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Advances in Mathematics 217 (2008) 537–560
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Kontsevich's formula and the WDVV equations in tropical geometry

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Received 15 November 2005; accepted 23 August 2007

Available online 29 October 2007

Communicated by Michael J. Hopkins

Abstract

Using Gromov–Witten theory the numbers of complex plane rational curves of degree d through $3d-1$ general given points can be computed recursively with Kontsevich's formula that follows from the so-called WDVV equations. In this paper we establish the same results entirely in the language of tropical geometry. In particular this shows how the concepts of moduli spaces of stable curves and maps, (evaluation and forgetful) morphisms, intersection multiplicities and their invariance under deformations can be carried over to the tropical world.

MSC: primary 14N35, 51M20; secondary 14N10

Keywords: Tropical geometry; Enumerative geometry; Gromov–Witten theory

1. Introduction

For $d \geqslant 1$ let N_d be the number of rational curves in the complex projective plane $\mathbb{P}^2$ that pass through $3d-1$ given points in general position. About 10 years ago Kontsevich has shown that these numbers are given recursively by the initial value $N_1 = 1$ and the equation

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[1] The second author has been funded by the DFG grant Ga 636/2.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2007.08.004

$$N_d = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right) N_{d_1} N_{d_2}$$

for $d > 1$ (see [3, Claim 5.2.1]). The main tool in deriving this formula is the so-called WDVV equations, i.e. the associativity equations of quantum cohomology. Stated in modern terms the idea of these equations is as follows: plane rational curves of degree d are parametrized by the moduli spaces of stable maps $\bar{M}_{0,n}(\mathbb{P}^2, d)$ whose points are in bijection to tuples $(C, x_1, \ldots, x_n, f)$ where $x_1, \ldots, x_n$ are distinct smooth points on a rational nodal curve C and $f : C \to \mathbb{P}^2$ is a morphism of degree d (with a stability condition). If $n \geqslant 4$ there is a "forgetful map" $\pi : \bar{M}_{0,n}(\mathbb{P}^2, d) \to \bar{M}_{0,4}$ that sends a stable map $(C, x_1, \ldots, x_n, f)$ to (the stabilization of) $(C, x_1, \ldots, x_4)$. The important point is now that the moduli space $\bar{M}_{0,4}$ of 4-pointed rational stable curves is simply a projective line. Therefore the two points

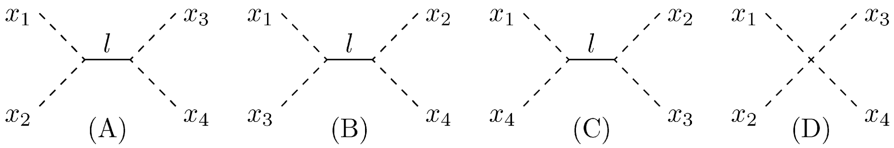
of $\bar{M}_{0,4}$ are linearly equivalent divisors, and hence so are their inverse images $D_{12|34}$ and $D_{13|24}$ under π. The divisor $D_{12|34}$ in $\bar{M}_{0,n}(\mathbb{P}^2, d)$ (and similarly of course $D_{13|24}$) can be described explicitly as the locus of all reducible stable maps with two components such that the marked points x_1, x_2 lie on one component and x_3, x_4 on the other. It is of course reducible since there are many combinatorial choices for such curves: the degree and the remaining marked points can be distributed onto the two components in an arbitrary way.

All that remains to be done now is to intersect the equation $[D_{12|34}] = [D_{13|24}]$ of divisor classes with cycles of dimension 1 in $\bar{M}_{0,n}(\mathbb{P}^2, d)$ to get some equations between numbers. Specifically, to get Kontsevich's formula one chooses $n = 3d$ and intersects the above divisors with the conditions that the stable maps pass through two given lines at x_1 and x_2 and through given points in $\mathbb{P}^2$ at all other x_i. The resulting equation can be seen to be precisely the recursion formula stated at the beginning of the introduction: the sum corresponds to the possible splittings of the degree of the curves onto their two components, the binomial coefficients correspond to the distribution of the marked points x_i with $i > 4$, and the various factors of d_1 and d_2 correspond to the intersection points of the two components with each other and with the two chosen lines (for more details see e.g. [1, Section 7.4.2]).

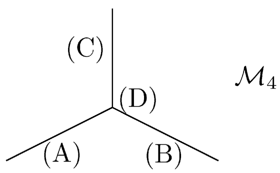
The goal of this paper is to establish the same results in tropical geometry. In contrast to most enumerative applications of tropical geometry known so far it is absolutely crucial for this to work that we pick the "correct" definition of (moduli spaces of) tropical curves even for somewhat degenerated curves.

To describe our definition let us start with abstract tropical curves, i.e. curves that are not embedded in some ambient space. An abstract tropical curve is simply an abstract connected graph Γ obtained by glueing closed (not necessarily bounded) real intervals together at their boundary points in such a way that every vertex has valence at least 3. In particular, every bounded edge of such an abstract tropical curve has an intrinsic length. Following an idea of Mikhalkin [5] the unbounded ends of Γ will be labeled and called the marked points of the curve. The most important example for our applications is the following:

Example 1.1. A 4-marked rational tropical curve (i.e. an element of the tropical analogue of $\bar{M}_{0,4}$ that we will denote by $\mathcal{M}_4$) is simply a tree graph with 4 unbounded ends. There are four possible combinatorial types for this:



(In this paper we will always draw the unbounded ends corresponding to marked points as dotted lines.) In the types (A) to (C) the bounded edge has an intrinsic length l; so each of these types leads to a stratum of $\mathcal{M}_4$ isomorphic to $\mathbb{R}_{>0}$ parametrized by this length. The last type (D) is simply a point in $\mathcal{M}_4$ that can be seen as the boundary point in $\mathcal{M}_4$ where the other three strata meet. Therefore $\mathcal{M}_4$ can be thought of as three unbounded rays meeting in a point—note that this is again a rational tropical curve!

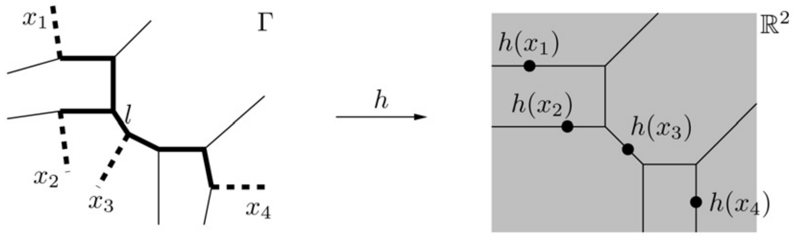


Let us now move on to plane tropical curves. As in the complex case we will adopt the "stable map picture" and consider maps from an abstract tropical curve to $\mathbb{R}^2$ rather than embedded tropical curves. More precisely, an n-marked plane tropical curve will be a tuple $(\Gamma, x_1, \ldots, x_n, h)$, where Γ is an abstract tropical curve, $x_1, \ldots, x_n$ are distinct unbounded ends of Γ, and $h : \Gamma \to \mathbb{R}^2$ is a continuous map such that

(a) on each edge of Γ the map h is of the form $h(t) = a + t \cdot v$ for some $a \in \mathbb{R}^2$ and $v \in \mathbb{Z}^2$ ("h is affine linear with integer direction vector v");
(b) for each vertex V of Γ the direction vectors of the edges around V sum up to zero (the "balancing condition");
(c) the direction vectors of all unbounded edges corresponding to the marked points are zero ("every marked point is contracted to a point in $\mathbb{R}^2$ by h").

Note that it is explicitly allowed that h contracts an edge E of Γ to a point. If this is the case and E is a bounded edge then the intrinsic length of E can vary arbitrarily without changing the image curve $h(\Gamma)$. This is of course the feature of "moduli in contracted components" that we know well from the ordinary complex moduli spaces of stable maps.

Example 1.2. The following picture shows an example of a 4-marked plane tropical curve of degree 2, i.e. of an element of the tropical analogue of $\bar{M}_{0,4}(\mathbb{P}^2, 2)$ that we will denote by $\mathcal{M}_{2,4}$. Note that at each marked point the balancing condition ensures that the two other edges meeting at the corresponding vertex are mapped to the same line in $\mathbb{R}^2$.

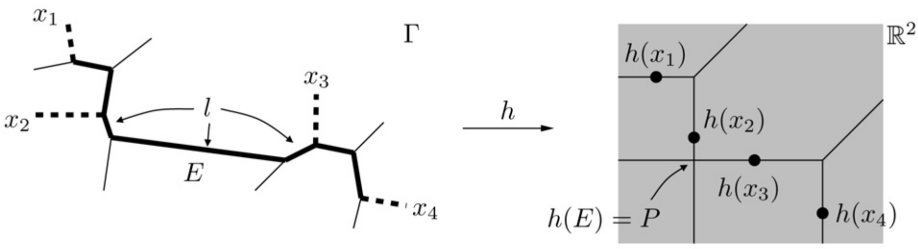


It is easy to see from this picture already that the tropical moduli spaces $\mathcal{M}_{d,n}$ of plane curves of degree d with $n \geqslant 4$ marked points admit forgetful maps to $\mathcal{M}_4$: given an n-marked plane tropical curve $(\Gamma, x_1, \dots, x_n, h)$ we simply forget the map h, take the minimal connected subgraph of Γ that contains $x_1, \dots, x_4$, and "straighten" this graph to obtain an element of $\mathcal{M}_4$. In the picture above we simply obtain the "straightened version" of the subgraph drawn in bold, i.e. the element of $\mathcal{M}_4$ of type (A) (in the notation of Example 1.1) with length parameter l as indicated in the picture.

The next thing we would like to do is to say that the inverse images of two points in $\mathcal{M}_4$ under this forgetful map are "linearly equivalent divisors." However, there is unfortunately no theory of divisors in tropical geometry yet. To solve this problem we will first impose all incidence conditions as needed for Kontsevich's formula and then only prove that the (suitably weighted) number of plane tropical curves satisfying all these conditions and mapping to a given point in $\mathcal{M}_4$ does not depend on this choice of point. The idea to prove this is precisely the same as for the independence of the incidence conditions in [2] (although the multiplicity with which the curves have to be counted has to be adapted to the new situation).

We will then apply this result to the two curves in $\mathcal{M}_4$ that are of type (A) respectively (B) above and have a fixed very large length parameter l. We will see that such very large lengths in $\mathcal{M}_4$ can only occur if there is a contracted bounded edge (of a very large length) somewhere as in the following example:

Example 1.3. Let C be a plane tropical curve with a bounded contracted edge E.



