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Renormalization of 3d quantum gravity from matrix models

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Abstract

Lorentzian simplicial quantum gravity is a non-perturbatively defined theory of quantum gravity which predicts a positive cosmological constant. Since the approach is based on a sum over space–time histories, it is perturbatively non-renormalizable even in three dimensions. By mapping the three-dimensional theory to a two-matrix model with ABAB interaction we show that both the cosmological and the (perturbatively) non-renormalizable gravitational coupling constant undergo additive renormalizations consistent with canonical quantization.

1. Introduction

Defining a theory of quantum gravity as a suitable sum over space–time histories is an appealing proposition, since it can in principle be done in a completely background-independent and non-perturbative way, with the structure of space–time being determined *dynamically*. In two space–time dimensions, such a program can be carried out successfully, although in this case—because of the absence of propagating gravitons—it may be more appropriate to talk about a theory of "quantum geometry" rather than one of quantum gravity. A well-known example is the non-perturbative lattice formulation of 2d (Euclidean) gravity which reproduces quantum Liouville theory in the limit of vanishing lattice spacing [1–3]. Attempts to use similar combinatorial and matrix-model techniques to extract information about the non-perturbative structure of higher-dimensional gravity have until recently met with little success. However, *if one performs the sum over geometries over space–times of Lorentzian* (*as opposed to Riemannian*) *signature*, matrix-model methods *can* be applied profitably in the non-perturbative quantization of three-dimensional quantum gravity, as was first shown in [4]. This line of investigation will be pursued further in the present work.

Quantum gravity in three space–time dimensions represents an interesting case in between dimensions two and four. On the one hand, it contains no propagating gravitational degrees of freedom and can be reduced classically to a finite-dimensional physical phase space, both in a metric [5] and a connection

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(Chern–Simons) formulation [6].[1] Nevertheless, the unreduced theory in terms of the metric $g_{\mu\nu}$ appears to be non-renormalizable when one tries to expand around a fixed background geometry, just as in four dimensions. A definition of three-dimensional quantum gravity via a "sum over geometries" therefore seems to require a genuinely non-perturbative construction, and in turn may shed light on the problem of non-renormalizability of the full, four-dimensional theory, where an explicit classical reduction is not available.

A non-perturbative definition of the sum over geometries in three- and four-dimensional quantum gravity was proposed in [7,8]. Unlike previous approaches, this method of "Lorentzian dynamical triangulations" or "Lorentzian simplicial quantum gravity" uses space–time geometries with physical, Lorentzian signature, rather than positive-definite Riemannian geometries as a fundamental input. Details on the classes of geometries included in the path sum and on earlier two-dimensional work that provided the motivation for this approach can be found in [8–10]. In view of the recent observational progress in cosmology (see [11] for a recent review) we should point out that the physical, renormalized cosmological constant in all of these models is necessarily positive.

In this Letter, we will present an explicit analysis of the renormalization behaviour of the 3d Lorentzian model, using a matrix-model formulation. This follows previous work which analyzed the phase structure of three-dimensional quantum gravity (for spherical spatial topology) with the help of computer simulations [12–14], and a demonstration [4] that 3d Lorentzian dynamical triangulations can be mapped to graph configurations generated by the so-called ABAB-matrix model [15].

Within *continuum* approaches to quantum gravity there have also been attempts to prove the non-perturbative renormalizability of gravity beyond dimension two, starting with an analysis of the theory in $2+\epsilon$ dimensions [16–18]. More recently, an effective average action approach has produced evidence of a non-trivial fixed point through an analysis of renormalization group flow equations [19–21].

[1] Whether and to what extent the associated *quantum* theories are related is still a contentious issue.

