}^{n} \sum_{u\in V(G)} \sum_{\substack{v\in V(H_i)\\ v\neq r_i}} q^{d(u,x_i)+d(r_i,v)} \\
&\quad + \sum_{1\leqslant i<j\leqslant n} \sum_{\substack{u\in V(H_i)\\ u\neq r_i}} \sum_{\substack{v\in V(H_j)\\ v\neq r_j}} q^{d(u,r_i)+d(x_i,x_j)+d(r_j,v)} \\
&\quad + \sum_{i=1}^{n} [W(H_i;q) - W_{r_i}(H_i;q) + 1] \\
&= W(G;q) + \sum_{i=1}^{n} \sum_{u\in V(G)} q^{d(u,x_i)} \sum_{\substack{v\in V(H_i)\\ v\neq r_i}} q^{d(r_i,v)} \\
&\quad + \sum_{1\leqslant i<j\leqslant n} q^{d(x_i,x_j)} \sum_{\substack{u\in V(H_i)\\ u\neq r_i}} q^{d(u,r_i)} \sum_{\substack{v\in V(H_j)\\ v\neq r_j}} q^{d(r_j,v)} \\
&\quad + n[W(H;q) - W_r(H;q) + 1] \\
&= W(G;q) + \sum_{i=1}^{n} \sum_{u\in V(G)} q^{d(x_i,u)}[W_{r_i}(H_i;q) - 1] \\
&\quad + \sum_{1\leqslant i<j\leqslant n} q^{d(x_i,x_j)}[W_{r_i}(H_i;q) - 1][W_{r_j}(H_j;q) - 1] \\
&\quad + n[W(H;q) - W_r(H;q) + 1] \\
&= W(G;q) + \left[\sum_{i=1}^{n} \sum_{u\in V(G)} q^{d(x_i,u)}\right][W_r(H;q) - 1] \\
&\quad + W(G|X;q)(W_r(H;q) - 1)^2 + n[W(H;q) - W_r(H;q) + 1] \\
&= W(G;q) + W_X(G;q)(W_r(H;q) - 1) \\
&\quad + W(G|X;q)(W_r(H;q) - 1)^2 + n[W(H;q) - W_r(H;q) + 1].
\end{aligned}
$$

To prove the identity involving the Wiener polynomial $W(\mathit{Link}[(G,X),(H,r)];q)$, we recall that $\mathit{Paste}[(G,X),(H^*(r),r^*)] = \mathit{Link}[(G,X),(H,r)]$, use the identity just proved to compute $W(\mathit{Paste}[(G,X),(H^*(r),r^*)];q)$, and apply Lemma 1 to eliminate references to $W_{r^*}(H^*(r);q)$ and $W(H^*(r);q)$. $\square$

If $X=V(G)$, then $W_X(G;q)=\overline{W}(G;q)=2W(G;q)+|V(G)|$ and $W(G|X;q)=W(G;q)$. Using these observations we can obtain Theorems 1 and 2 as corollaries of Theorem 3. At the other extreme for the size of X, we see that if X contains only one vertex, say x, then $W_X(G;q)=W_x(G;q)$ and $W(G|X;q)=0$, and we get the following corollary.

Corollary 2. *Let G and H be connected graphs, let x be a vertex of G, and let r be a specified vertex of H.*

(i) *If F is the graph obtained from G and H by identifying the vertices x and r as a single vertex (i.e., if $F=\mathit{Paste}[(G,\{x\}),(H,r)]$, then*

$$W(F;q) = W(G;q) - W_x(G;q) + W_x(G;q)W_r(H;q) + W(H;q) - W_r(H;q) + 1.$$

(ii) *If J is the graph obtained from G and H by adding a single edge from x to r, (i.e., if $J = Link[(G, \{x\}), (H, r)]$, then*

$$W(J; q) = W(G; q) + qW_x(G; q)W_r(H; q) + W(H; q).$$

The preceding corollary can be used repeatedly to find the distance distributions of arbitrary articulated networks that are constructed by linking one perinet at a time to an existing articulated network. In effect, this allows one to both preview and update the distance distribution generating functions when a new perinet is to be added to an existing network. More generally, if a network can be constructed by a finite sequence of *Paste* and *Link* operations, then Theorem 3 and/or Corollary 2 can be used repeatedly to find the Wiener polynomial of the network.