In this picture the parameter l is the sum of the intrinsic lengths of the three marked edges, in particular it is very large if the intrinsic length of E is. By the balancing condition it follows that locally around $P = h(E)$ the tropical curve must be a union of two lines through P, i.e. that the tropical curve becomes "reducible" with two components meeting in P (in the picture above we have a union of two tropical lines).

Hence we get the same types of splitting of the curves into two components as in the complex picture—and thus the same resulting formula for the (tropical) numbers N_d.

Our result shows once again quite clearly that it is possible to carry many concepts from classical complex geometry over to the tropical world: moduli spaces of curves and stable maps, morphisms, divisors and divisor classes, intersection multiplicities, and so on. Even if we only make these constructions in the specific cases needed for Kontsevich's formula we hope that our paper will be useful to find the correct definitions of these concepts in the general tropical setting. It should also be quite easy to generalize our results to other cases, e.g. to tropical curves of other degrees (corresponding to complex curves in toric surfaces) or in higher-dimensional spaces. Work in this direction is in progress.

This paper is organized as follows: in Section 2 we define the moduli spaces of abstract and plane tropical curves that we will work with later. They have the structure of (finite) polyhedral complexes. For morphisms between such complexes we then define the concepts of multiplicity and degree in Section 3. We show that these notions specialize to Mikhalkin's well-known "multiplicities of plane tropical curves" when applied to the evaluation maps. In Section 4 we apply the same techniques to the forgetful maps described above. In particular, we show that the numbers of tropical curves satisfying given incidence conditions and mapping to a given point in $\mathcal{M}_4$ do not depend on this choice of point in $\mathcal{M}_4$. Finally, we apply this result to two different points in $\mathcal{M}_4$ to derive Kontsevich's formula in Section 5.

2. Abstract and plane tropical curves

In this section we will mainly define the moduli spaces of (abstract and plane) tropical curves that we will work with later. Our definitions here differ slightly from our earlier ones in [2]. A common feature of both definitions is that we will always consider a plane curve to be a "parametrized tropical curve," i.e. a graph Γ with a map h to the plane rather than an embedded tropical curve. In contrast to our earlier work however it is now explicitly allowed (and crucial for our arguments to work) that the map h contracts some edges of Γ to a point. Moreover, following Mikhalkin [5] marked points will be contracted unbounded ends instead of just markings. For simplicity we will only give the definitions here for rational curves.

Definition 2.1 *(Graphs).*

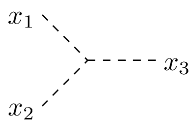
(a) Let $I_1, \dots, I_n \subset \mathbb{R}$ be a finite set of closed, bounded or half-bounded real intervals. We pick some (not necessarily distinct) boundary points $P_1, \dots, P_k, Q_1, \dots, Q_k \in I_1 \cup \dots \cup I_n$ of these intervals. The topological space Γ obtained by identifying P_i with Q_i for all $i = 1, \dots, k$ in $I_1 \cup \dots \cup I_n$ is called a *graph*. As usual, the *genus* of Γ is simply its first Betti number $\dim H_1(\Gamma, \mathbb{R})$.

(b) For a graph Γ the boundary points of the intervals $I_1, \dots, I_n$ are called the *flags*, their image points in Γ the *vertices* of Γ. If F is such a flag then its image vertex in Γ will be denoted ∂F. For a vertex V the number of flags F with $\partial F = V$ is called the *valence* of V and denoted $\operatorname{val} V$. We denote by Γ^0 and Γ' the sets of vertices and flags of Γ, respectively.

(c) The open intervals $I_1^\circ, \dots, I_n^\circ$ are naturally open subsets of Γ; they are called the *edges* of Γ. An edge will be called bounded (respectively unbounded) if its corresponding open interval is. We denote by Γ^1 (respectively Γ_0^1 and Γ_∞^1) the set of edges (respectively bounded and unbounded edges) of Γ. Every flag $F \in \Gamma'$ belongs to exactly one edge that we will denote by $[F] \in \Gamma^1$. The unbounded edges will also be called the *ends* of Γ.

Definition 2.2 *(Abstract tropical curves).* A (rational, abstract) tropical curve is a connected graph Γ of genus 0 all of whose vertices have valence at least 3. An *n-marked tropical curve* is a tuple $(\Gamma, x_1, \ldots, x_n)$ where Γ is a tropical curve and $x_1, \ldots, x_n \in \Gamma^1_\infty$ are distinct unbounded edges of Γ. Two such marked tropical curves $(\Gamma, x_1, \ldots, x_n)$ and $(\tilde{\Gamma}, \tilde{x}_1, \ldots, \tilde{x}_n)$ are called isomorphic (and will from now on be identified) if there is a homeomorphism $\Gamma \to \tilde{\Gamma}$ mapping x_i to $\tilde{x}_i$ for all i and such that every edge of Γ is mapped bijectively onto an edge of $\tilde{\Gamma}$ by an affine map of slope ± 1, i.e. by a map of the form $t \mapsto a \pm t$ for some $a \in \mathbb{R}$. The space of all n-marked tropical curves (modulo isomorphisms) with precisely n unbounded edges will be denoted $\mathcal{M}_n$. (It can be thought of as a tropical analogue of the moduli space $\bar{M}_{0,n}$ of n-pointed stable rational curves.)

Example 2.3. We have $\mathcal{M}_n = \emptyset$ for $n < 3$ since any graph of genus 0 all of whose vertices have valence at least 3 must have at least 3 unbounded edges. For $n = 3$ unbounded edges there is exactly one such tropical curve, namely



(in this paper we will always draw the unbounded edges corresponding to the markings x_i as dotted lines). Hence $\mathcal{M}_3$ is simply a point.

Remark 2.4. The isomorphism condition of Definition 2.2 means that every edge of a marked tropical curve has a parametrization as an interval in $\mathbb{R}$ that is unique up to translations and sign. In particular, every bounded edge E of a tropical curve has an intrinsic *length* that we will denote by $l(E) \in \mathbb{R}_{>0}$.

One way to fix this translation and sign ambiguity is to pick a flag F of the edge E: there is then a unique choice of parametrization such that the corresponding closed interval is $[0, l(E)]$ (or $[0, \infty)$ for unbounded edges), with the chosen flag F being the zero point of this interval. We will call this the *canonical parametrization* of E with respect to the flag F.

Example 2.5. The moduli space $\mathcal{M}_4$ is simply a rational tropical curve with 3 ends—see Example 1.1.

Definition 2.6 *(Plane tropical curves).*

(a) Let $n \geqslant 0$ be an integer. An *n-marked plane tropical curve* is a tuple $(\Gamma, x_1, \ldots, x_n, h)$, where Γ is an abstract tropical curve, $x_1, \ldots, x_n \in \Gamma^1_\infty$ are distinct unbounded edges of Γ, and $h : \Gamma \to \mathbb{R}^2$ is a continuous map, such that:
   (i) On each edge of Γ the map h is of the form $h(t) = a + t \cdot v$ for some $a \in \mathbb{R}^2$ and $v \in \mathbb{Z}^2$ (i.e. "h is affine linear with rational slope"). The integral vector v occurring in this equation if we pick for E the canonical parametrization with respect to a chosen flag F of E (see Remark 2.4) will be denoted $v(F)$ and called the *direction* of F.
   (ii) For every vertex V of Γ we have the *balancing condition*

$$\sum_{F \in \Gamma' \colon \partial F = V} v(F) = 0.$$

(iii) Each of the unbounded edges $x_1, \dots, x_n \in \Gamma^1_\infty$ is mapped to a point in $\mathbb{R}^2$ by h (i.e. $v(F) = 0$ for the corresponding flags).

(b) Two n-marked plane tropical curves $(\Gamma, x_1, \dots, x_n, h)$ and $(\tilde{\Gamma}, \tilde{x}_1, \dots, \tilde{x}_n, \tilde{h})$ are called isomorphic (and will from now on be identified) if there is an isomorphism $\varphi : (\Gamma, x_1, \dots, x_n) \to (\tilde{\Gamma}, \tilde{x}_1, \dots, \tilde{x}_n)$ of the underlying abstract curves as in Definition 2.2 such that $\tilde{h} \circ \varphi = h$.

(c) The *degree* of an n-marked plane tropical curve is defined to be the multiset $\Delta = \{v(F); [F] \in \Gamma^1_\infty \backslash \{x_1, \dots, x_n\}\}$ of directions of its non-marked unbounded edges. If this degree consists of the vectors $(-1, 0)$, $(0, -1)$, $(1, 1)$ each d times then we simply say that the degree of the curve is d. The space of all n-marked plane tropical curves of degree Δ (respectively d) will be denoted $\mathcal{M}_{\Delta,n}$ (respectively $\mathcal{M}_{d,n}$). It can be thought of as a tropical analogue of the moduli spaces of stable maps to toric surfaces (respectively the projective plane).

Remark 2.7. For a concrete example of a marked plane tropical curve see Example 1.2.

Note that the map h of a marked plane tropical curve $(\Gamma, x_1, \dots, x_n, h)$ need not be injective on the edges of Γ: it may happen that $v(F) = 0$ for a flag F, i.e. that the corresponding edge is contracted to a point. Of course it follows then in such a case that the remaining flags around the vertex ∂F satisfy the balancing condition themselves. If ∂F is a 3-valent vertex this means that the other two flags around this vertex are negatives of each other, i.e. that the image $h(\Gamma)$ in $\mathbb{R}^2$ is just a straight line locally around this vertex.

This applies in particular to the marked unbounded edges $x_1, \dots, x_n$ as they are required to be contracted by h. They can therefore be seen as tropical analogues of marked points in the ordinary complex moduli spaces of stable maps. By abuse of notation we will therefore often refer to these marked unbounded edges as “marked points” in the rest of the paper.

Note also that contracted bounded edges lead to “hidden moduli parameters” of plane tropical curves: if we vary the length of a contracted bounded edge then we arrive at a continuous family of different plane tropical curves whose images in $\mathbb{R}^2$ are all the same. This feature of moduli in contracted components is of course well-known from the complex moduli spaces of stable maps.

Remark 2.8. If the direction $v(F) \in \mathbb{Z}^2$ of a flag F of a plane tropical curve is not equal to zero then it can be written uniquely as a positive integer times a primitive integral vector. This positive integer is what is usually called the *weight* of the corresponding edge. In this paper we will not use this notation however since it seems more natural for our applications not to split up the direction vectors in this way.

