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Filtrations, weights and quiver problems

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ABSTRACT

We prove the relationship between stability of (generalized) linear dynamical systems and their reachability by using tools of linear algebra.

1. Introduction

A major problem related to the action of a connected reductive complex Lie group on a finite dimensional complex vector space is the construction of geometric quotients, which are usually associated to the restriction of the action to suitable subsets of the whole vector space. By phrasing Mumford's GIT-stability [1], it is possible to construct such open sets, which are associated to the characters of the group (see also the paper by King [2]). It is not our aim to present here these results in full generality, since we will pay attention to two particular cases. Consider first the vector space W of pairs (A, A_1), where A is a square matrix of order r and A_1 is a $r \times r_1$ matrix. The general linear group $\mathrm{GL}_r(\mathbb{C})$ acts in a natural fashion on W. Applying the general definitions, one can construct the set of χ-stable points $W^{s,\chi}$, associated to the character $\chi = \det$. On the other hand, such pairs of matrices arise in control theory as linear dynamical systems. An important rôle is played by those systems which are reachable (i.e. satisfy a certain rank condition), we denote by $\mathcal{M}$ the set of reachable pairs. The second particular case considered in this paper is that of the action of the group $\mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}_r(\mathbb{C})$ on the space $\widetilde{W}$ of

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triples (E, A, A_1), where E, A are square matrices of order r and A_1 is a $r \times r_1$ matrix. Consider the character $\widetilde{\chi}$ given by $\det^{r+1} \cdot \det^{-r}$ and let $\widetilde{W}^{s,\widetilde{\chi}}$ be the set of associated stable points. In the framework of control theory, such triples are called generalized linear systems and one may consider the subset $\widetilde{\mathcal{M}}$ of admissible reachable triples. These four sets are apparently independent of each other, but it was step-by-step proved that one has a nice commutative diagram whose vertical maps are bijections:

$$\begin{array}{ccc} W^{s,\chi} & \overset{j}{\hookrightarrow} & \widetilde{W}^{s,\widetilde{\chi}} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ \mathcal{M} & \underset{i}{\hookrightarrow} & \widetilde{\mathcal{M}}. \end{array} \tag{1}$$

The one-to-one correspondence α between reachable linear systems and χ-stable pairs was proved by Byrnes and Hurt [3]. They used tools from algebraic geometry, strengthening the already existing algebro-geometric methods in control theory (see e.g. Tannenbaum's monograph [4] for a survey). Later, Helmke and Shayman [5] and Helmke [6] constructed the natural inclusion i, showing the relationship between reachable linear dynamical systems and admissible reachable generalized linear systems. The next construction was that of the map j: the actions on W, respectively, on $\widetilde{W}$ were interpreted in [7] by using partial quiver factorization problems and, in that framework, the construction of j arises in a natural fashion. Finally, the diagram is closed by the map β, and it was shown by Bader (see [8,9]) that this map is one-to-one. We notice that, since all these maps are compatible with the corresponding actions, they induce maps between the associated quotients. For instance, i provides a smooth compactification of the moduli space of reachable linear systems.

The aim of the present paper is to give alternative proofs of the results mentioned above, by remaining completely in the framework of linear algebra and by using a basis-free approach. Specifically, the bijectivity of the maps α and β is proved in Theorem 1, respectively, Theorem 3, while the construction of the map j is detailed in Theorem 2. Moreover, the actions mentioned above are regarded as partial quiver factorization problems. Particularly, the techniques presented in the paper could be adapted, in order to prove similar results for arbitrary quivers.

2. Preliminaries

This section is preparatory in nature and is divided in three subsections. We aim to recall some general definitions, to present several explicit examples and to prove lemmas which will be used in the main part of the paper.

2.1. Elements of Hermitian type: stability

In this section we present, in a general framework, concepts and notation that will be used throughout the paper. Specifically, we recall the definition of elements of Hermitian type and we fix some notation for the spaces spanned by eigenvectors corresponding to nonpositive eigenvalues. Finally, we give the definition of (semi)stable elements.