4. Generalizing the Wiener polynomials to weighted graphs

In mathematical modeling, graphs with positive edge weights are often useful. For example, if a graph G models a computer network, the weight on an edge may represent the time required to send a message of unit length along that edge. Similarly, the sum of the weights on the edges of a path represents the time required to send a unit length message along the path. It seems natural then to generalize the polynomials discussed in previous sections to the weighted graph setting. Further support for such a generalization is given by the fact that a Wiener index for weighted graphs has been studied in [7,4,9,11] in relation to chemical graph theory.

For weighted graphs, we define the *weight* of a path to be the sum of the weights on the path's edges. A path of minimum weight from a vertex u to a vertex v (lying in a common component of the graph) will be called a *u–v weight-geodesic*; its weight is termed the *weighted distance* from u to v and denoted by $wd(u, v)$. We will find it convenient to reserve the notation $d(u, v)$ for the usual unweighted distance from u to v in the underlying unweighted graph. If a connected graph is uniformly weighted with each edge having weight c, then, for each pair (u, v) of vertices, a u–v path is a u–v geodesic if and only if it is a u–v weight-geodesic; furthermore, $wd(u, v) = c \cdot d(u, v)$.

By analogy with the varieties of Wiener polynomials discussed earlier, we define five generating functions for distance distributions in weighted graphs. The (unordered) *weighted Wiener function* of a connected weighted graph G is

$$wW(G; q) = \sum_{\{u,v\}} q^{wd(u,v)},$$

where the sum is over all unordered pairs $\{u, v\}$ of distinct vertices in G. The *ordered weighted Wiener function* is

$$\overline{wW}(G; q) = \sum_{(u,v)} q^{wd(u,v)},$$

where the sum is over all ordered pairs (u, v) of not necessarily distinct vertices. If u is a vertex of G the *weighted Wiener function* of G *relative to* u is

$$wW_u(G; q) = \sum_{v \in V(G)} q^{wd(u,v)}.$$

For each nonempty subset X of $V(G)$, we define the *weighted Wiener function of G restricted to X* as

$$wW(G|X; q) = \sum_{\{u,v\} \subseteq X} q^{wd(u,v)},$$

and the *weighted Wiener function of G relative to X* as

$$wW_X(G; q) = \sum_{x \in X} wW_x(G; q).$$

The first three of these five functions are generating functions for weighted distance distributions, ordered weighted distance distributions, and weighted distance degree sequences, respectively; e.g., the coefficient of q^k in $\overline{wW}(G; q)$ is the number of ordered pairs of vertices at distance k from each other. These distance frequency functions are polynomials if and only if the edge weights in G are integers. They equal the corresponding Wiener polynomials if and only if all edge weights are 1. In a manner analogous to the Wiener polynomials, the weighted distance frequency function and the ordered weighted distance frequency function are related by the equation

$$\overline{wW}(G;q) = 2wW(G;q) + |V(G)|.$$

Furthermore,

$$\overline{wW}(G;q) = \sum_{u \in V(G)} wW_u(G;q).$$

Relative weighted distance frequency functions can be easily found for specific graphs by employing Dijkstra's algorithm. Also, in a manner similar to the methods used to obtain Facts 1 and 2, we can use standard elementary probability arguments to obtain the following facts about weighted distance statistics for connected weighted graphs; these are the weighted analogues to Facts 1 and 2 that were stated earlier.