The following results about the structure of the spaces $\mathcal{M}_n$ and $\mathcal{M}_{\Delta,n}$ are very similar to those in [2], albeit much simpler.

Definition 2.9 *(Combinatorial types).* The *combinatorial type* of a marked tropical curve $(\Gamma, x_1, \dots, x_n)$ is defined to be the homeomorphism class of Γ relative $x_1, \dots, x_n$ (i.e. the data of $(\Gamma, x_1, \dots, x_n)$ modulo homeomorphisms of Γ that map each x_i to itself). The combinatorial type of a marked plane tropical curve $(\Gamma, x_1, \dots, x_n, h)$ is the data of the combinatorial type of the marked tropical curve $(\Gamma, x_1, \dots, x_n)$ together with the direction vectors $v(F)$ for all flags $F \in \Gamma'$. In both cases the codimension of such a type α is defined to be

$$\operatorname{codim}\alpha := \sum_{V\in\Gamma^0} (\operatorname{val} V - 3).$$

We denote by $\mathcal{M}_n^\alpha$ (respectively $\mathcal{M}_{\Delta,n}^\alpha$) the subset of $\mathcal{M}_n$ (respectively $\mathcal{M}_{\Delta,n}$) that corresponds to marked tropical curves of type α.

Lemma 2.10. *For all n and Δ there are only finitely many combinatorial types occurring in the spaces $\mathcal{M}_n$ and $\mathcal{M}_{\Delta,n}$.*

Proof. The statement is obvious for $\mathcal{M}_n$. For $\mathcal{M}_{\Delta,n}$ we just note in addition that by [4, Proposition 3.11] the image $h(\Gamma)$ is dual to a lattice subdivision of the polygon associated to Δ. In particular, this means that the absolute value of the entries of the vectors $v(F)$ is bounded in terms of the size of Δ, i.e. that there are only finitely many choices for the direction vectors. □

Proposition 2.11. *For every combinatorial type α occurring in $\mathcal{M}_n$ (respectively $\mathcal{M}_{\Delta,n}$) the space $\mathcal{M}_n^\alpha$ (respectively $\mathcal{M}_{\Delta,n}^\alpha$) is naturally an (unbounded) open convex polyhedron in a real vector space, i.e. a subset of a real vector space given by finitely many linear strict inequalities. Its dimension is as expected, i.e.*

$$\dim \mathcal{M}_n^\alpha = n - 3 - \operatorname{codim}\alpha,$$

$$\textit{respectively}\quad \dim \mathcal{M}_{\Delta,n}^\alpha = |\Delta| - 1 + n - \operatorname{codim}\alpha.$$

Proof. The first formula follows immediately from the combinatorial fact that a 3-valent tropical curve with n unbounded edges has exactly $n-3$ bounded edges: the space $\mathcal{M}_n^\alpha$ is simply parametrized by the lengths of all bounded edges, i.e. it is given as the subset of $\mathbb{R}^{n-3-\operatorname{codim}\alpha}$ where all coordinates are positive.

The statement about $\mathcal{M}_{\Delta,n}^\alpha$ follows in the same way, noting that a plane tropical curve in $\mathcal{M}_{\Delta,n}$ has $|\Delta| + n$ unbounded edges and that we need two additional (unrestricted) parameters to describe translations, namely the coordinates of the image of a fixed "root vertex" $V \in \Gamma^0$. □

Ideally, one would of course like to make the spaces $\mathcal{M}_n$ and $\mathcal{M}_{\Delta,n}$ into tropical varieties themselves. Unfortunately, there is however no general theory of tropical varieties yet. We will therefore work in the category of polyhedral complexes, which will be sufficient for our purposes.

Definition 2.12 *(Polyhedral complexes).* Let $X_1, \dots, X_N$ be (possibly unbounded) open convex polyhedra in real vector spaces. A *polyhedral complex* with cells $X_1, \dots, X_N$ is a topological space X together with continuous inclusion maps $i_k : \overline{X_k} \to X$ such that X is the disjoint union of the sets $i_k(X_k)$ and the "coordinate changing maps" $i_k^{-1} \circ i_l$ are linear (where defined) for all $k \neq l$. We will usually drop the inclusion maps i_k in the notation and say that the cells X_k are contained in X.

The *dimension* $\dim X$ of a polyhedral complex X is the maximum of the dimensions of its cells. We say that X is of *pure dimension* $\dim X$ if every cell is contained in the closure of a cell of dimension $\dim X$. A point of X is said to be *in general position* if it is contained in a cell of dimension $\dim X$.

Example 2.13. The moduli spaces $\mathcal{M}_n$ and $\mathcal{M}_{\Delta,n}$ are polyhedral complexes of pure dimensions $n-3$ and $|\Delta| - 1 + n$, respectively, with the cells corresponding to the combinatorial types.

In fact, this follows from Lemma 2.10 and Proposition 2.11 together with the obvious remark that the boundaries of the cells $\mathcal{M}_n^\alpha$ (and $\mathcal{M}_{\Delta,n}^\alpha$) can naturally be thought of as subsets of $\mathcal{M}_n$ (respectively $\mathcal{M}_{\Delta,n}$) as well: they correspond to tropical curves where some of the bounded edges acquire zero length and finally vanish, leading to curves with vertices of higher valence. A tropical curve in $\mathcal{M}_n$ or $\mathcal{M}_{\Delta,n}$ is in general position if and only if it is 3-valent.

3. Tropical multiplicities

Having defined moduli spaces of abstract and plane tropical curves as polyhedral complexes we will now go on and define morphisms between them. Important properties of such morphisms will be their "tropical" multiplicities and degrees.

Definition 3.1.

(a) A *morphism* between two polyhedral complexes X and Y is a continuous map $f : X \to Y$ such that for each cell $X_i \subset X$ the image $f(X_i)$ is contained in only one cell of Y, and $f|_{X_i}$ is a linear map (of polyhedra).

(b) Let $f : X \to Y$ be a morphism of polyhedral complexes of the same pure dimension, and let $P \in X$ be a point such that both P and $f(P)$ are in general position (in X respectively Y). Then locally around P the map f is a linear map between vector spaces of the same dimension. We define the *multiplicity* $\operatorname{mult}_f(P)$ of f at P to be the absolute value of the determinant of this linear map. Note that the multiplicity depends only on the cell of X in which P lies. We will therefore also call it the multiplicity of f in this cell.

(c) Again let $f : X \to Y$ be a morphism of polyhedral complexes of the same pure dimension. A point $P \in Y$ is said to be *in f-general position* if P is in general position in Y and all points of $f^{-1}(P)$ are in general position in X. Note that the set of points in f-general position in Y is the complement of a subset of Y of dimension at most $\dim Y - 1$; in particular it is a dense open subset. Now if $P \in Y$ is a point in f-general position we define the *degree* of f at P to be

$$\deg_f(P) := \sum_{Q \in f^{-1}(P)} \operatorname{mult}_f(Q).$$

Note that this sum is indeed finite: first of all there are only finitely many cells in X. Moreover, in each cell (of maximal dimension) of X where f is not injective (i.e. where there might be infinitely many inverse image points of P) the determinant of f is zero and hence so is the multiplicity for all points in this cell.

Moreover, since X and Y are of the same pure dimension, the cones of X on which f is not injective are mapped to a locus of codimension at least 1 in Y. Thus the set of points in f-general position away from this locus is also a dense open subset of Y, and for all points in this locus we have that not only the sum above but indeed the fiber of P is finite.

Remark 3.2. Note that the definition of multiplicity in Definition 3.1(b) depends on the choice of coordinates on the cells of X and Y. For the spaces $\mathcal{M}_n$ and $\mathcal{M}_{\Delta,n}$ (with cells $\mathcal{M}_n^\alpha$ and $\mathcal{M}_{\Delta,n}^\alpha$) there were several equally natural choices of coordinates in the proof of Proposition 2.11: for graphs of a fixed combinatorial type we had to pick an ordering of the bounded edges and a root vertex. We claim that the coordinates for two different choices will simply differ by a linear isomorphism with determinant ± 1. In fact, this is obvious for a relabeling of the bounded edges. As for a change of root vertex simply note that the difference $h(V_2) - h(V_1)$ of the images of two vertices is given by $\sum_F l([F]) \cdot v(F)$, where the sum is taken over the (unique) chain of flags leading from V_1 to V_2. This is obviously a linear combination of the lengths of the bounded edges, i.e. of the other coordinates in the cell. As these length coordinates themselves remain unchanged it is clear that the determinant of this change of coordinates is 1. We conclude that *the multiplicities and degrees of a morphism of polyhedral complexes whose source and/or target is a moduli space of abstract or plane tropical curves do not depend on any choices* (*of a root vertex or a labeling of the bounded edges*).

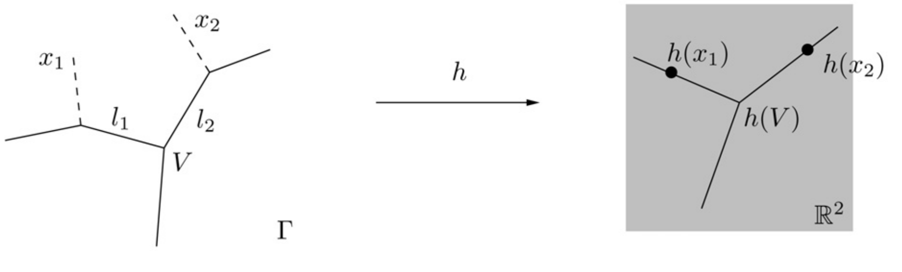
Example 3.3. For $i \in \{1, \ldots, n\}$ the *evaluation maps*

$$\mathrm{ev}_i : \mathcal{M}_{\Delta,n} \to \mathbb{R}^2, \quad (\Gamma, x_1, \ldots, x_n, h) \mapsto h(x_i)$$

are morphisms of polyhedral complexes. We denote the two coordinate functions of ev_i by $\mathrm{ev}_i^1, \mathrm{ev}_i^2 : \mathcal{M}_{\Delta,n} \to \mathbb{R}$ and the total evaluation map by $\mathrm{ev} = \mathrm{ev}_1 \times \cdots \times \mathrm{ev}_n : \mathcal{M}_{\Delta,n} \to \mathbb{R}^{2n}$. Of course these maps are morphisms of polyhedral complexes as well.