Let G be a connected reductive complex Lie group with Lie algebra $\mathfrak{g}$ (throughout this paper the Lie algebra of a Lie group will be denoted by the corresponding 'german' character). We denote by Z the center of G and by $Z_{\mathbb{R}}$ its unique maximal compact subgroup. We also set:

$$\mathcal{T}_G := \left\{ \tau \in \mathfrak{g}^\vee \mid \tau_{|[\mathfrak{g},\mathfrak{g}]} = 0,\ \tau(\mathfrak{z}_{\mathbb{R}}) \subseteq \mathbb{R} \right\},$$

which is a real vector space, naturally isomorphic to $(\mathfrak{z}_{\mathbb{R}})^\vee$ (the dual of the Lie algebra of $Z_{\mathbb{R}}$). Moreover, let K be an arbitrary maximal compact subgroup of G. Then its Lie algebra $\mathfrak{k}$ can be decomposed as

$\mathfrak{k} = \mathfrak{z}_{\mathbb{R}} \oplus [\mathfrak{k}, \mathfrak{k}]$. Since K is connected, the following relations hold

$$\left\{\tau \in \mathfrak{k}^\vee | \langle \tau, \mathrm{ad}_g(\xi)\rangle = \langle \tau, \xi\rangle, \ \forall g \in K, \xi \in \mathfrak{k}\right\} = \{\tau \in \mathfrak{k}^\vee | \tau_{|[\mathfrak{k},\mathfrak{k}]} = 0\} \simeq \mathcal{T}_G. \tag{2}$$

Throughout this paper we will tacitly use this identification and we will call *weights* the elements of $\mathcal{T}_G$.

An element s of $\mathfrak{g}$ is called *of Hermitian type* if there exists a compact subgroup K of G such that $s \in i\mathfrak{k}$; the set of elements of Hermitian type will be denoted by $H(G)$. The previous definition and equivalent characterizations of the elements of $H(G)$ can be found in [10, Definition 3.1].

Let further $\rho : G \to \mathrm{GL}(W)$ be a representation of G on a finite dimensional complex vector space W; its kernel will be denoted by H. In particular, if $s \in H(G)$ is an element of Hermitian type, the endomorphism $\rho_*(s)$ has only real eigenvalues and is diagonalizable (see [10]). For an eigenvalue λ of $\rho_*(s)$, we denote by $W(\lambda)$ the corresponding subspace of eigenvectors. Consider the spaces spanned by eigenvectors corresponding to nonpositive, respectively, to negative eigenvalues

$$W^{\leqslant 0}(s) := \bigoplus_{\lambda \leqslant 0} W(\lambda), \quad W^{<0}(s) := \bigoplus_{\lambda<0} W(\lambda).$$

We notice that, if $W = \bigoplus_{i=1}^p W_i$ is the direct sum of the representations W_i of the group G, then for any $s \in H(G)$ it holds

$$W^{\leqslant 0}(s) = \bigoplus_{i=1}^p W_i^{\leqslant 0}(s), \quad W^{<0}(s) = \bigoplus_{i=1}^p W_i^{<0}(s). \tag{3}$$

We end this section by introducing, in this rather general framework, the condition of stability which will be used throughout the paper. As we already pointed out, stability was firstly introduced by adapting Mumford's Geometric Invariant Theory. There are two alternative approaches to define stability: symplectic and analytic and the three definitions are equivalent (see [10] for details and further references). Furthermore, it turns out that, in the case of linear actions, they can be reformulated by using only notions which are specific to linear algebra (see [10–12]). For the purposes of this paper, it is useful to use the following equivalent characterization [13, p. 18] as definition:

Definition 1. Fix τ in $\mathcal{T}_G$. An element $w \in W$ is called:

(i) *τ-semistable* if for any $s \in H(G)$ such that $w \in W^{\leqslant 0}(s)$, it holds $\langle \tau, is\rangle \geqslant 0$;
(ii) *τ-stable* if it is τ-semistable and for any $s \in H(G)\setminus\mathfrak{h}$ such that $w \in W^{\leqslant 0}(s)$ it holds $\langle \tau, is\rangle > 0$.

