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On Boundary Value Problem for a Class of Retarded Nonlinear Partial Differential Equations

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1. INTRODUCTION

This paper is devoted to a problem of nonlinear oscillations of an elastic plate in a potential supersonic gas flow. The case when the inertial forces are essentially weaker than the resistance ones is called a quasistatic case. This problem can be described by a class of retarded quasilinear partial differential equations

$$\gamma \dot{u} + \Delta^2 u - f\left(\int_\Omega |\nabla u(x,t)|^2 \, dx \right) \Delta u + \rho \frac{\partial u}{\partial x_1} - q(u_t) = d_0(x), \qquad x \in \Omega, \quad t > 0 \quad (1)$$

with the boundary conditions

$$u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0. \quad (2)$$

Here Ω is a bounded domain in R^2, $x = (x_1, x_2)$, γ, ρ, are positive parameters of the system, $\dot{u} = \frac{\partial u}{\partial t}$, and Δ is the Laplace operator. Assumptions on the scalar function $f(s)$ will be given below. We rely here on the Berger approach to large deflection [1], (in [1] $f(s)$ is a linear function).

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The retarded term has the form

$$q(u_t; x) = \frac{1}{2\pi k}\int_{-\infty}^{x_1} d\xi \int_0^{2\pi} d\theta \left[\left(a_\theta \frac{\partial}{\partial x_1} + b_\theta \frac{\partial}{\partial x_2}\right)^2 u\right]^* \times\left(\xi, x_2 - \frac{x_1 - \xi}{k}\cos\theta, t - \kappa_\theta(x_1 - \xi)\right)$$

$$a_\theta = \frac{\nu \sin\theta - 1}{\nu - \sin\theta}, \qquad b_\theta = \frac{k\cdot\cos\theta}{\nu - \sin\theta},$$

$$\kappa_\theta(\xi) = \frac{\xi}{k^2}(\nu - \sin\theta), \tag{3}$$

where $\Psi^*(x)$ is the extension of $\Psi(x)$ by zero outside of Ω, and the parameter $\nu > 1$ represents the gas velocity, $k = \sqrt{\nu^2 - 1}$. Formula (3) shows that the value of retarded term at time t uses values of $u(s)$ for $s \in (t - t_*, t)$, where $t_* = l(\nu - 1)^{-1}$ is a time retardation and l is the length of Ω along x_1 axis. That is why here and below we use the notation $u_t = u_t(\theta) = u(t + \theta)$, $\theta \in (-t_*, 0)$.

The investigation of the considered problem with a Cauchy initial conditions was begun in second order in a time nonretarded setting ($q(u_t) \equiv 0$). The existence and uniqueness theorems have been obtained in [2]; the long-time behaviour for the one dimensional case was investigated by various authors (see, e.g., [3–5] and the references therein). The analysis of influence of potential supersonic flow carried out in [6, 7] leads to the retarded equation (1). The Cauchy problem for second order in the time retarded case has been investigated in [8, 9], where the existence and properties of solutions in different spaces were studied. For a quasistatic formulations see [10, 11]. In [11] the author considered problem (1)–(3) with the following initial conditions (cf. [12]):

$$u|_{t=0+} = u_o, \qquad u|_{t\in(-t_*,0)} = \varphi(x,t). \tag{4}$$

In general we do not assume any compatibility conditions between $\varphi(s)$ and u_o. So even if φ is continuous or piecewise continuous, we do not assume that $\lim_{s\to 0}\varphi(s) = u_o$. For mechanical models (as in our case, the model of oscillations of a plate) such initial conditions can describe a strike (shock) at time moment $t = 0$ (see the discussion of such initial conditions in the finite dimensional case, e.g., in [13]). It was proved [11, Theorem 2.1] that (1)–(4) have an unique solution which is continuous for all $t > 0$. From the point of view of applications it is not convenient to evaluate a function in a moment of the strick (a point of discontinuity of solution); it is more convenient to evaluate it in some moment $t = b > 0$ when the solution is continuous. To this end, in finite dimensional problems, i.e.,