As a concrete example consider plane tropical curves of the following combinatorial types:

(a) For the combinatorial type

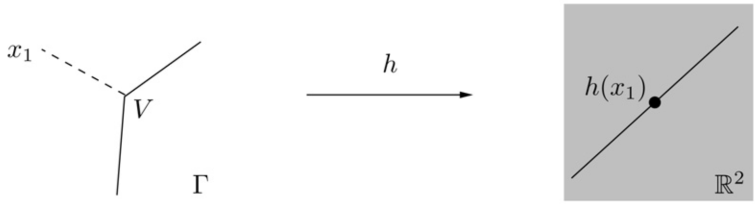


we choose V as the root vertex, say its image has coordinates $h(V) = (a, b)$. There are two bounded edges with lengths l_i and direction vectors $v_i = (v_{i,1}, v_{i,2})$ (counted from the root vertex) for $i = 1, 2$. Then a, b, l_1, l_2 are the coordinates of $\mathcal{M}^{\alpha}_{\Delta,2}$, and the evaluation maps are given by $h(x_i) = h(V) + l_i \cdot v_i = (a + l_i v_{i,1}, b + l_i v_{i,2})$. In particular, the total evaluation map $\mathrm{ev} = \mathrm{ev}_1 \times \mathrm{ev}_2$ is linear, and in the coordinates above its matrix is

$$\begin{pmatrix} 1 & 0 & v_{1,1} & 0 \\ 0 & 1 & v_{1,2} & 0 \\ 1 & 0 & 0 & v_{2,1} \\ 0 & 1 & 0 & v_{2,2} \end{pmatrix}.$$

An easy computation shows that the absolute value of the determinant of this matrix is $\mathrm{mult}_{\mathrm{ev}}(\alpha) = |\det(v_1, v_2)|$. This is in fact the definition of the multiplicity $\mathrm{mult}(V)$ of the vertex V in [4, Definition 4.15].

(b) For the combinatorial type



the computation is even simpler: with the same reasoning as above the matrix of the evaluation map is just the 2×2 unit matrix, and thus we get $\mathrm{mult}_{\mathrm{ev}}(\alpha) = 1$.

Note that the entries of the matrices of evaluation maps will always be integers since the direction vectors of plane tropical curves lie in $\mathbb{Z}^2$ by definition. In particular, multiplicities and degrees of evaluation maps will always be non-negative integers.

Example 3.4. Let $n = |\Delta| - 1$, and consider the evaluation map $\mathrm{ev} : \mathcal{M}_{\Delta,n} \to \mathbb{R}^{2n}$. Since both source and target of this map have dimension $2n$ we can consider the numbers

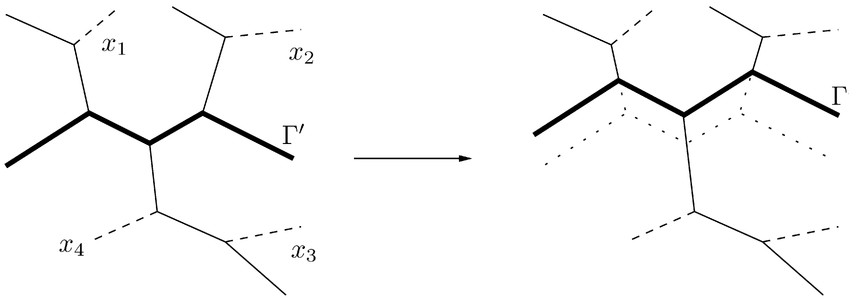
$$N_\Delta(\mathcal{P}) := \deg_{\mathrm{ev}}(\mathcal{P}) \in \mathbb{Z}_{\geqslant 0}$$

for all points $\mathcal{P} \in \mathbb{R}^{2n}$ in ev-general position. Note that these numbers are obviously just counting the tropical curves of degree Δ through the points $\mathcal{P}$, where each such curve C is counted with a certain multiplicity $\mathrm{mult}_{\mathrm{ev}}(C)$. In the remaining part of this section we want to show how this multiplicity can be computed easily and that it is in fact the same as in Definitions 4.15 and 4.16 of [4].

Definition 3.5. Let $C = (\Gamma, x_1, \ldots, x_n, h) \in \mathcal{M}_{\Delta,n}$ be a 3-valent plane tropical curve.

(a) A *string* of C is a subgraph of Γ homeomorphic to $\mathbb{R}$ (i.e. a "path in Γ with two unbounded ends") that does not intersect the closures $\overline{x_i}$ of the marked points.
(b) We say that (the combinatorial type of) C is *rigid* if Γ has no strings.
(c) The *multiplicity* $\mathrm{mult}(V)$ of a vertex V of C is defined to be $|\det(v_1, v_2)|$, where v_1 and v_2 are two of the three direction vectors around V (by the balancing condition it does not matter which ones we take here). The *multiplicity* $\mathrm{mult}(C)$ of C is the product of the multiplicities of all its vertices that are not adjacent to any marked point.

Remark 3.6. If $C = (\Gamma, x_1, \ldots, x_n, h)$ is a plane curve that contains a string $\Gamma' \subset \Gamma$ then there is a 1-parameter deformation of C that moves the position of the string in $\mathbb{R}^2$, but changes neither the images of the marked points nor the lines in $\mathbb{R}^2$ on which the edges of $\Gamma \backslash \Gamma'$ lie. The following picture shows an example of (the image of) a plane 4-marked tropical curve with exactly one string Γ' together with its corresponding deformation:



Remark 3.7. If $C = (\Gamma, x_1, \ldots, x_n, h)$ is an n-marked plane tropical curve of degree Δ then the connected subgraph $\Gamma \backslash \bigcup_i x_i$ has exactly $|\Delta|$ unbounded ends. So if $n < |\Delta| - 1$ there must be at least two unbounded ends that are still connected in $\Gamma \backslash \bigcup_i \overline{x_i}$, i.e. there must be a string in C. If $n = |\Delta| - 1$ then C is rigid if and only if every connected component of $\Gamma \backslash \bigcup_i \overline{x_i}$ has *exactly* one unbounded end.

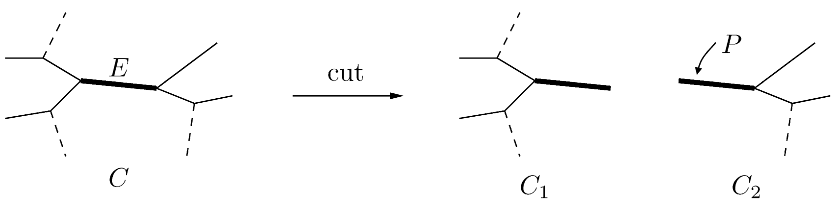
Proposition 3.8. *Let* $n = |\Delta| - 1$. *For any* n*-marked* 3*-valent plane tropical curve* C *we have*

$$\operatorname{mult}_{\mathrm{ev}}(C) = \begin{cases} \operatorname{mult}(C) & \text{if } C \text{ is rigid,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\operatorname{mult}(C)$ *is as in Definition* 3.5(c).

Proof. If C is not rigid then by Remark 3.6 it can be deformed with the images of the marked points fixed in $\mathbb{R}^2$. This means that the evaluation map cannot be a local isomorphism and thus $\operatorname{mult}_{\mathrm{ev}}(C) = 0$. We will therefore assume from now on that C is rigid.

We prove the statement by induction on the number $k = 2n - 2$ of bounded edges of C. The first cases $k = 0$ and $k = 2$ have been considered in Example 3.3. So we can assume that $k \geqslant 4$. Choose any bounded edge E so that there is at least one bounded edge of C to both sides of E. We cut C along this edge into two halves C_1 and C_2. By extending the cut edge to infinity on both sides we can make C_1 and C_2 into plane tropical curves themselves:



(note that in this picture we have not drawn the map h to $\mathbb{R}^2$ but only the underlying abstract tropical curves). For $i \in \{1, 2\}$ we denote by n_i and k_i the number of marked points and bounded edges of C_i, respectively. Of course we have $n_1 + n_2 = n$ and $k_1 + k_2 = k - 1 = 2n - 3$.

Assume first that $k_1 \leqslant 2n_1 - 3$. As C_1 is 3-valent the total number of unbounded edges of C_1 is $k_1 + 3 \leqslant 2n_1$; the number of unmarked unbounded edges is therefore at most n_1. This means that there must be at least one bounded connected component when we remove the closures of the marked points from C_1. The same is then true for C, i.e. by Remark 3.7 C is not rigid in contradiction to our assumption. By symmetry the same is of course true if $k_2 \leqslant 2n_2 - 3$.

The only possibility left is therefore $k_1 = 2n_1 - 2$ and $k_2 = 2n_2 - 1$ (or vice versa). If we pick a root vertex in C_1 then in the matrix representation of the evaluation map we have $2n_1$ coordinates in $\mathbb{R}^{2n}$ (namely the images of the marked points on C_1) that depend on only $2 + k_1 = 2n_1$ coordinates (namely the root vertex and the lengths of the k_1 bounded edges in C_1). Hence the matrix has the form

$$\left(\begin{array}{c|c} A_1 & 0 \\ \hline * & A_2 \end{array}\right)$$

where A_1 and A_2 are square matrices of size $2n_1$ and $2n_2$, respectively. Note that A_1 is precisely the matrix of the evaluation map for C_1. As for A_2 its columns correspond to the lengths of E and the k_2 bounded edges of C_2, and its rows to the image points of the n_2 marked points on C_2. So if we consider the plane curve $\tilde{C}_2$ obtained from C_2 by adding a marked point at a point P on E (see the picture above) and pick the vertex P as the root vertex then the matrix for the evaluation map of $\tilde{C}_2$ is of the form

$$\left(\begin{array}{c|c} I_2 & 0 \\ \hline * & A_2 \end{array}\right)$$

where I_2 denotes the 2×2 unit matrix and the two additional rows and columns correspond to the position of the root vertex. In particular this matrix has the same determinant as A_2. So we conclude that

$$\mathrm{mult}_{\mathrm{ev}}(C) = |\det A_1 \cdot \det A_2| = \mathrm{mult}_{\mathrm{ev}_1}(C_1) \cdot \mathrm{mult}_{\mathrm{ev}_2}(\tilde{C}_2),$$

where ev_1 and ev_2 denote the evaluation maps on C_1 and $\tilde{C}_2$, respectively. The proposition now follows by induction, noting that C_1 and C_2 are rigid if C is. □

Remark 3.9. By Proposition 3.8 our numbers $N_\Delta(\mathcal{P})$ are the same as the ones in [4], and thus by the Correspondence Theorem (Theorem 1 in [4]) the same as the corresponding complex numbers of stable maps. In particular they do not depend on $\mathcal{P}$ (as long as the points are in general position), and it is clear that the numbers $N_d := N_d(\mathcal{P})$ must satisfy Kontsevich's formula stated in the introduction. It is the goal of the rest of the paper to give an entirely tropical proof of this statement.