For a fixed τ, the set of τ-semistable elements will be denoted by $W^{ss,\tau}$, analogously $W^{s,\tau}$ represents the set of τ-stable elements. These sets are unions of orbits. This is due to the fact that (semi)stability is a property of the *orbits*. Thus, an element is (semi)stable if and only if all the elements in its G-orbit are (semi)stable, too. This fact is not at all clear from Definition 1, but can be proved by using alternative approaches (e.g. GIT or symplectic). On the other hand, for the linear problems that will be discussed in this paper, this property can be proved by using specific arguments.

2.2. Examples: partial quiver factorization problems

The aim of this section is to present four basic examples of spaces spanned by eigenvectors corresponding to nonpositive eigenvalues. These examples correspond to four basic types of quiver factorization problems (see [14] or [7] for the terminology) and, by using (3), they can be extended to a large class of such problems. For instance, Example 4 below can be used in the study of (generalized) Kronecker quivers.

In the remaining of the paper, we consider a r-dimensional complex vector space V. Let $s \in H(\mathrm{GL}(V))$ be an element of Hermitian type. We denote by $\lambda_1 < \cdots < \lambda_q$ the (real) eigenvalues of s and by $V(\lambda_i)$, $1 \leqslant i \leqslant q$, the corresponding eigenspaces. For any $i = 1, \ldots, q$ we define

$$V_i := \bigoplus_{\lambda_j \leqslant \lambda_i} V(\lambda_j)$$

and we put $V_0 := \{0\}$, obtaining a filtration of V, denoted by $\mathcal{F}_s$,

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_q = V.$$

We also define

$$\mathcal{F}_s^{\leqslant 0} := \bigoplus_{\lambda_j \leqslant 0} V(\lambda_j), \quad \mathcal{F}_s^{<0} := \bigoplus_{\lambda_j < 0} V(\lambda_j).$$

In particular, there exist indices j_1, j_2 such that $\mathcal{F}_s^{\leqslant 0} = V_{j_1}, \mathcal{F}_s^{<0} = V_{j_2}$.

Remark 1. Fix a basis of eigenvectors $\{v_1, \ldots, v_r\}$ corresponding to the set of eigenvalues of s (ordered increasingly). For an element $g \in \mathrm{GL}(V)$ one can construct an new element of Hermitian type, denoted by $g * s$, as follows: its eigenvalues are those of s and $\{g(v_1), \ldots, g(v_r)\}$ represents a basis of eigenvectors. Although the filtrations associated to s and to $(g * s)$ do not coincide, the dimensions of the subspaces arising in these filtrations are the same and, particularly, one has $\mathrm{tr}(s) = \mathrm{tr}(g * s)$.

Example 1. Consider the $\mathrm{GL}(V)$-action by conjugation on $W := \mathrm{End}_{\mathbb{C}}(V)$, which is a factorization problem associated to the quiver

$$\bullet \circlearrowleft$$

Then $\varphi \in W^{\leqslant 0}(s)$ if and only if the filtration $\mathcal{F}_s$ is φ-invariant.

Example 2. Take the $\mathrm{GL}(V)$-action on the space $W := \mathrm{Hom}_{\mathbb{C}}(V', V)$ given by $(g, \varphi) \mapsto g \circ \varphi$ (here V' is another finite dimensional complex vector space). This action is a partial quiver factorization problem associated to the quiver

$$\circ \longrightarrow \bullet$$

The tail of the vertex is represented as 'unmarked' ($\circ$), while its head is represented as 'marked' ($\bullet$), that is only the symmetry group corresponding to the head acts on the representation space of the quiver (see [7]).

Then one has $\varphi \in W^{\leqslant 0}(s)$ if and only if $\mathrm{im}(\varphi) \subseteq \mathcal{F}_s^{\leqslant 0}$.

Example 3. Take the $\mathrm{GL}(V)$-action on the space $W := \mathrm{Hom}_{\mathbb{C}}(V, V')$ given by $(g, \varphi) \mapsto \varphi \circ g^{-1}$, which is a partial quiver factorization problem associated to the quiver

$$\bullet \longrightarrow \circ$$

Then one has $\varphi \in W^{\leqslant 0}(s)$ if and only if $\ker(\varphi) \supseteq \mathcal{F}_s^{<0}$.

The last example takes into account the case when two symmetry groups are acting on the representation space of the quiver.