2. Quantum gravity and the ABAB-matrix model

We start out with a brief description of the three-dimensional Lorentzian simplicial space–times appearing in the sum over geometries, and the construction of the partition function. In the standard formulation of the model, the spatial hypersurfaces of constant integer proper time t are given by two-dimensional equilateral triangulations, each corresponding to a unique piecewise flat 2d geometry. These are the same geometries as appear in the construction of 2d Euclidean quantum gravity, which is known to be rather robust with regard to changes in both the types of building blocks used and their gluing rules [22]. We exploited this universality in [4] by using 2d spatial geometries made up of equilateral squares instead of triangles, and accordingly changing the 3d building blocks from tetrahedra only to a set of tetrahedra and pyramids.

Any two neighbouring spatial quadrangulations at times t and $t+1$ can be connected (in many inequivalent ways) by a three-dimensional "sandwich" geometry constructed from these building blocks, as indicated in Fig. 1. The square base of a pyramid (or an upside-down pyramid) coincides with a square of the spatial slice at time t (or $t+1$), whereas the tetrahedral building block is needed to connect between the two types of pyramids within the same sandwich.

The amplitude for propagation from an initial quadrangulation g_1 to a final one g_2 in n proper-time steps is obtained by summing over all geometrically distinct ways of stacking n sandwich geometries $\Delta t = 1$ in between g_1 and g_2, in such a way that their 2d boundary geometries match pairwise at integer times. The weight of each geometry is given by a discretized version of the Einstein action, here conveniently taken as the Regge action for piecewise linear geometries [23]. After Wick-rotating, the partition function (or proper-time propagator) can be written as

$$Z(\kappa, \lambda; g_1, g_2, n) = \sum_{\mathcal{T}, \partial\mathcal{T} = g_1 \cup g_2} \frac{1}{C_{\mathcal{T}}} \mathrm{e}^{-S(\mathcal{T})}, \tag{1}$$

where $C_{\mathcal{T}}$ is the order of the automorphism group of the (generalized) triangulation $\mathcal{T}$, and the sum is over all $\mathcal{T}$ with fixed boundaries g_1 and g_2 of the kind just described. The gravitational action, including a

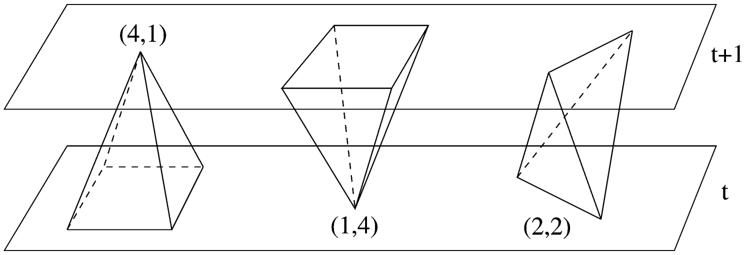

Fig. 1. The fundamental building blocks of 3d Lorentzian quantum gravity interpolate between adjacent spatial slices of integer times t and $t+1$, and are labelled according to the numbers (i_t, i_{t+1}) of their vertices lying in the two slices.

cosmological term, is given by

$$S(\mathcal{T}) = -\kappa\big(N_{14}(\mathcal{T}) + N_{41}(\mathcal{T}) - N_{22}(\mathcal{T})\big) + \lambda\left(N_{14}(\mathcal{T}) + N_{41}(\mathcal{T}) + \frac{1}{2}N_{22}(\mathcal{T})\right), \qquad (2)$$

where $N_{41}(\mathcal{T})$ and $N_{14}(\mathcal{T})$ count the numbers of pyramids and upside-down pyramids and $N_{22}(\mathcal{T})$ the number of tetrahedra contained in a given triangulation $\mathcal{T}$. The simplicity of the Regge action in our case stems from the fact that we use only two types of building blocks, and contributions to volumes and curvatures (in the form of deficit angles) occur only in terms of a few basic units (see [4,8] for further details). The simplicial action contains two dimensionless coupling constants κ and λ, related to their continuum counterparts by[2]

$$\kappa = \frac{a}{4\pi G^{(0)}}\left(-\pi + 3\cos^{-1}\frac{1}{3}\right), \qquad \lambda = \frac{a^3 \Lambda^{(0)}}{24\sqrt{2}\pi}, \qquad (3)$$

where a is a geodesic lattice cut-off with the dimension of length. It should be emphasized that these are "naive" relations between the dimensionless lattice coupling constants and those of the continuum theory, which will *not* be valid in the quantum theory. As we shall see in due course, additive renormalizations of both coupling constants will be needed in that case.