4. The forgetful maps

We will now introduce the forgetful maps that have already been mentioned in the introduction. As for the complex moduli spaces of stable maps there are many such maps: given an n-marked plane tropical curve we can forget the map to $\mathbb{R}^2$, or some of the marked points, or both.

Definition 4.1 *(Forgetful maps).* Let $n \geqslant m$ be integers, and let $C = (\Gamma, x_1, \dots, x_n, h) \in \mathcal{M}_{\Delta,n}$ be an n-marked plane tropical curve.

(a) (Forgetting the map and some points.) Let $C(m)$ be the minimal connected subgraph of Γ that contains the unbounded edges $x_1, \dots, x_m$. Note that $C(m)$ cannot contain vertices of valence 1. So if we "straighten" the graph $C(m)$ at all 2-valent vertices (i.e. we replace the two adjacent edges and the vertex by one edge whose length is the sum of the lengths of the original edges) then we obtain an element of $\mathcal{M}_m$ that we denote by $\mathrm{ft}_m(C)$.

(b) (Forgetting some points only.) Let $\tilde{C}(m)$ be the minimal connected subgraph of Γ that contains all unmarked ends as well as the marked points $x_1, \ldots, x_m$. Again $\tilde{C}(m)$ cannot have vertices of valence 1. If we straighten $\tilde{C}(m)$ as in (a) we obtain an abstract tropical curve $\tilde{\Gamma}$ with $|\Delta| + m$ markings. Note that the restriction of h to $\tilde{\Gamma}$ still satisfies the requirements for a plane tropical curve, i.e. $(\tilde{\Gamma}, x_1, \ldots, x_m, h|_{\tilde{\Gamma}})$ is an element of $\mathcal{M}_{\Delta,m}$. We denote it by $\tilde{\mathrm{ft}}_m(C)$.

It is obvious that the maps $\mathrm{ft}_m : \mathcal{M}_{\Delta,n} \to \mathcal{M}_m$ and $\tilde{\mathrm{ft}}_m : \mathcal{M}_{\Delta,n} \to \mathcal{M}_{\Delta,m}$ defined in this way are morphisms of polyhedral complexes. We call them the *forgetful maps* (that keep only the first m marked points respectively the first m marked points and the map). Of course there are variations of the above maps: we can forget a given subset of the n marked points that are not necessarily the last ones, or we can forget some points of an abstract tropical curve to obtain maps $\mathcal{M}_n \to \mathcal{M}_m$.

Example 4.2. For the plane tropical curve C of Example 1.2 the graph $C(4)$ is simply the subgraph drawn in bold, and $\mathrm{ft}_4(C)$ is the "straightened version" of this graph, i.e. the 4-marked tropical curve of type (A) in Example 1.1 with length parameter l as indicated in the picture. Of course this length parameter is then also the local coordinate of $\mathcal{M}_4$ if we want to represent the morphism ft_4 of polyhedral complexes by a matrix, i.e. the matrix describing ft_4 is the matrix with one row that has a 1 precisely at the column corresponding to the bounded edge marked l (and zeroes otherwise).

The map that we need to consider for Kontsevich's formula is the following:

Definition 4.3. Fix $d \geqslant 2$, and let $n = 3d$. We set

$$\pi := \mathrm{ev}_1^1 \times \mathrm{ev}_2^2 \times \mathrm{ev}_3 \times \cdots \times \mathrm{ev}_n \times \mathrm{ft}_4 : \mathcal{M}_{d,n} \to \mathbb{R}^{2n-2} \times \mathcal{M}_4,$$

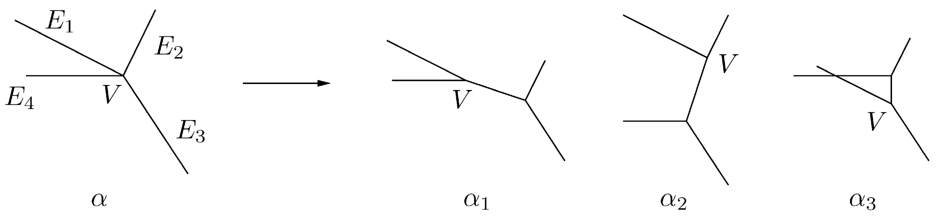
i.e. π describes the first coordinate of the first marked point, the second coordinate of the second marked point, both coordinates of the other marked points, and the point in $\mathcal{M}_4$ defined by the first four marked points. Obviously, π is a morphism of polyhedral complexes of pure dimension $2n - 1$.

The central result of this section is the following proposition showing that the degrees $\deg_\pi(\mathcal{P})$ of π do not depend on the chosen point $\mathcal{P}$. Ideally this should simply follow from π being a "morphism of tropical varieties" (and not just a morphism of polyhedral complexes). As there is no such theory yet however we have to prove the independence of $\mathcal{P}$ directly.

Proposition 4.4. *The degrees* $\deg_\pi(\mathcal{P})$ *do not depend on* $\mathcal{P}$ (*as long as* $\mathcal{P}$ *is in* π*-general position*).

Proof. It is clear that the degree of π is *locally* constant on the subset of $\mathbb{R}^{2n-2} \times \mathcal{M}_4$ of points in π-general position since at any curve that counts for $\deg_\pi(\mathcal{P})$ with a non-zero multiplicity the map π is a local isomorphism. Recall that the points in π-general position are the complement of a polyhedral complex of codimension 1, i.e. they form a finite number of top-dimensional regions separated by "walls" that are polyhedra of codimension 1. Hence to show that $\deg_\pi$ is globally constant it suffices to consider a general point on such a wall and to show that $\deg_\pi$ is locally constant at these points too. Such a general point on a wall is simply the image under π of a general plane tropical curve C of a combinatorial type of codimension 1. So we simply have to check that $\deg_\pi$ is locally constant around such a point $C \in \mathcal{M}_{\Delta,n}$.

By definition a combinatorial type α of codimension 1 has exactly one 4-valent vertex V, with all other vertices being 3-valent. Let $E_1, \ldots, E_4$ denote the four (bounded or unbounded) edges around V. There are precisely 3 combinatorial types $\alpha_1, \alpha_2, \alpha_3$ that have α in their boundary, as indicated in the following local picture:



Let us assume first that all four edges E_i are bounded. We denote their lengths by l_i and their directions (pointing away from V) by v_i. To set up the matrices of π we choose the root vertex V in α_i as in the picture. We denote its image by $w \in \mathbb{R}^2$.

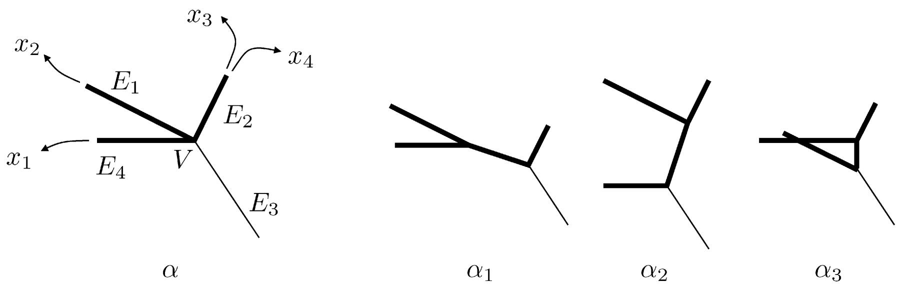
The following table shows the relevant parts of the matrices A_i of π for the three combinatorial types α_i. Each matrix contains the first block of columns (corresponding to the image w of the root vertex and the lengths l_i of the edges E_i) and the ith of the last three columns (corresponding to the length of the newly added bounded edge). The columns corresponding to the other bounded edges are not shown; it suffices to note here that they are the same for all three matrices. All rows but the last one correspond to the images in $\mathbb{R}^2$ of the marked points; we get different types of rows depending on via which edge E_i this marked point can be reached from V. For the marked points x_i with $i \geqslant 3$ we use both coordinates in $\mathbb{R}^2$ (hence one row in the table below corresponds to two rows in the matrix), for x_1 only the first and for x_2 only the second coordinate. The last row corresponds to the coordinate in $\mathcal{M}_4$ as in Example 4.2. In the following table I_2 denotes the 2×2 unit matrix, and each $*$ and $**$ stands for 0 or 1 (see below).

	w	l_1	l_2	l_3	l_4	l^{α_1}	l^{α_2}	l^{α_3}
points behind E_1	I_2	v_1	0	0	0	0	0	0
points behind E_2	I_2	0	v_2	0	0	v_2+v_3	0	v_2+v_4
points behind E_3	I_2	0	0	v_3	0	v_2+v_3	v_3+v_4	0
points behind E_4	I_2	0	0	0	v_4	0	v_3+v_4	v_2+v_4
coordinate of $\mathcal{M}_4$	0	$*$	$*$	$*$	$*$	$**$	$**$	$**$

To look at these matrices (in particular at the entries marked $*$) further we will distinguish several cases depending on how many of the edges $E_1, \ldots, E_4$ of C are contained in the subgraph $C(4)$ of Definition 4.1:

(a) 4 edges: Then $\mathrm{ft}_4(C)$ is the curve (D) of Example 1.1, and the three types $\alpha_1, \alpha_2, \alpha_3$ are mapped precisely to the three other types (A), (B), (C) of $\mathcal{M}_4$ by ft_4, i.e. to the three cells of $\mathbb{R}^{2n-2} \times \mathcal{M}_4$ around the wall by π. For these three types the length parameter in $\mathcal{M}_4$ is simply the one newly inserted edge. Hence the entries $*$ in the matrix above are all 0, whereas the entries $**$ are all 1. It follows that the three matrices A_1, A_2, A_3 have a 1 as the bottom right entry and all zeroes in the remaining places of the last row. Their determinants therefore do not depend on the last column. But this is the only column that differs for the three matrices, i.e. A_1, A_2, and A_3 all have the same determinant. It follows by definition that $\deg_\pi$ is locally constant around C. This completes the proof of the proposition in this case.