$$\dot{u} = f(u_t), \qquad u(t) \in R^n, \quad t \geq 0,$$

one can consider the boundary value problem with conditions (see, e.g., [13])

$$u|_{t \in (-t_*, 0)} = \varphi(x, t), \qquad u|_{t=b} = u_b, \quad b > 0.$$

In the present paper we formulate an analogous boundary value problem for the infinite dimensional system (1)–(3) such that Cauchy conditions (4) are a partial case. As an another motivation of the considered infinite dimensional boundary value problem we note that our investigation of systems with impulses has a common background with investigations of continuous dynamical systems based on the recently introduced concept of inertial manifold with delay [15]. However, our methods are different from [13, 15].

Let us first introduce the function spaces we need.

2. FORMULATION OF THE PROBLEM AND RESULT

Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis of $L^2(\Omega)$ consisting of the eigenfunctions of the Dirichlet problem for Ω:

$$\Delta e_k + \lambda_k e_k = 0, \qquad e_k(x) = 0 \quad \text{if } x \in \partial\Omega, \qquad 0 < \lambda_1 \leq \lambda_2 \leq \cdots.$$

We use the following scale of spaces:

$$\mathscr{F}_s = \left\{ u = \sum_{k=1}^{\infty} u_k e_k : \|u\|_s^2 \equiv \sum_{k=1}^{\infty} \lambda_k^s u_k^2 < \infty \right\}, \qquad s \in \Re. \tag{5}$$

We denote by $\|.\|$ and $(.\,,.)$ the norm and the inner product in $\mathscr{F}_o = L^2(\Omega)$. Let us define by P_N the orthoprojection in the space $\mathscr{F}_s$ on the subspace spanned by $\{e_1, \ldots, e_N\}$ and we set $Q_N \equiv I - P_N$.

Now we are in a position to give our initial boundary conditions

$$u|_{t \in (-t_*, 0)} = \varphi(x, t), \qquad Q_N u|_{t=0+} = q_0, \qquad P_N u|_{t=b} = p_b, \qquad b > 0, \tag{6}$$

where N is some nonnegative integer.

Remark 2.1. The Cauchy conditions (4) are a partial case of (6) when $N = 0$.

DEFINITION. A strong solution of problem (1)–(3), (6) on an interval $[0,T]$ is a vector-function $u(t) \in C(0,T;\mathscr{F}_1) \cap L^2(-t_*,T;\mathscr{F}_2)$ with derivative $\dot{u}(t) \in L^2(0,T;\mathscr{F}_{-2})$ if Eq. (1) is satisfied almost everywhere in t on $[0,T]$ as an equality in $\mathscr{F}_{-2}$ and conditions (6) hold.

Our result is the following

THEOREM. *Let* $q_0 \in Q_N\mathscr{F}_1$, $p_b \in P_N\mathscr{F}_0$, $\varphi \in L^2(-t_*,0;\mathscr{F}_2)$, $d_0 \in \mathscr{F}_0$, *and let f be a local Lipschitz and satisfying the condition*

$$\inf \lim_{s\to\infty} f(s) \ge -C_f,$$

with some constant C_f. *Then there exists* b_0 *such that for any* $b \le b_0$ *the problem* (1)–(3), (6) *has a strong solution on any interval* $[0,T]$. *This solution is unique and satisfies the property* $u(t) \in L^2(0,T;\mathscr{F}_3)$.