We can rewrite the partition function (1) as

$$Z(\kappa,\lambda; g_1, g_2, n) = \sum_N \mathrm{e}^{-\lambda N} \sum_{\mathcal{T}_N} \frac{1}{C_{\mathcal{T}_N}} \mathrm{e}^{\kappa(N_{14}(\mathcal{T}_N)+N_{41}(\mathcal{T}_N)-N_{22}(\mathcal{T}_N))}, \qquad (4)$$

where the sum over the total space–time volume $N = N_{14} + N_{41} + \frac{1}{2}N_{22}$ has been pulled out, together with the accompanying Boltzmann weight $\mathrm{e}^{-\lambda N}$, and the remaining sum runs over all triangulations $\mathcal{T}_N$ of fixed volume N, whose Boltzmann weights depend on the curvature term multiplying κ. To leading order, the number of triangulations at fixed volume grows exponentially with the volume, leading to the asymptotic behaviour

$$f(N; g_1, g_2)\mathrm{e}^{\lambda_c(\kappa)N}, \qquad (5)$$

for the second sum in (4), where $f(N; g_1, g_2)$ indicates subleading terms in N. It follows immediately that for a given κ the regularized quantum gravity model is only well defined (that is, its state sum converges) for $\lambda > \lambda_c(\kappa)$, corresponding to the region above the critical line in the phase diagram of Fig. 2. The critical line limits the region of convergence of the partition function Z. Taking $\lambda \to \lambda_c(\kappa)$ from *inside* this region of convergence, the average value of (suitable powers of) N will diverge, corresponding to the limit of infinite lattice volume. Such a limit is clearly necessary if a continuum limit in any conventional sense is to be achieved.

The continuum limit is obtained by scaling the lattice spacing a to zero while keeping the continuum time $T = n \cdot a$ fixed (and therefore, increasing the number n of discrete time steps at a rate $1/a$). Different, non-canonical scaling relations between T and a are in principle possible,[3] but the computer simula-

[2] Note that our cosmological constant $\Lambda^{(0)}$ is defined as the quantity that multiplies the volume term $\int \mathrm{d}^3x\,\sqrt{g}$. More conventionally this term would be called $\Lambda^{(0)}/(8\pi G^{(0)})$.

[3] In two-dimensional *Euclidean* quantum gravity the proper time T scales anomalously and one has to keep $n\sqrt{a}$ fixed [24]. By contrast, the scaling in two-dimensional Lorentzian simplicial quantum gravity is canonical [9]. The relation between the two formulations is well understood [25].

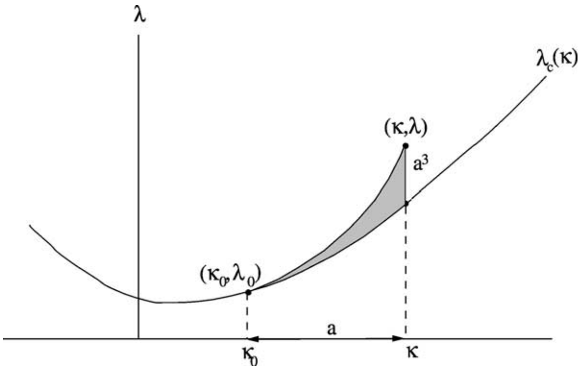

Fig. 2. The phase diagram of 3d Lorentzian quantum gravity in the plane spanned by the bare inverse gravitational coupling κ and the bare cosmological constant λ, together with the canonical approach to a point $(\kappa_0, \lambda_c(\kappa_0))$ on the critical line.