Example 4. Take the representation ρ of the group $\mathrm{GL}(V) \times \mathrm{GL}(V)$ on the space $W := \mathrm{Hom}_{\mathbb{C}}(V, V)$ given by $\rho(g, h) \cdot \varphi = g \circ \varphi \circ (h)^{-1}$, which is the (full) quiver factorization problem associated to the quiver

$$\bullet \longrightarrow \bullet$$

Fix $(s', s'') \in H(\mathrm{GL}(V) \times \mathrm{GL}(V)) \simeq H(\mathrm{GL}(V)) \times H(\mathrm{GL}(V))$ an element of Hermitian type. Let $\lambda'_1 < \lambda'_2 < \cdots < \lambda'_a$, respectively, $\lambda''_1 < \lambda''_2 < \cdots < \lambda''_b$ be the eigenvalues of s', respectively, s''. Then $\varphi \in W^{\leqslant 0}(s', s'')$ if and only if for any real number m it holds

$$\varphi\left(\bigoplus_{\lambda''_k \leqslant m} V(\lambda''_k)\right) \subseteq \bigoplus_{\lambda'_j \leqslant m} V(\lambda'_j). \tag{4}$$

We will now prove the latter statement (the assertions of Examples 1, 2 and 3 can be proved by using similar arguments). In the case of the action ρ considered in Example 4, the eigenvalues of $\rho_*(s', s'')$ are $\{\lambda'_p - \lambda''_q\}_{p,q}$ and, according to our notations, let $(W(\lambda'_p - \lambda''_q))_{p,q}$ be the corresponding eigenspaces. Thus, any $\varphi \in W$ can be decomposed as $\varphi = \sum \varphi_{\lambda'_p - \lambda''_q}$, where for any $p = 1, \ldots, a$ and $q = 1, \ldots, b$ one has

$$s' \circ \varphi_{\lambda'_p - \lambda''_q} - \varphi_{\lambda'_p - \lambda''_q} \circ s'' = (\lambda'_p - \lambda''_q)\varphi_{\lambda'_p - \lambda''_q}. \tag{5}$$

Using the equality (5) one can prove that for any eigenvector v corresponding to an eigenvalue λ'' and for any p and q it holds $\varphi_{\lambda'_p - \lambda''_q}(v) \in V(\lambda'' + \lambda'_p - \lambda''_q)$. We deduce that, writing $\varphi = \varphi_{-,0} + \varphi_+$, with $\varphi_{-,0} \in W^{\leqslant 0}(s', s'')$, respectively, $\varphi_+ \in \bigoplus_{\nu > 0} W(\nu)$ the following inclusions are verified for any λ''

$$\varphi_{-,0}(V(\lambda'')) \subseteq \bigoplus_{\lambda' \leqslant \lambda''} V(\lambda'), \quad \varphi_+(V(\lambda'')) \subseteq \bigoplus_{\lambda' > \lambda''} V(\lambda'). \tag{6}$$

The inclusions (6) and the fact that the relation (4) holds if and only if $\varphi(V(\lambda'')) \subseteq \bigoplus_{\lambda' \leqslant \lambda''} V(\lambda')$, for any λ'', yield now the desired conclusion.

We notice that the proof above can be easily generalized to the case of the natural $\mathrm{GL}(V'') \times \mathrm{GL}(V')$-action on the space $W = \mathrm{Hom}_{\mathbb{C}}(V'', V')$, where V'' and V' are two finite dimensional vector spaces. We focused our attention to the case $V'' = V'$, since this situation will be relevant in the remaining of the paper.

Remark 2. Suppose that $\varphi \in W^{\leqslant 0}(s', s'')$, with W as in Example 4. Fix an element $(g, h) \in \mathrm{GL}(V) \times \mathrm{GL}(V)$. Then $g \circ \varphi \circ h^{-1} \in W^{\leqslant 0}(g * s', h * s'')$ and it holds $\mathrm{tr}(s') = \mathrm{tr}(g * s')$, respectively, $\mathrm{tr}(s'') = \mathrm{tr}(h * s'')$. This compatibility property can be easily generalized to arbitrary quiver factorization problems.