(b) 3 edges: The following picture shows what the combinatorial types $\alpha, \alpha_1, \alpha_2, \alpha_3$ look like locally around the vertex V in this case. As in Example 1.2 we have drawn the edges belonging to $C(4)$ in bold.



We see that exactly one edge E_i (namely E_2 in the example above) counts towards the length parameter in $\mathcal{M}_4$, and that the newly inserted edge counts towards this length parameter in exactly one of the combinatorial types α_i (namely α_1 in the example above). Hence in the table showing the matrices A_i above exactly one of the entries $*$ and exactly one of the entries $**$ is 1, whereas the others are 0.

(c) 2 edges: There are two possibilities in this case. If V is a point in $C(4)$ corresponding to an interior point of the bounded edge in $\mathrm{ft}_4(C)$ then an analysis completely analogous to that in (b) shows that exactly 2 of the entries $*$ and also 2 of the entries $**$ above are 1, whereas the others are zero. If on the other hand V corresponds to an interior point of an unbounded edge in $\mathrm{ft}_4(C)$ then all entries $*$ and $**$ above are 0.

(d) fewer than 2 edges: As it is not possible that exactly one of the edges E_i is contained in $C(4)$ we must then have that there is no such edge, and consequently that all entries $*$ and $**$ above are 0.

Summarizing, we see in all remaining cases (b), (c), and (d) that there are equally many entries $**$ equal to 1 as there are entries $*$ equal to 1. So using the linearity of the determinant in the column corresponding to the new bounded edge we get that $\det A_1 + \det A_2 + \det A_3$ is equal to the determinant of the matrix whose corresponding entries are

	w	l_1	l_2	l_3	l_4	l
points behind E_1	I_2	v_1	0	0	0	0
points behind E_2	I_2	0	v_2	0	0	$2v_2 + v_3 + v_4$
points behind E_3	I_2	0	0	v_3	0	$2v_3 + v_2 + v_4$
points behind E_4	I_2	0	0	0	v_4	$2v_4 + v_2 + v_3$
coordinate of $\mathcal{M}_4$	0	$*$	$*$	$*$	$*$	$**$

where $**$ is now the sum of the four entries marked $*$. If we now subtract the four l_i-columns and add v_1 times the w-columns from the last one then all entries in the last column vanish (note that $v_1 + v_2 + v_3 + v_4 = 0$ by the balancing condition). So we conclude that

$$\det A_1 + \det A_2 + \det A_3 = 0. \tag{1}$$

For a given $i \in \{1, 2, 3\}$ let us now determine whether the combinatorial type α_i occurs in the inverse image of a fixed point $\mathcal{P}$ near the wall. We may assume without loss of generality that the multiplicity of α_i is non-zero since other types are irrelevant for the statement of the proposition. So the restriction π_i of π to $\mathcal{M}^{\alpha_i}_{\Delta,n}$ is given by the invertible matrix A_i. There is therefore at most one inverse image point in $\pi_i^{-1}(\mathcal{P})$, which would have to be the point with coordinates $A_i^{-1} \cdot \mathcal{P}$. In fact, this point exists in $\mathcal{M}^{\alpha_i}_{\Delta,n}$ if and only if all coordinates of $A_i^{-1} \cdot \mathcal{P}$ corresponding to lengths of bounded edges are positive. By continuity this is obvious for all edges except the newly added one since in the boundary curve C all these edges had positive length. We conclude that there is a point in $\pi_i^{-1}(\mathcal{P})$ if and only if the last coordinate (corresponding to the length of the newly added edge) of $A_i^{-1} \cdot \mathcal{P}$ is positive. By Cramer's rule this last coordinate is $\det \tilde{A}_i / \det A_i$, where $\tilde{A}_i$ denotes the matrix A_i with the last column replaced by $\mathcal{P}$. But note that $\tilde{A}_i$ does not depend on i since the last column was the only one where the A_i differ. Hence whether there is a point in $\pi_i^{-1}(\mathcal{P})$ or not depends solely on the sign of $\det A_i$: either there are such inverse image points for exactly those i where $\det A_i$ is positive, or exactly for those i where $\det A_i$ is negative. But by (1) the sum of the absolute values of the determinants satisfying this condition is the same in both cases. This means that $\deg_\pi$ is locally constant around C.

Strictly speaking we have assumed in the above proof that all edges E_i are bounded. It is very easy however to adapt these arguments to the other cases: if an edge E_i is not bounded then there is no coordinate l_i corresponding to its length, but neither are there marked points that can be reached from V via E_i. We leave it as an exercise to check that the above proof still holds in this case with essentially no modifications. □

5. Kontsevich's formula

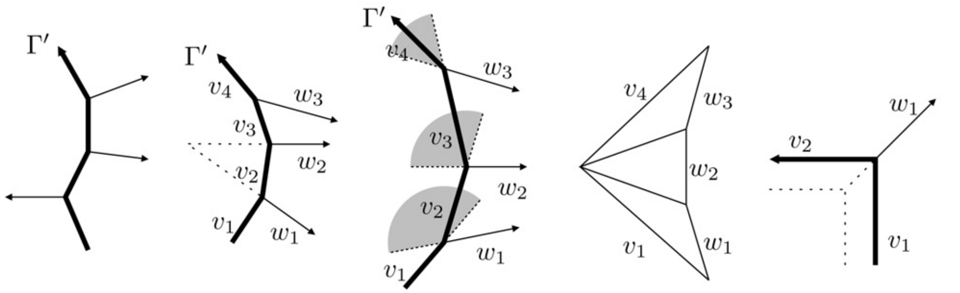
We have just shown that the degrees of the morphism $\pi : \mathcal{M}_{d,n} \to \mathbb{R}^{2n-2} \times \mathcal{M}_4$ of Definition 4.3 do not depend on the point chosen in the target. We now want to apply this result by equating the degrees for two different points, namely two points where the $\mathcal{M}_4$-coordinate is very large, but corresponds to curves of type (A) or (B) in Example 1.1. We will first prove that a very large length in $\mathcal{M}_4$ requires the curves to acquire a contracted bounded edge.

Proposition 5.1. *Let $d \geqslant 2$ and $n = 3d$, and let $\mathcal{P} \in \mathbb{R}^{2n-2} \times \mathcal{M}_4$ be a point in π-general position whose $\mathcal{M}_4$-coordinate is very large* (*i.e. it corresponds to a* 4*-marked curve of type* (A), (B)*, or* (C) *as in Example* 1.1 *with a very large length l*)*. Then every plane tropical curve $C \in \pi^{-1}(\mathcal{P})$ with* $\operatorname{mult}_\pi(C) \neq 0$ *has a contracted bounded edge.*

Proof. We have to show that the set of all points $\operatorname{ft}_4(C) \in \mathcal{M}_4$ is bounded in $\mathcal{M}_4$, where C runs over all curves in $\mathcal{M}_{d,n}$ with non-zero π-multiplicity that have no contracted bounded edge and satisfy the given incidence conditions at the marked points. As there are only finitely many combinatorial types by Lemma 2.10 we can restrict ourselves to curves of a fixed (but arbitrary) combinatorial type α. Since $\mathcal{P}$ is in π-general position we can assume that the codimension of α is 0, i.e. that the curve is 3-valent.

Let $C' \in \mathcal{M}_{d,n-2}$ be the curve obtained from C by forgetting the first two marked points as in Definition 4.1. We claim that C' has exactly one string (see Definition 3.5(a)). In fact, C' must have at least one string by Remark 3.7 since C' has less than $3d-1=n-1$ marked points. On the other hand, if C' had at least two strings then by Remark 3.6 C' would move in an at least 2-dimensional family with the images of $x_3, \dots, x_n$ fixed. It follows that C moves in an at least 2-dimensional family as well with the first coordinate of x_1, the second of x_2, and both of $x_3, \dots, x_n$ fixed. As $\mathcal{M}_4$ is one-dimensional this means that C moves in an at least 1-dimensional family with the image point under π fixed. Hence π is not a local isomorphism, i.e. $\mathrm{mult}_\pi(C)=0$ in contradiction to our assumptions.

So let Γ' be the unique string in C'. The deformations of C' with the given incidence conditions fixed are then precisely the ones of the string described in Remark 3.6. Note that the edges adjacent to Γ' must be bounded since otherwise we would have two strings. So if there are edges adjacent to Γ' to both sides of Γ' as in picture (a) below (note that there are no contracted bounded edges by assumption) then the deformations of C' with the combinatorial type and the incidence conditions fixed are bounded on both sides. For the deformations of C with its combinatorial type and the incidence conditions fixed this means that the lengths of all inner edges are bounded except possibly the edges adjacent to x_1 and x_2. This is sufficient to ensure that the image of these curves under ft_4 is bounded in $\mathcal{M}_4$ as well.



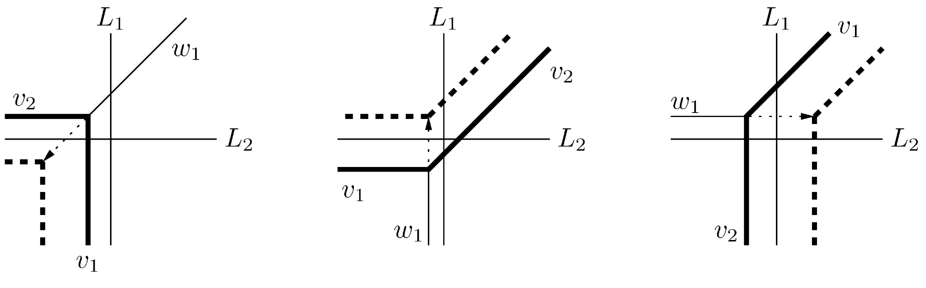
(a) (b) (c) (d) (e)

Hence we are only left with the case when all adjacent edges of Γ' are on the same side of Γ', say after picking an orientation of Γ on the right side as in picture (b) above. Label the edges (respectively their direction vectors) of Γ' by $v_1, \dots, v_k$ and the adjacent edges of the curve by $w_1, \dots, w_{k-1}$ as in the picture. As above the movement of C' to the right within its combinatorial type is bounded. If one of the directions w_{i+1} is obtained from w_i by a left turn (as it is the case for $i=1$ in the picture) then the edges w_i and w_{i+1} meet to the left of Γ'. This restricts the movement of C' to the left within its combinatorial type too since the corresponding edge v_{i+1} then shrinks to zero. We can then conclude as in case (a) above that the image of these curves under ft_4 is bounded.