Proof of Theorem. Introduce the space

$$Y \equiv \left\{C(-t_*,b;\mathscr{F}_1) \cap L^2(-t_*,b;\mathscr{F}_2) : y|_{t\in(-t_*,0)} \equiv 0\right\} \tag{7}$$

with the norm

$$\|y\|_Y^2 \equiv \max_{s\in[0,b]} \|y(s)\|_1^2 + \int_0^b \|y(t)\|_2^2\, dt.$$

In the space $Y \times P_N L^2(\Omega)$ we will use the following norm:

$$|(y;p_0)|_{YN}^2 \equiv \max_{s\in[0,b]} \|y(s)\|_1^2 + \int_0^b \|y(s)\|_2^2\, ds + \|p_0\|_1^2.$$

The next assertion is of importance to us (see [7, 8]):

LEMMA 2.1. *If* $u(t) \in L^2(-t_*,T;\mathscr{F}_{2+2\sigma})$ *then*

$$\|q(u_t)\|_{2\sigma}^2 \le Ct_* \int_{t-t_*}^t \|u(\tau)\|_{2+2\sigma}^2\, d\tau, \qquad 0 \le \sigma < \tfrac{1}{4},$$

and the map $u \to q(u,t)$ *is linear and continuous from* $L^2(-t_*,T;\mathscr{F}_{2+2\sigma})$ *to* $L^2(0,T;\mathscr{F}_{2\sigma})$.

Rewrite (1) in the form

$$\dot{u}(t) + Au(t) + M(u_t) = 0. \tag{8}$$

Here $A \equiv (-\Delta_D)^2\gamma^{-1}$ and

$$M(u_t) \equiv \left[-f\left(\|\nabla u(t)\|^2\right)\Delta u(t) - \rho\frac{\partial u(t)}{\partial x_1} - q(u_t) + d_o\right]\gamma^{-1}.$$

Write the variation of constants formula for the solution of (8):

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-\tau)A}M(u_\tau)\,d\tau.$$

If we write $u(t) = y(t) + v(t)$, where

$$v(t) \equiv \begin{cases} e^{-tA}u_0 & \text{if } t \geq 0, \\ \varphi(t), & \text{if } t \in (-t_*, 0), \end{cases} \tag{9}$$

then $y(t)$ should satisfy

$$y(t) = \begin{cases} -\int_0^t e^{-(t-\tau)A}M(y_\tau + v_\tau)\,d\tau, & \text{if } t \geq 0, \\ 0 & \text{if } t \in (-t_*, 0). \end{cases}$$

LEMMA 2.2. *Let $y^i \in Y$ such that $\|y^i\|_Y \leq C_T$ and let $p_0^i \in P_N L^2(\Omega)$, $i = 1, 2$. Then there exists constant $D_T > 0$ such that for all $\tau \in [0, b]$ one has*

$$\left\|A^{-1/4}\left(M\left(y_\tau^1 + v_\tau^1\right) - M\left(y_\tau^2 + v_\tau^2\right)\right)\right\| \leq D_T\left\|\left(y^1; p_0^1\right) - \left(y^2; p_0^2\right)\right\|_{YN}.$$

Here D_T is some constant; v^i *is defined by* (9) *with* $u_0^i = p_0^i + q_0$.

Proof of Lemma 2.2. Using Lemma 2.1 we obtain

$$\begin{aligned} &\left\|A^{-1/4}\left(M\left(y_\tau^1 + v_\tau^1\right) - M\left(y_\tau^2 + v_\tau^2\right)\right)\right\| \\ &\quad\leq C_T\left\|A^{1/4}\left(y^1(\tau) - y^2(\tau)\right)\right\| + C_T\left\|A^{1/4}\left(v^1(\tau) - v^2(\tau)\right)\right\| \\ &\qquad + \left(Ct_*\int_{\tau-t_*}^{\tau}\left\|A^{1/2}\left(y^1(s) - y^2(s)\right)\right\|^2 ds\right)^{1/2} \\ &\qquad + \left(Ct_*\int_{\tau-t_*}^{\tau}\left\|A^{1/2}\left(v^1(s) - v^2(s)\right)\right\|^2 ds\right)^{1/2}. \end{aligned}$$