tions of [12] supported the presence of canonical scaling in 3d quantum gravity. More precisely, we expect to leading order in a a scaling of the form

$$\frac{a}{G} = \kappa - \kappa_0, \qquad a^3 \Lambda = \lambda(\kappa) - \lambda_c(\kappa), \tag{6}$$

as illustrated in Fig. 2. The approach to the critical line is governed by the dimensionless combination $G^3\Lambda$ which serves as the true, "observable" coupling constant of 3d quantum gravity. The physics underlying (6) is as follows: for a given value of the bare inverse gravitational coupling κ the average discrete space–time volume $\langle N\rangle$ and its dimensionful counterpart $\langle V\rangle$ behave like

$$\langle N\rangle \sim \frac{1}{\lambda - \lambda_c(\kappa)} \quad \Rightarrow \quad \langle V\rangle := a^3\langle N\rangle \sim \frac{a^3}{\lambda - \lambda_c(\kappa)}, \tag{7}$$

that is, the number of building blocks diverges in the limit as $\lambda \to \lambda_c(\kappa)$. The physical requirement that the continuum volume $\langle V\rangle$ remain finite and be proportional to the inverse *renormalized* cosmological constant $1/\Lambda$ fixes the second scaling relation in (6). The first relation is then determined by demanding that $G^3\Lambda$ be a dimensionless coupling constant of the theory. This is precisely achieved by approaching a given point $(\kappa_0, \lambda_c(\kappa_0))$ on the critical curve according to the canonical scaling assignment (6). Note in passing that there is no way of obtaining a renormalized cosmological coupling $\Lambda \leqslant 0$, in agreement with our earlier remarks. Also, we choose the approach to the critical line such that the sign of the renormalized Newton constant is standard and positive.

Our construction raises the question of whether or not physics depends on the choice of κ_0. Indications from the computer simulations of the model are that the final result is independent of the value of κ_0 in the range probed [12]. We will discuss in the following how this question can be addressed analytically.

Let g_t and g_{t+1} be two spatial quadrangulations at t and $t+1$, and $\langle g_{t+1}|\hat{T}|g_t\rangle$ the transition amplitude or proper-time propagator for the single time step from t to $t+1$. By definition, $\hat{T}$ is the transfer matrix in the sense of Euclidean lattice theory, and can be shown to satisfy the usual properties of a transfer matrix [8]. The propagator for n time steps is obtained by an n-fold iteration,

$$Z(\kappa, \lambda; g_1, g_2, n) = \langle g_2|\hat{T}^n|g_1\rangle. \tag{8}$$

Consider now the matrix model of two hermitian $M \times M$-matrices with partition function

$$Z(\alpha_1, \alpha_2, \beta) = \int \mathrm{d}A\,\mathrm{d}B\,\mathrm{e}^{-M\,\mathrm{tr}(A^2+B^2-\frac{\alpha_1}{4}A^4-\frac{\alpha_2}{4}B^4-\frac{\beta}{2}ABAB)}. \tag{9}$$

In the context of the large-M expansion the free energy F can be expressed as

$$M^2F(\alpha_1, \alpha_2, \beta) \equiv -\log Z(\alpha_1, \alpha_2, \beta) = \sum_{h=0}^{\infty} M^{\chi(h)} F_h(\alpha_1, \alpha_2, \beta), \tag{10}$$

where $\chi(h) = 2 - 2h$ is the Euler number of the quadrangulations dual to the four-valent graphs generated by the matrix model. It was argued in [4] that the transfer matrix for transitions between two spatial geometries g_t and g_{t+1} of genus h is related to $F_h(\alpha_1, \alpha_2, \beta)$ according to

$$F_h(\alpha_1, \alpha_2, \beta) = \sum_{N_t, N_{t+1}} \mathrm{e}^{-z_t N_t - z_{t+1}N_{t+1}} \times \sum_{g_{t+1}(N_{t+1}),\, g_t(N_t)} \big\langle g_{t+1}(N_{t+1})\big|\hat{T}\big|g_t(N_t)\big\rangle_h, \tag{11}$$