2.3. Technical lemmas

In this section, we remain in the framework of Example 4 and we aim to prove several results which will be used in the proofs of the main results. They focus on the situation when the identity belongs to the space spanned by eigenvectors corresponding to (non)positive eigenvalues of a pair (s', s''). We will show that in this case the vector of (ordered weakly increasingly) eigenvalues of s' is, component-wise, less or equal to the vector corresponding to s'' (Lemma 1 and Remark 3). Moreover, if a certain inequality concerning the traces of s' and s'' is verified, then s' (and automatically s'') have at least one positive eigenvalue and, in the associated filtrations, one can find proper subspaces, corresponding to positive eigenvalues, which have the same dimension (Lemma 2).

Let $s', s'' \in H(\mathrm{GL}(V))$ be elements of Hermitian type with eigenvalues $\lambda'_1 < \lambda'_2 < \cdots < \lambda'_a$, respectively, $\lambda''_1 < \lambda''_2 < \cdots < \lambda''_b$. Let further

$$\{0\} = V'_0 \subset V'_1 \subset \cdots \subset V'_a = V, \quad \{0\} = V''_0 \subset V''_1 \subset \cdots \subset V''_b = V$$

be the corresponding filtrations. We denote by $d'_i := \dim_{\mathbb{C}}(V'_i)$ $(i = 1, \ldots, a)$, respectively, $d''_i := \dim_{\mathbb{C}}(V''_i)$ $(i = 1, \ldots, b)$ the dimensions of the vector spaces arising in the filtrations $\mathcal{F}_{s'}$, respectively,

$\mathcal{F}_{s''}$. For any $k = 1, \dots, b$ we put

$$j(k) := \begin{cases} \max\{l \mid \lambda'_l \leqslant \lambda''_k\}, & \text{if } \{l \mid \lambda'_l \leqslant \lambda''_k\} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}.$$

Lemma 1. *Suppose that* id $\in W^{\leqslant 0}(s', s'')$*. Then for any* $k = 1, \dots, b$ *it holds* $d''_k \leqslant d'_{j(k)}$*. Moreover, if* s' *and* s'' *have the same eigenvalues with the same multiplicities, then* s' *and* s'' *have the same associated filtrations, i.e.* $\mathcal{F}_{s'} = \mathcal{F}_{s''}$*.*

Proof. Using Example 4, we deduce that for any k it holds

$$V''_k = \mathrm{id}\left(\bigoplus_{\lambda''_j \leqslant \lambda''_k} V(\lambda''_j)\right) \subseteq \bigoplus_{\lambda'_j \leqslant \lambda''_k} V(\lambda'_j) = \bigoplus_{\lambda'_j \leqslant \lambda'_{j(k)}} V(\lambda'_j) = V'_{j(k)}$$

and hence $d''_k \leqslant d'_{j(k)}$ (we notice that this assertion remains true if we replace id with $\psi \in \mathrm{Hom}_{\mathbb{C}}(V, V)$ invertible). Let us now suppose that $a = b$, that for any $k = 1, \dots, a$ we have $\lambda'_k = \lambda''_k$ and that the corresponding multiplicities are equal. In particular, we deduce that for any k one has $j(k) = k$ and $\dim_{\mathbb{C}}(V'_k) = \dim_{\mathbb{C}}(V''_k)$. Since $V''_k \subseteq V'_{j(k)} = V'_k$, we conclude that the two filtrations coincide. □

Remark 3. Consider the sequences of eigenvalues $(\lambda'(1), \dots, \lambda'(r))$, respectively, $(\lambda''(1), \dots, \lambda''(r))$ of s', respectively, s'' ordered weakly increasingly and such that each eigenvalue occurs as many times as its multiplicity is. Then the condition in Lemma 1 that for any k one has $d''_k \leqslant d'_{j(k)}$ is equivalent to the condition that for any $i = 1, \dots, r$ it holds $\lambda'(i) \leqslant \lambda''(i)$. Moreover, under the assumption that $d''_k \leqslant d'_{j(k)}$, it holds $d''_k = d'_{j(k)}$ if and only if $\lambda''(d''_k) < \lambda'(d''_k + 1)$.