We can therefore assume that for all i the direction w_{i+1} is either the same as w_i or obtained from w_i by a right turn as in picture (c). The balancing condition then ensures that for all i both the directions v_{i+1} and $-w_{i+1}$ lie in the angle between v_i and $-w_i$ (shaded in the picture above). It follows that all directions v_i and $-w_i$ lie within the angle between v_1 and $-w_1$. In particular the string Γ' cannot have any self-intersections in $\mathbb{R}^2$. We can therefore pass to the (local) dual picture (d) (see e.g. [4, Section 3.4]) where the edges dual to w_i correspond to a concave side of the polygon whose other two edges are the ones dual to v_1 and v_k. In other words, the intersection points of the edges dual to w_{i-1} and w_i must be in the interior of the triangle spanned by the edges dual to v_1 and v_k for all $1 < i < k$.

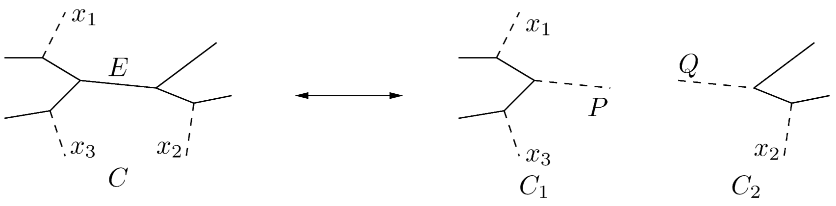
But note that both v_1 and v_k must be $(-1,0)$, $(0,-1)$, or $(1,1)$ since they are outer directions of a curve of degree d. Consequently, their dual edges have to be among the vectors $\pm(1,0)$, $\pm(0,1)$, $\pm(1,-1)$. But any triangle spanned by two of these vectors has area (at most) $\frac{1}{2}$ and thus does not admit any integer interior points. It follows that intersection points of the dual edges of w_{i-1} and w_i as above cannot exist and therefore that $k = 2$, i.e. that the string consists just of the two unbounded ends v_1 and v_2 that are connected to the rest of the curve by exactly one internal edge w_1. It must therefore look as in picture (e).

In this case the movement of the string is indeed not bounded to the left. Note that then w_1 is the only internal edge whose length is not bounded within the deformations of C' since the rest of the curve (not shown in picture (e)) does not move at all. But we will show that this unbounded length of w_1 cannot count towards the length parameter in $\mathcal{M}_4$ for the deformations of C: first of all this would require two of the marked points $x_1, \ldots, x_4$ to lie on v_1 or v_2 for all curves in the deformation, but of course with v_1 and v_2 forming a string we cannot have x_3 or x_4 (where we impose point conditions) on them. Hence we would have to have x_1 and x_2 (that we require to lie on a vertical line L_1 respectively a horizontal line L_2) somewhere on v_1 and v_2. But the following picture shows that for all three possibilities for v_1 and v_2 the union of the edges v_1 and v_2 (drawn in bold) finally becomes disjoint from at least one of the lines L_1 and L_2 as the length of w_1 increases:



This means that we cannot have both x_1 and x_2 on the union of v_1 and v_2 as the length of w_1 increases. Consequently, we cannot get unbounded length parameters in $\mathcal{M}_4$ in this case either. This finishes the proof of the proposition. $\square$

Remark 5.2. Let $C = (\Gamma, x_1, \ldots, x_n, h)$ be a plane tropical curve with a contracted bounded edge E, and assume that there is at least one more bounded edge to both sides of E. Then in the same way as in the proof of Proposition 3.8 we can split Γ at E into two graphs Γ_1 and Γ_2, making the edge E into a contracted *unbounded* edge on both sides. By restricting h to these graphs we obtain two new plane tropical curves C_1 and C_2. The marked points $x_1, \ldots, x_n$ obviously split up onto C_1 and C_2; in addition there is one more marked point P respectively Q on both curves that corresponds to the newly added contracted unbounded edge. If C is a curve of degree d then (by the balancing condition) C_1 and C_2 are of some degrees d_1 and d_2 with $d_1 + d_2 = d$.



We will say in this situation that C is obtained by *glueing* C_1 and C_2 along the identification $P = Q$, and that C is a *reducible* plane tropical curve that can be decomposed into C_1 and C_2. For the image we obviously have $h(\Gamma) = h(\Gamma_1) \cup h(\Gamma_2)$, so when considering embedded plane tropical curves C is in fact just the union of the two curves C_1 and C_2 of smaller degree (see Example 1.3).

Lemma 5.3. *Let $\mathcal{P} = (a, b, p_3, \dots, p_n, z) \in \mathbb{R}^{2n-2} \times \mathcal{M}_4$ be a point in π-general position such that $z \in \mathcal{M}_4$ is of type* (A) (*see Example* 1.1) *with a very large length parameter. Then for every plane tropical curve C in $\pi^{-1}(\mathcal{P})$ with non-zero π-multiplicity we have exactly one of the following cases*:

(a) *x_1 and x_2 are adjacent to the same vertex* (*that maps to* (a, b) *under h*);
(b) *C is reducible and decomposes uniquely into two components C_1 and C_2 of some degrees d_1 and d_2 with $d_1 + d_2 = d$ such that the marked points x_1 and x_2 are on C_1, the points x_3 and x_4 are on C_2, and exactly* $3d_1 - 1$ *of the other points $x_5, \dots, x_n$ are on C_1.*

Proof. By Proposition 5.1 any curve $C \in \pi^{-1}(\mathcal{P})$ with non-zero π-multiplicity has at least one contracted bounded edge. In fact, C must have *exactly* one such edge: if C had at least 2 contracted bounded edges then there would be $2n - 2$ coordinates in the target of π (namely the evaluation maps) that depend on only $2n - 3$ variables (namely the root vertex and the lengths of all but 2 of the $2n - 3$ bounded edges), hence we would have $\mathrm{mult}_\pi(C) = 0$.

So let E be the unique contracted bounded edge of C. Note that E must be contained in the subgraph $C(4)$ of Definition 4.1(a) since otherwise we could not have a very large length parameter in $\mathcal{M}_4$. As the point z is of type (A) we conclude that x_1 and x_2 must be to one side, and x_3 and x_4 to the other side of E. Denote these sides by C_1 and C_2, respectively.

If there are no bounded edges in C_1 then C is not reducible as in Remark 5.2. Instead C_1 consists only of E, x_1, and x_2, i.e. we are then in case (a). The evaluation conditions then require that all of C_1 must be mapped to the point (a, b). Note that it is not possible that there are no bounded edges in C_2 since this would require x_3 and x_4 to map to the same point in $\mathbb{R}^2$.

We are left with the case when there are bounded edges to both sides of E. In this case C is reducible as in Remark 5.2, so we are in case (b). In this case x_1 and x_2 cannot be adjacent to the same vertex since this would require another contracted edge by the balancing condition. Now let n_1 and n_2 be the number of marked points $x_5, \dots, x_n$ on C_1 respectively C_2; we have to show that $n_1 = 3d_1 - 1$ and $n_2 = 3d_2 - 3$. So assume that $n_1 \geqslant 3d_1$. Then at least $2n_1 + 2 \geqslant 3d_1 + n_1 + 2$ of the coordinates of π (the images of the n_1 marked points as well as the first image coordinate of x_1 and the second of x_2) would depend on only $3d_1 + n_1 + 1$ coordinates (2 for the root vertex and one for each of the $3d_1 + (n_1 + 2) - 3$ bounded edges), leading to a zero π-multiplicity. Hence we conclude that $n_1 \leqslant 3d_1 - 1$. The same argument shows that $n_2 \leqslant 3d_2 - 3$, so as the total number of points is $n_1 + n_2 = n - 4 = (3d_1 - 1) + (3d_2 - 3)$ it follows that we must have equality. $\square$

Remark 5.4. In fact, the following “converse” of Lemma 5.3 is also true: as above let $\mathcal{P} = (a, b, p_3, \dots, p_n, z) \in \mathbb{R}^{2n-2} \times \mathcal{M}_4$ be a point in π-general position such that $z \in \mathcal{M}_4$ is of type (A) (see Example 1.1) with a very large length parameter. Now let C_1 and C_2 be two (unmarked) plane tropical curves of degrees d_1 and d_2 with $d_1 + d_2 = d$ such that the image of C_1 passes through $L_1 := \{(x, y);\ x = a\}$, $L_2 := \{(x, y);\ y = b\}$, and $3d_1 - 1$ of the points $p_5, \dots, p_n$, and the image of C_2 through p_3, p_4, and the other $3d_2 - 3$ of the points $p_5, \dots, p_n$.

Then for each choice of points $P \in C_1$ and $Q \in C_2$ that map to the same image point in $\mathbb{R}^2$, and for each choice of points $x_1, \dots, x_n$ on C_1 and C_2 that map to $L_1, L_2, p_3, \dots, p_n$, respectively, we can make C_1 and C_2 into marked plane tropical curves and glue them together to a single reducible n-marked curve C in $\pi^{-1}(\mathcal{P})$ as in Remark 5.2 (the length of the one contracted edge is determined by z).

As $\mathcal{P}$ was assumed to be in π-general position we can never construct a curve C in this way that is not 3-valent. In particular this means for example that C_1 and C_2 are guaranteed to be 3-valent themselves. Moreover, a point that is in the image of both C_1 and C_2 cannot be a vertex of either curve. In particular, it is not possible that C_1 and C_2 share a common line segment in $\mathbb{R}^2$. In the same way we see that the image of C_1 cannot meet L_1 or L_2 in a vertex or have a line segment in common with L_1 or L_2, and cannot meet $L_1 \cap L_2$ at all.

Summarizing, we see that after choosing the two curves C_1 and C_2 as well as the points $x_1, \dots, x_n, P, Q$ on them there is a unique curve in $\pi^{-1}(\mathcal{P})$ obtained from this data. So if we want to compute the degree of π and have to sum over all points in $\pi^{-1}(\mathcal{P})$ then for the curves of type (b) in Lemma 5.3 we can as well sum over all choices of $C_1, C_2, x_1, \dots, x_n, P, Q$ as above.