where N_t and N_{t+1} denote the numbers of squares of the quadrangulations defining the spatial geometries at times t and $t+1$, both of Euler number $\chi(h)$. Pulling out the double-sum over discrete boundary volumes is convenient when studying the transfer matrix per se (see [9,26] for an analogous procedure in two space–time dimensions). The two dimensionless boundary constants z_t and z_{t+1} can be viewed as cosmological coupling constants for the boundary areas. For the purposes of the present Letter we will choose particular values for z_t and z_{t+1}, in such a way that the relations

$$\alpha_1 = \alpha_2 = e^{\kappa-\lambda}, \qquad \beta = e^{-(\frac{1}{2}\lambda+\kappa)}, \tag{12}$$

hold between the matrix model coupling constants α_i, β, and the bare gravitational and cosmological coupling constants $1/\kappa$ and λ of three-dimensional gravity. The relations (12) were derived previously in [4], and we will use them in the next section to translate the canonical approach (6) to the matrix model and draw conclusions about the renormalization behaviour of the theory.

The derivation of Eq. (12) requires some explanation. Generic matrix elements of $\hat{T}$ in (11) grow exponentially with the total discrete three-volume $N = N_t + N_{t+1} + N_{22}/2$, reflecting the fact that there are exponentially many three-geometries which interpolate between two given two-geometries g_t and g_{t+1}. This exponential growth is taken care of by the combined *additive renormalizations* of the cosmological and gravitational constants, as discussed earlier in this section.

There is a completely analogous entropy for the boundary two-geometries, since the number of quadrangulations of a given topology and a given discrete two-volume N_t grows exponentially with N_t. Just as in the case of the three-volume, this exponential growth can be cancelled by an *additive renormalization*, in this case of the boundary cosmological constant z_t, leading to a renormalized boundary cosmological constant multiplying a continuum area. Assume that the second sum in (11) grows like $e^{z_c(N_t+N_{t+1})}$ to leading order in the boundary two-volumes, and renormalize z_t and z_{t+1} canonically according to

$$z_t = z_c + a^2 Z_t, \qquad z_{t+1} = z_c + a^2 Z_{t+1}. \tag{13}$$

Defining the continuum area A_t of a quadrangulation of N_t squares by $A_t := N_t a^2$, the total area contribution in the exponential in (11) becomes

$$\begin{aligned}(z_c - z_t)N_t + (z_c - z_{t+1})N_{t+1} \\ = -(Z_t A_t + Z_{t+1} A_{t+1}),\end{aligned} \tag{14}$$

as anticipated. In this Letter, we set $Z_t = Z_{t+1} = 0$, corresponding to $z_t = z_{t+1} = z_c$ in (11), since we are only interested in the bulk coupling constants Λ and G. This implies the symmetry $\alpha_1 = \alpha_2$, as well as the relation (12). From a technical point of view it means that we have to deal only with the symmetric ABAB-matrix model which, contrary to the asymmetric model, has been solved explicitly [15].

3. Renormalization of 3d gravity

The canonical approach (6) to a critical point (κ_0, λ_0) on the critical line of the (κ, λ)-coupling constant plane, Fig. 2, can be mapped via (12) to the (β, α)-plane, as shown in Fig. 3. Let $F(\alpha, \beta)$ denote the free energy of the symmetric ABAB-matrix model, and set $\alpha_1 = \alpha_2 \equiv \alpha$. It is convenient to change variables from (β, α) to (s, r), where

$$s = \frac{\beta}{\alpha}, \qquad r = \sqrt{\alpha^2 + \beta^2}. \tag{15}$$

The upper right-hand quadrant of the α-β-plane corresponds to $r, s \in [0, \infty]$. Approaching a point $(\beta_c(s), \alpha_c(s))$ on the critical line from below along a line segment of constant s, the coordinate r will vary between 0 and $r_c(s) = \sqrt{\alpha_c(s)^2 + \beta_c(s)^2}$. According to [15], $F(\alpha, \beta)$ or $F(s, r)$ are analytic functions of their arguments *below* the critical line. Moreover, approaching the critical line along $s = \text{const}$, $F(r, s)$ has an expansion