Denote $\Lambda_i = \lambda_i^2$ eigenvalues of the operator A, and using estimate (see, e.g., [14])

$$\|A^{\alpha} e^{-tA} u\| \le \left(\frac{\alpha - \beta}{t} + \Lambda_1 \right)^{\alpha - \beta} e^{-t\Lambda_1} \|A^{\beta} u\| \tag{10}$$

and definition of $v^i(t)$ by (9), we get

$$\begin{aligned}
\int_{\tau - t_*}^{\tau} & \left\| A^{1/2}\big(v^1(s) - v^2(s)\big)\right\|^2 ds \\
&\le \int_0^{\tau} \left\| A^{1/2} e^{-sA}\big(p_0^1 - p_0^2\big)\right\|^2 ds \\
&\le \int_0^{\tau} \left(\frac{1}{4s} + \Lambda_1 \right)^{1/4} e^{-s\Lambda_1} \|p_0^1 - p_0^2\|_1^2 \, ds \\
&\le \|p_0^1 - p_0^2\|_1^2 \int_0^{\tau} \left(\frac{1}{4s} + \Lambda_1 \right)^{1/4} e^{-s\Lambda_1} \, ds = \|p_0^1 - p_0^2\|_1^2 \, \alpha\left(\tau; \frac{1}{4}\right).
\end{aligned}$$

Here we have denoted

$$\alpha(b; k) \equiv \int_0^b \left(\frac{k}{s} + \Lambda_1 \right)^k e^{-s\Lambda_1} \, ds. \tag{11}$$

Note that for $k < 1$, $\alpha(b; k) \to 0$ when $b \to 0$. So we have

$$\begin{aligned}
&\left\| A^{-1/4}\big(M\big(y_\tau^1 + v_\tau^1\big) - M\big(y_\tau^2 + v_\tau^2\big)\big)\right\| \\
&\quad \le C_T' \Bigg(\max_{[0,b]} \|y^1 - y^2\|_1 + \|p_0^1 - p_0^2\|_1 + \left(\int_0^b \|y^1(s) - y^2(s)\|_2^2 \, ds \right)^{1/2} \\
&\qquad + \|p_0^1 - p_0^2\|_1 \, \alpha\big(b; \tfrac{1}{4}\big)^{1/2} \Bigg).
\end{aligned}$$

This completes the proof of Lemma 2.2.

Let us fix $\varphi \in L^2(-t_*, 0; \mathscr{F}_2)$, $q_0 \in Q_N \mathscr{F}_1$, and $p_b \in P_N L^2(\Omega)$. Consider the map $F(y; p_0)$: $Y \times P_N L^2(\Omega) \to Y \times P_N L^2(\Omega)$ defined as follows:

$$F(y; p_0) \equiv \begin{pmatrix} \begin{cases} -\int_0^t e^{-(t-\tau)A} M(y_\tau + v_\tau)\, d\tau, & \text{if } t \ge 0; \\ 0 & \text{if } t \in (-t_*, 0) \end{cases} \\ e^{Ab} p_b + \int_0^b e^{\tau A} P_N M(y_\tau + v_\tau)\, d\tau. \end{pmatrix}.$$

Here the function $v(t)$ is defined by (9) with $u_0 = p_0 + q_0 \in \mathcal{F}_1$. Note that q_0 is fixed but p_0 is a variable of F. We deduce the second coordinate of F from the equation

$$p_b \equiv P_N u(b) = e^{-Ab} p_0 - \int_0^b e^{-(b-\tau)A} P_N M(u_\tau)\, d\tau.$$

Hence $p_0 = e^{Ab} p_b + \int_0^b e^{\tau A} P_N M(u_\tau)\, d\tau$.

A fixed point of operator F gives unknown coordinates $P_N u(0)$ of initial data $u(0)$ and the solution on the interval $[0, b]$. Having φ and full $u(0)$ we arrive at initial conditions (4) and hence one can use all results on the continuation of the solution and its properties obtained in [11].