Fact 3. *Let G be a connected weighted graph having p vertices, and assume that the probability of selecting any unordered pair of distinct vertices in G is* $1/\binom{p}{2}$. *Then*

(i) *the mean weighted distance between pairs of distinct vertices in G is*

$$\mu = \frac{wW'(G;1)}{\binom{p}{2}} = \frac{\overline{wW'}(G;1)}{p(p-1)} \quad \textit{and}$$

(ii) *the variance of the weighted distance distribution of G is*

$$\sigma^2 = \frac{wW''(G;1)}{\binom{p}{2}} + \mu - \mu^2 = \frac{\overline{wW''}(G;1)}{p(p-1)} + \mu - \mu^2.$$

Fact 4. *Let u be a vertex of a connected weighted graph G having p vertices, and assume that* $1/(p-1)$ *is the probability of selecting any vertex of G other than u. Then*

(i) *the mean weighted distance from u to any other vertex of G is*

$$\mu_u = \frac{wW'_u(G;1)}{p-1} \quad \textit{and}$$

(ii) *the variance of the set of weighted distances from u to distinct vertices of G is*

$$\sigma_u^2 = \frac{wW''_u(G;1)}{p-1} + \mu_u - \mu_u^2.$$

As one would expect, many of the theorems, corollaries and lemmas of the preceding sections have analogues for weighted graphs. For each such result, the proof of the analogue is very similar to the proof of the corresponding statement for unweighted graphs, so we omit the proofs. For example, by making minor changes in the proof of Theorem 1 to account for weighted distances, we can obtain the following result that is analogous to Theorem 1.

Theorem 4. *If G and H are connected weighted graphs, and r is a vertex of H, then*

$$wW(Art[G,(H,r)];q) = wW(G;q)[wW_r(H;q)]^2 + |V(G)|wW(H;q).$$

Equivalently,

$$\overline{wW}(Art[G,(H,r)];q) = (\overline{wW}(G;q) - |V(G)|)[wW_r(H;q)]^2 + |V(G)| \cdot \overline{wW}(H;q).$$

Formulations of the weighted distance frequency functions for suspended products $Susp[G, (H, r)]$ depends on the weight assigned to the edge joining r^* to r in $H^*(r)$. The following lemma is analogous to Lemma 1.

Lemma 3. *Suppose that r is a specified vertex of a connected weighted graph H, and r^* and $H^*(r)$ are as given in the definition of the suspended product $Susp[G, (H, r)]$, for some connected graph G. Furthermore, suppose that the edge from r^* to r has been given the weight γ. Then*

(i) $wW_{r^*}(H^*(r); q) = 1 + q^{\gamma} wW_r(H; q)$, *and*
(ii) $wW(H^*(r); q) = wW(H; q) + q^{\gamma} wW_r(H; q)$, *or equivalently*, $\overline{wW}(H^*(r); q) = \overline{wW}(H; q) + 2q^{\gamma} wW_r(H; q) + 1$.

By direct application of Theorem 4, Lemma 3 and the definition of a suspended product, we obtain the following analogue to Theorem 2.

Theorem 5. *Let G be a connected weighted graph and r be a specified vertex of a connected weighted graph H. Let r^* and $H^*(r)$ be as given in the definition of the suspended product $Susp[G, (H, r)]$, and suppose that the edge from r^* to r has been given the weight γ. Then*

$$wW(Susp[G, (H, r)]; q) = wW(G; q)(1 + q^{\gamma} wW_r(H; q))^2 + |V(G)|(wW(H; q) + q^{\gamma} wW_r(H; q))$$

and

$$\begin{aligned}\overline{wW}(Susp[G, (H, r)]; q) = {} & (\overline{wW}(G; q) - |V(G)|)(1 + q^{\gamma} wW_r(H; q))^2 \\ & + |V(G)|(\overline{wW}(H; q) + 2q^{\gamma} wW_r(H; q) + 1).\end{aligned}$$

Minor adjustments to the proof of Theorem 3 yield that theorem's weighted distance analogue, which follows.