$$\begin{aligned}&F(s, r) - F\big(s, r_c(s)\big) \\ &\quad = c_1(s)\delta r + c_2(s)\delta r^2 + c_{5/2}(s)\delta r^{5/2} \\ &\qquad + c_3(s)\delta r^3 + \cdots\end{aligned} \tag{16}$$

in the vicinity of the critical point $(s, r_c(s))$, where $\delta r = r_c(s) - r$ and where the coefficients $c_i(s)$ are analytic functions of s for both $0 < s < 1$ and $1 < s < \infty$. Around the special point $(s, r_c(s)) = (1, r_c(1))$ which separates the so-called A-phase ($s < 1$) from the B-phase ($s > 1$), the behaviour is more complicated than the one given in (16). As discussed in [4], phase A is the one relevant for canonical quantum gravity and we will consider only coupling constant variations inside phase A.

The straight approach along $s = \text{const}$ to the critical line underlying (16) is not the one relevant for three-dimensional quantum gravity, since it would translate to a curve in the (κ, λ)-plane which approaches the

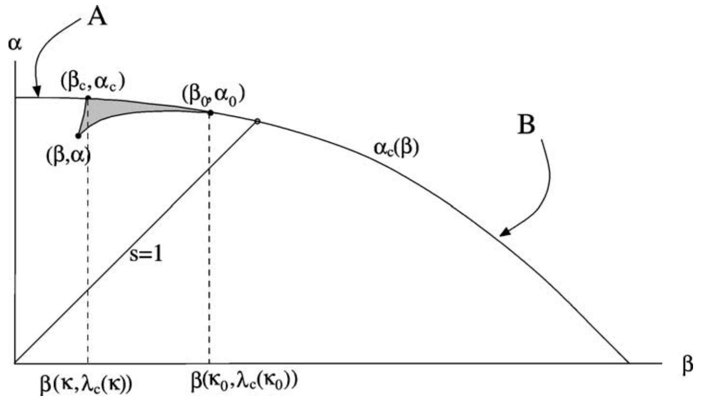

Fig. 3. The phase diagram of 3d Lorentzian quantum gravity in the plane spanned by the two coupling constants β and α of the matrix model, together with the canonical approach to a point (β_0, α_0) on the critical line. The end point $(\beta_c, \alpha_c = \beta_c)$ of the diagonal $s = 1$ separates phase A from phase B.

corresponding critical point (κ_0, λ_0) non-tangentially. In the notation of (6), this would imply $\kappa - \kappa_0 \propto \lambda(\kappa) - \lambda_c(\kappa)$, in contradiction with the scaling relations (6). Stated differently, insisting on canonical dimensions for G and Λ and a finite Λ, the gravitational coupling G would have to go to infinity like $1/a^2$ when the cut-off is removed.

One can of course repeat the analysis of [15] for an arbitrary approach to the critical line. However, rather than giving the technical details of this, let us just state the final result for the case at hand. We can approach a critical point (β_0, α_0) along any curve $(\beta(a), \alpha(a))$, where for convenience we have identified the curve parameter a with the lattice cut-off. For the canonical gravitational interpretation to be valid, the scaling must follow (6), that is, both the tangent and the curvature of the curve $(\beta(a), \alpha(a))$ must agree with those of the critical line $(\beta_c(s), \alpha_c(s))$ at the point (β_0, α_0). The difference between the two curves will only appear in their third-order derivatives, as indicated by Fig. 3. In order to investigate the analyticity properties of the free energy, we perform a decomposition