Before we can actually do the summation we still have to compute the multiplicity of π at the curves in $\pi^{-1}(\mathcal{P})$:

Proposition 5.5. *With notations as in Lemma* 5.3 *and Remark* 5.4 *let C be a point in $\pi^{-1}(\mathcal{P})$. Then*

(a) *if C is of type* (a) *as in Lemma* 5.3 *its π-multiplicity is* $\mathrm{mult}_{\mathrm{ev}}(C')$, *where C' denotes the curve obtained from C by forgetting x_1, and* ev *is the evaluation at the* $3d - 1$ *points* $x_2, \dots, x_n$;
(b) *if C is of type* (b) *as in Lemma* 5.3 *its π-multiplicity is*

$$\mathrm{mult}_\pi(C) = \mathrm{mult}_{\mathrm{ev}}(C_1) \cdot \mathrm{mult}_{\mathrm{ev}}(C_2) \cdot (C_1 \cdot C_2)_{P=Q} \cdot (C_1 \cdot L_1)_{x_1} \cdot (C_1 \cdot L_2)_{x_2},$$

where $\mathrm{mult}_{\mathrm{ev}}(C_i)$ *denotes the multiplicities of the evaluation map at the* $3d_i - 1$ *points of* $x_3, \dots, x_n$ *that lie on the respective curve, and* $(C' \cdot C'')_P$ *denotes the intersection multiplicity of the tropical curves C' and C'' at the point $P \in C' \cap C''$ (see* [6, *Section* 4]*), i.e.* $|\det(v', v'')|$ *where v' and v'' are the direction vectors of C' and C'' at P. In particular,* $(C_1 \cdot L_i)_{x_i}$ *is simply the first respectively second coordinate of the direction vector of C_1 at x_i for $i \in \{1, 2\}$.*

Proof. We simply have to set up the matrix for π and compute its determinant. First of all note that in both cases (a) and (b) the length of the contracted bounded edge is irrelevant for all evaluation maps and contributes with a factor of 1 to the $\mathcal{M}_4$-coordinate of π. Hence the column of π corresponding to the contracted bounded edge has only one entry 1 and all others zero. To compute its determinant we may therefore drop both the $\mathcal{M}_4$-row and the column corresponding to the contracted bounded edge.

In case (a) the matrix obtained this way is then exactly the same as if we had only one marked point instead of x_1 and x_2 and evaluate this point for both coordinates in $\mathbb{R}^2$ (instead of evaluating x_1 for the first and x_2 for the second). This proves (a).

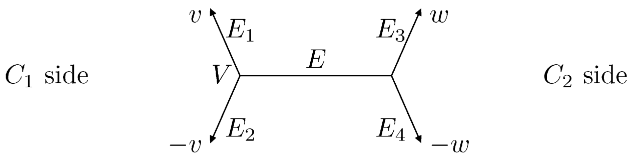
For (b) let us first consider the marked point x_1 where we only evaluate the first coordinate. Let E_1 and E_2 be the two adjacent edges and assume first that both of them are bounded. Denote their common direction vector by $v = (v^1, v^2)$ and their lengths by l_1, l_2. Assume that the root vertex is on the E_1-side of x_1. Then the entries of the matrix for π corresponding to l_1 and l_2 are

$\downarrow$ evaluation at...	l_1	l_2
x_1 (1 row)	v^1	0
points reached via E_1 from x_1 (2 rows each, except only 1 for x_2)	0	0
points reached via E_2 from x_1 (2 rows each, except only 1 for x_2)	v	v

We see that after subtracting the l_2-column from the l_1-column we again get one column with only one non-zero entry v^1. So for the determinant we get $v^1 = (C_1 \cdot L_1)_{x_1}$ as a factor, dropping the corresponding row and column (which simply means forgetting the point x_1 as in Definition 4.1(b)). Essentially the same argument holds if one of the adjacent edges—say E_2—is unbounded: in this case there is only an l_1-column which has zeroes everywhere except in the one x_1-row where the entry is v^1.

The same is of course true for x_2 and leads to a factor of $(C_1 \cdot L_2)_{x_2}$.

Next we consider again the contracted bounded edge E at which we split the curve C into the two parts C_1 and C_2. Choose one of its boundary points as root vertex V, say the one on the C_1 side. Denote the adjacent edges and their directions as in the following picture:



If we set $l_i = l(E_i)$ the matrix of π (of size $2n - 4$) reads

		root	lengths in C_1 ($2n_1 - 3$ cols)	l_1	l_2	l_3	l_4	lengths in C_2 ($2n_2 + 1$ cols)
($2n_1$ rows)	pts behind E_1	I_2	$*$	v	0	0	0	0
	pts behind E_2	I_2	$*$	0	$-v$	0	0	0
($2n_2 + 4$ rows)	pts behind E_3	I_2	0	0	0	w	0	$*$
	pts behind E_4	I_2	0	0	0	0	$-w$	$*$

where n_1 and n_2 are as in the proof of Lemma 5.3, I_2 is the 2×2 unit matrix, and $*$ denotes arbitrary entries. Now add v times the root columns to the l_2-column, subtract the l_1-column from the l_2-column and the l_4-column from the l_3-column to obtain the following matrix with the same determinant:

			lengths in C_1					lengths in C_2
		root	$(2n_1 - 3$ cols)	l_1	l_2	l_3	l_4	$(2n_2 + 1$ cols)
$(2n_1$ rows)	pts behind E_1	I_2	$*$	v	0	0	0	0
	pts behind E_2	I_2	$*$	0	0	0	0	0
$(2n_2 + 4$ rows)	pts behind E_3	I_2	0	0	v	w	0	$*$
	pts behind E_4	I_2	0	0	v	w	$-w$	$*$

Note that this matrix has a block form with a zero block at the top right. Denote the top left block (of size $2n_1$) by A_1 and the bottom right (of size $2n_2 + 4$) by A_2, so that the multiplicity that we are looking for is $|\det A_1 \cdot \det A_2|$.

The matrix A_1 is precisely the matrix for the evaluation map of C_1 if we forget the marked point corresponding to E and choose the other end point of E_2 as the root vertex. Hence $|\det A_1| = \text{mult}_{\text{ev}}(C_1)$. In the same way the matrix for the evaluation map of C_2, if we again forget the marked point corresponding to E and now choose the other end point of E_3 as the root vertex, is the matrix A_2' obtained from A_2 by replacing v and w in the first two columns by the first and second unit vector, respectively. But A_2 is simply obtained from A_2' by right multiplication with the matrix

$$\begin{pmatrix} v & w & 0 \\ 0 & 0 & I_{2n_2+2} \end{pmatrix}$$

which has determinant $\det(v, w)$. So we conclude that

$$|\det A_2| = \left|\det(v, w)\right| \cdot \left|\det A_2'\right| = (C_1 \cdot C_2)_{P=Q} \cdot \text{mult}_{\text{ev}}(C_2).$$

Collecting these results we now obtain the formula stated in the proposition. □

Of course there are completely analogous statements to Lemma 5.3, Remark 5.4, and Proposition 5.5 if the $\mathcal{M}_4$-coordinate of the curves in question is of type (B) instead of type (A) (see Example 1.1). Note however that there are no curves of type (a) in Lemma 5.3 in this case since x_1 and x_3 would have to map to $L_1 \cap p_3$, which is empty.

We can now collect our results to obtain the final theorem. The idea of this final step is actually the same as in the case of complex curves.

Theorem 5.6 *(Kontsevich's formula). The numbers N_d of Example* 3.4 *and Remark* 3.9 *satisfy the recursion formula*

$$N_d = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right) N_{d_1} N_{d_2}$$

for $d > 1$.

Proof. We compute the degree of the map π of Definition 4.3 at two different points. First consider a point $\mathcal{P} = (a, b, p_3, \ldots, p_n, z) \in \mathbb{R}^{2n-2} \times \mathcal{M}_4$ in π-general position with $\mathcal{M}_4$-coordinate

z of type (A) (see Example 1.1) with a very large length. We have to count the points in $\pi^{-1}(\mathcal{P})$ with their respective π-multiplicity. Starting with the curves of type (a) in Lemma 5.3 we see by Proposition 5.5 that they simply count curves of degree d through $3d-1$ points with their ordinary (ev-)multiplicity, so this simply gives us a contribution of N_d. For the curves of type (b) Remark 5.4 tells us that we can as well count tuples $(C_1, C_2, x_1, \dots, x_n, P, Q)$, where

(a) C_1 and C_2 are tropical curves of degrees d_1 and d_2 with $d_1 + d_2 = d$;
(b) x_1, x_2 are marked points on C_1 that map to L_1 and L_2, respectively;
(c) x_3, x_4 are marked points on C_2 that map to p_3 and p_4, respectively;
(d) $x_5, \dots, x_n$ are marked points that map to $p_5, \dots, p_n$ and of which exactly $3d_1 - 1$ lie on C_1 and $3d_2 - 3$ on C_2;
(e) $P \in C_1$ and $Q \in C_2$ are points with the same image in $\mathbb{R}^2$;

where each such tuple has to be counted with the multiplicity computed in Proposition 5.5.

There are $\binom{3d-4}{3d_1-1}$ choices to split up the points $x_5, \dots, x_n$ as in (d). After fixing d_1 and d_2 we then have $N_{d_1} \cdot N_{d_2}$ choices for C_1 and C_2 in (a) if we count each of them with their ev-multiplicity (which we have to do by Proposition 5.5). By Bézout's theorem (see [6, Theorem 4.2]) there are d_1 possibilities for x_1 in (b)—namely the intersection points of C_1 with L_1—if we count each of them with its local intersection multiplicity $(C_1 \cdot L_1)_{x_1}$ as required by Proposition 5.5. In the same way there are again d_1 choices for x_2 and $d_1 \cdot d_2$ choices for the glueing point $P = Q$. (Note that we can apply Bézout's theorem without problems since we have seen in Remark 5.2 that C_1 intersects L_1, L_2, and C_2 in only finitely many points.)

Altogether we see that the degree of π at $\mathcal{P}$ is

$$\deg_\pi(\mathcal{P}) = N_d + \sum_{d_1+d_2=d} d_1^3 d_2 \binom{3d-4}{3d_1-1} N_{d_1} N_{d_2}.$$

Repeating the same arguments for a point $\mathcal{P}'$ with $\mathcal{M}_4$-coordinate of type (B) as in Example 1.1 we get

$$\deg_\pi(\mathcal{P}') = \sum_{d_1+d_2=d} d_1^2 d_2^2 \binom{3d-4}{3d_1-2} N_{d_1} N_{d_2}.$$

Equating these two expressions by Proposition 4.4 now gives the desired result. $\square$

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