Theorem 6. *Let G and H be connected weighted graphs, let $X = \{x_1, x_2, \ldots, x_n\}$ be a nonempty subset of $V(G)$, and let r be a specified vertex of H. Then*

$$\begin{aligned}wW(Paste[(G, X), (H, r)]) = {} & wW(G; q) + wW_X(G; q)[wW_r(H; q) - 1] \\ & + wW(G|X; q)[wW_r(H; q) - 1]^2 + n[wW(H; q) - wW_r(H; q) + 1].\end{aligned}$$

Furthermore, if for each $i = 1, 2, \ldots, n$, the edge joining x_i to r_i, as given in the definition of $Link[(G, X), (H, r)]$, has weight γ, then

$$\begin{aligned}W(Link[(G, X), (H, r)]; q) = {} & wW(G; q) + q^{\gamma} wW_X(G; q) wW_r(H; q) \\ & + q^{2\gamma} wW(G|X; q)[wW_r(H; q)]^2 + n wW(H; q).\end{aligned}$$

As a corollary to this theorem, we have an analogue to Corollary 2. Its proof follows immediately from the previous theorem and the observations that if X contains only one vertex, say x, then $wW_X(G; q) = wW_x(G; q)$ and $wW(G|X; q) = 0$.

Corollary 3. *Let G and H be connected weighted graphs, let x be a vertex of G, and let r be a specified vertex of H.*

(i) *If F is the weighted graph obtained from G and H by identifying the vertices x and r as a single vertex (i.e., if $F = Paste[(G, \{x\}), (H, r)]$, then*

$$wW(F; q) = wW(G; q) - wW_x(G; q) + wW_x(G; q) wW_r(H; q) + wW(H; q) - wW_r(H; q) + 1.$$

(ii) *If J is the graph obtained from G and H by adding a single edge of weight γ, from x to r, (i.e., if $J = Link[(G, \{x\}), (H, r)]$, then*

$$wW(J; q) = wW(G; q) + q^{\gamma} wW_x(G; q) wW_r(H; q) + wW(H; q).$$

Finally, we remark that if each edge of a weighted graph G has the same weight, say c, then the five functions discussed in this section can be expressed in terms of the corresponding polynomials for unweighted graphs. Specifically, we have the following:

Remark. If each edge of a connected weighted graph G has the same weight, say c, then

1. $wW(G; q) = W(G; q^c)$,
2. $\overline{wW}(G; q) = \overline{W}(G; q^c)$,
3. $wW_r(G; q) = W_r(G; q^c)$, for each vertex r of G,
4. $wW(G|X; q) = W(G|X; q^c)$, for each subset X of $V(G)$, and
5. $wW_X(G; q) = W_X(G; q^c)$, for each subset X of $V(G)$.

Consequently, a weighted Wiener function (and variants) of a uniformly weighted graph can be easily computed if the corresponding (unweighted) polynomial is known; e.g., if each edge of the n-dimensional hypercube has weight c, then, $\overline{wW}(Q_n; q) = \overline{W}(Q_n; q^c) = 2^n(1 + q^c)^n$. Furthermore, each of Theorems 4, 5 and 6 can be modified to a slightly simpler form in the cases when one or both of the connected weighted graphs G and H are uniformly weighted. As an illustration, we state the following corollary to Theorem 5.

Corollary 4. *Let G be a connected weighted graph and r be a specified vertex of a connected weighted graph H. Furthermore, suppose that every edge of G has weight α and every edge of H has weight β. Let r^* and $H^*(r)$ be as given in the definition of the suspended product $Susp[G, (H, r)]$, and suppose that the edge from r^* to r has weight γ. Then*

$$wW(Susp[G, (H, r)]; q) = W(G; q^{\alpha})(1 + q^{\gamma} W_r(H; q^{\beta}))^2 + |V(G)|(W(H; q^{\beta}) + q^{\gamma} W_r(H; q^{\beta}))$$

5. Conclusion

In this paper we have presented a relatively simple and useful way, via generating functions, to find and store distance distributions for graphs modeling the very large class of computer networks we call articulated networks. By assigning appropriate weights to the edges of the graph modeling an articulated network and using these generating function methods with a symbolic computation engine, one can quickly generate data and statistics related to the communication time between distinct processors during distributed computing. In this manner, the generating function approach can be used as a design and analysis tool for large articulated networks. These methods are especially simple when the articulated networks are homogeneous and/or when the backbone and the perinets are modeled by distance transitive graphs with uniformly weighted edges. The authors are currently preparing a paper in which properties of the Wiener functions are exploited to analyze existing networks and design new networks. We will also present new results of special interest to computer scientists.

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