$$\begin{aligned} &F\big(\alpha(a), \beta(a)\big) - F(\alpha_0, \beta_0) \\ &\quad = \big(F(\alpha, \beta) - F(\alpha_c, \beta_c)\big) \\ &\qquad + \big(F(\alpha_c, \beta_c) - F(\alpha_0, \beta_0)\big), \end{aligned} \tag{17}$$

where, in the notation of Fig. 3, the approaching curve $(\kappa(a), \lambda(a))$ translates into $(\beta(a), \alpha(a))$, (β_c, α_c) corresponds to the point $(\kappa, \lambda_c(\kappa))$, and (β_0, α_0) to (κ_0, λ_0) on the critical line. To evaluate the first difference in (17) we can use

$$\alpha - \alpha_c \sim \Lambda a^3 + \cdots, \qquad \beta - \beta_c \sim \Lambda a^3 + \cdots, \tag{18}$$

as well as the expansion (16). In the second difference we can use

$$\alpha_c - \alpha_0 \sim -a/G + \cdots, \qquad \beta_c - \beta_0 \sim -a/G + \cdots, \tag{19}$$

without any reference to the renormalized cosmological constant Λ, defined by (6). This happens because both (β_0, α_0) and (β_c, α_c) lie on the critical line, whereas Λ is a measure of the *distance from* the critical line. The important point is that—as long as we stay in phase A—the difference $F(\alpha_c, \beta_c) - F(\alpha_0, \beta_0)$ is entirely analytic in $\alpha_c - \alpha_0$. We conclude that the non-analytic behaviour of the free energy occurs as a function of the cosmological coupling constant alone. This non-analyticity ensures the existence of an infinite-volume limit of 3d quantum gravity in the sense of (7). *The renormalized gravitational coupling constant G plays no role in taking the continuum limit*, which is entirely dictated by the non-analytic part of $F(\alpha, \beta)$.

Let us discuss this behaviour in some more detail. The free energy $F(\alpha, \beta)$ of the matrix model serves as the *partition function* of the sum over sandwich configurations of the three-dimensional Lorentzian gravity model, as described above. Its continuum limit is associated with a limit where the number N of 3d building blocks diverges, and $a \to 0$, while keeping the continuum three-volume $V = Na^3$ finite. This large-N behaviour is related to the expansion

$$F(\alpha,\beta) = \sum_{N_{14},N_{41},N_{22}} \mathcal{N}(N_{14},N_{41};N_{22}) \times \alpha^{N_{14}+N_{41}} \beta^{N_{22}}, \qquad (20)$$

of $F(\alpha,\beta)$ into large powers of α and β, where $\mathcal{N}(N_{14},N_{41};N_{22})$ denotes the number of three-geometries constructed from (N_{14},N_{41},N_{22}) building blocks between neighbouring spatial surfaces at t and $t+1$ (see [4] for details). The non-analytic part of $F(\alpha,\beta)$ is associated with simultaneous infinitely large powers of α and β, which in turn is reflected in a finite radius of convergence of the power expansion.

We will denote the non-analytic part of $F(\alpha,\beta)$ by $F_{\text{singular}}(\alpha,\beta)$, and it is only this part that should be kept when discussing the continuum limit. Thus, returning to the expansion (16), the first two terms on the right-hand side are irrelevant to a potential continuum limit dictated by the non-analytic term $(r_c - r)^{5/2}$. Likewise, the term $F(\alpha_c,\beta_c) - F(\alpha_0,\beta_0)$ in Eq. (17) can be ignored when discussing continuum physics. The term $F(\alpha,\beta) - F(\alpha_c,\beta_c)$ in that relation is similar to the quantity (16) which characterizes the non-tangential approach to a critical point. The continuum expression which survives is therefore

$$F_{\text{singular}}(\Lambda, G) \sim \left(\Lambda a^3\right)^{5/2}. \qquad (21)$$

One would obtain the same expression in the 2d (Euclidean) quantum gravity interpretation given in [15], except that the power of the lattice cut-off would be different. This is due to the tangential approach to the critical point in the present case, reflecting the different physical properties of the higher-dimensional gravity theory.