Let us show that F is contaction in $Y \times P_N L^2(\Omega)$. We will denote two coordinates of F as $(F_1; F_2)^T$. Let us estimate F_1, using Lemma 2.2, (10), and (11):

$$\begin{aligned}
&\left\| A^{1/4}\left(F_1\left(y^1; p_0^1\right)(t) - F_1\left(y^2; p_0^2\right)(t)\right)\right\| \\
&\quad \le \int_0^t \left\| A^{1/4} e^{-(t-\tau)A}\left(M\left(y_\tau^1 + v_\tau^1\right) - M\left(y_\tau^2 + v_\tau^2\right)\right)\right\| d\tau \\
&\quad \le \int_0^t \left(\frac{1}{2(t-\tau)} + \Lambda_1\right)^{1/2} e^{-(t-\tau)\Lambda_1} \left\| A^{-1/4}\left(M\left(y_\tau^1 + v_\tau^1\right)\right.\right. \\
&\qquad\qquad \left.\left. - M\left(y_\tau^2 + v_\tau^2\right)\right)\right\| d\tau \\
&\quad \le \alpha\left(b, \frac{1}{2}\right) D_T \left|\left(y^1; p_0^1\right) - \left(y^2; p_0^2\right)\right|_{YN}.
\end{aligned}$$

In the same way, we deduce

$$\begin{aligned}
&\left\| A^{1/2}\left(F_1\left(y^1; p_0^1\right)(t) - F_1\left(y^2; p_0^2\right)(t)\right)\right\| \\
&\quad \le \alpha\left(b, \frac{3}{4}\right) D_T \left|\left(y^1; p_0^1\right) - \left(y^2; p_0^2\right)\right|_{YN}.
\end{aligned}$$

Remark 2.2. We need an a priori estimate $\|y^i\|_Y \le C_T$ to use Lemma 2.2. It can be obtained as in [11, Theorem 2.1].

Now let us estimate the second coordinate F_2:

$$\begin{aligned}
&\left\| A^{1/4}\left(F_2\left(y^1; p_0^1\right)(t) - F_2\left(y^2; p_0^2\right)(t)\right)\right\| \\
&\quad \le \int_0^b \left\| A^{1/4} e^{A\tau} P_N\left(M\left(y_\tau^1 + v_\tau^1\right) - M\left(y_\tau^2 + v_\tau^2\right)\right)\right\| d\tau \\
&\quad \le \Lambda_N^{1/2} \int_0^b e^{\Lambda_N \tau} \left\| A^{-1/4}\left(M\left(y_\tau^1 + v_\tau^1\right) - M\left(y_\tau^2 + v_\tau^2\right)\right)\right\| d\tau \\
&\quad \le D_T \frac{e^{\Lambda_N b} - 1}{\Lambda_N^{1/2}} \left|\left(y^1; p_0^1\right) - \left(y^2; p_0^2\right)\right|_{YN}.
\end{aligned}$$

Combine the last three estimates to get

$$\begin{aligned}
&\left|F\left(y^1; p_0^1\right) - F\left(y^2; p_0^2\right)\right|_{YN}^2 \\
&\quad \le \left[\alpha\left(b, \tfrac{1}{2}\right)^2 + \alpha\left(b, \tfrac{3}{4}\right)^2 + \Lambda_N^{-1}\left(e^{\Lambda_N b} - 1\right)^2\right] \\
&\quad\quad \times D_T^2 \left|\left(y^1; p_0^1\right) - \left(y^2; p_0^2\right)\right|_{YN}^2.
\end{aligned}$$

Choosing b small enough, we obtain a contraction of the map F. The proof of the theorem is complete.

COROLLARY. *We consider here the same class of strong solutions as in* [11] *and under the same conditions as nonlinear function f. So all results on long-time asymptotic behaviour of solutions to* (1)–(3), (6), *including the existence of a finite dimensional attractor*, *are valid*.

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