One should keep in mind that F_{singular} is not identical with the partition function (4) for three-dimensional quantum gravity for $n = 1$, but rather is a particular sum of matrix elements of the transfer matrix between two adjacent constant proper-time slices, which are separated by one lattice unit a. However, as was also argued in [4], the study of this sum is sufficient to exhibit the renormalization behaviour of the bare gravitational and cosmological coupling constants.[4] The only way in which the (perturbatively) non-renormalizable gravitational coupling constant G makes an appearance in 3d Lorentzian quantum gravity is by fixing the approach to the chosen critical point κ_0, and thereby defining the dimensionless quantity

$$\frac{\lambda - \lambda_c(\kappa)}{(\kappa - \kappa_0)^3} = \text{const} = \Lambda G^3. \qquad (22)$$

Consequently, all observables we may think of calculating in this formulation will be of the form

$$\mathcal{O}(\Lambda, G) = \Lambda^{\dim/3} F\left(\Lambda G^3\right) \qquad (23)$$

after the continuum limit has been performed, where “dim” refers to the mass dimension of the observable $\mathcal{O}$.

4. Discussion

Three-dimensional simplicial Lorentzian quantum gravity gives an explicit realization of the summation over three-geometries. As in all quantum theories with a cut-off, a prescription must be given of how to remove the cut-off and recover the underlying continuum quantum field theory; we did this by specifying the renormalization of the bare coupling constants of the theory. The relation of the model to the ABAB-matrix model allowed us to give a detailed discussion of a possible renormalization of the gravitational and cosmological coupling constants, consistent both with the existence of an infinite-volume limit of the model and with a canonical scaling of the renormalized coupling constants.

The bare gravitational and the bare cosmological coupling constants turned out to be subject to additive renormalizations. The perturbative non-renormalizability of the gravitational coupling constant is resolved in this non-perturbative approach by the fact that the *renormalized* gravitational coupling constant only appears in the particular combination (22), defined by the canonical approach to the critical line.

One way to obtain more detailed information about the continuum limit would be by analyzing the full

[4] In an analogous analysis of two-dimensional simplicial Lorentzian quantum gravity one also can deduce the renormalization of the cosmological constant from the study of the same restricted combination of matrix elements.

transfer matrix, instead of the contracted version we have studied in the present work. From the transfer matrix one can extract the *continuum* proper-time Hamiltonian $\hat{H}$ by virtue of the relation

$$\hat{T} = \mathrm{e}^{-a\hat{H}} \approx \hat{I} - a\hat{H}. \tag{24}$$

This can be done explicitly in both two-dimensional Lorentzian and Euclidean simplicial quantum gravity, where the Hamiltonian is a differential operator in a single variable, the one-volume of the spatial universe. Three-dimensional quantum gravity is more involved since the spatial geometries at a fixed time constitute an infinite-dimensional field space, spanned by the conformal factor and a finite number of Teichmüller parameters. However, from our knowledge of the classical, canonical structure of the theory we do not expect the conformal part of the geometry to play a dynamical role. From this point of view—in addition to any Teichmüller parameters—at most the constant mode of the conformal factor (equivalently, the two-dimensional total area) of the spatial geometry should appear in the Hamiltonian.

We know that at the discretized level there are transitions between any pair of two-geometries of the same topology, that is, all matrix elements of $\hat{T}$ are non-vanishing. It would be very interesting to understand in detail how the matrix elements lose their sensitivity to anything but the Teichmüller parameters and the total area in the continuum limit. Although the ABAB-matrix model cannot be used to address the issue of how the dependence of the transfer matrix on the conformal factor drops out, solving its *asymmetric* version (with $\alpha_1 \neq \alpha_2$) would determine the dependence of the transfer matrix (and thus the quantum Hamiltonian) on the area of the spatial boundaries. We hope to return to this issue in the near future.

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