Lemma 2. *Suppose that* id $\in W^{\leqslant 0}(s', s'')$ *for a pair* $(s', s'') \neq (0, 0)$*. Suppose also that the inequality* $(r+1)\mathrm{tr}(s') - r\mathrm{tr}(s'') \geqslant 0$ *is fulfilled. Then:*

(i) s' *has at least one positive eigenvalue;*
(ii) let q *be such that* $\mathcal{F}_{s'}^{\leqslant 0} = V'_q$*; in particular, if* $q \neq 0$, λ'_q *is the greatest nonpositive eigenvalue of* s'*. There exists* t *such that* $q \leqslant j(t) < a$ *and such that* $d''_t = d'_{j(t)}$*.*

Proof. We first notice that, using the notation in Remark 3, the inequality in the hypothesis can be rewritten as

$$\sum_{i=1}^{r} \lambda'(i) + r\left(\sum_{i=1}^{r} (\lambda'(i) - \lambda''(i))\right) \geqslant 0. \tag{7}$$

(i) If all the eigenvalues of s' were nonpositive, then, for any i, we would have $\lambda'(i) \leqslant 0$. On the other hand, by Remark 3, for any i we have $\lambda'(i) - \lambda''(i) \leqslant 0$. Using the inequality (7), we deduce that all the numbers $\lambda'(i), \lambda''(i)$ must be zero, which contradicts our assumption that $(s', s'') \neq (0, 0)$.

(ii) We first claim that there exists $d'_q \leqslant l < r$ such that $\lambda''(l) < \lambda'(l+1)$ (if $d'_q = 0$, we set $\lambda'(0) = \lambda''(0) := 0$). Indeed, assume that for any $d'_q \leqslant l \leqslant r-1$ we have $\lambda''(l) \geqslant \lambda'(l+1)$. Then, we get

$$\sum_{l=d'_q}^{r-1} (r - l + 1)(\lambda'(l+1) - \lambda''(l)) \leqslant 0.$$

To this inequality we add $(r - d'_q + 1)\lambda'(d'_q)$ (which is nonpositive) and $(-\lambda''(r))$ (which is negative). By suitable changing the summing index for λ', we obtain

$$\sum_{l=d'_q+1}^{r} \lambda'(l) + \sum_{l=d'_q}^{r} (r - l + 1)(\lambda'(l) - \lambda''(l)) < 0.$$

Since $\lambda'(1), \ldots, \lambda'(d'_q), \lambda'(1) - \lambda''(1), \ldots, \lambda'(r) - \lambda''(r)$ are nonpositive, this would imply

$$\sum_{i=1}^{r} \lambda'(i) + r \sum_{i=1}^{r} (\lambda'(i) - \lambda''(i)) < 0,$$

and this contradicts the relation (7).

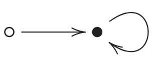
In conclusion, there exists l such that $d'_q \leqslant l < r$ and $\lambda''(l) < \lambda'(l+1)$. Applying Remark 3, we get the inequality $\lambda''(l) < \lambda''(l+1)$, which means that there exists $t < b$ such that $l = d''_t$. Again by Remark 3, we obtain the relations $d'_q \leqslant d''_t = d'_{j(t)} < r = d'_a$, which yield the desired statement. □

3. Main results

We will start the discussion by considering a basic partial factorization of a quiver, namely that one that involves the quiver with one loop and one arrow. We firstly describe the set of (semi)stable elements in this case, pointing out the relationship to linear systems (Section 3.1). We further relate, in a natural fashion, this quiver to another one, namely to an augmented Kronecker quiver and we show the relationship between the associated stability conditions (Section 3.2). The analysis of the stability for the latter quiver factorization problem will be deepened in Section 3.3.

3.1. The one-arrow-one-loop-quiver and linear systems

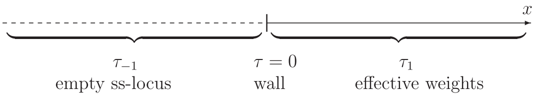
Consider the marked quiver Q represented below, consisting of two vertices, one loop and one arrow



To this diagram, one can associate in natural fashion a quiver factorization problem as follows. Take an r-dimensional complex vector space V as in Section 2 and another complex vector space V_1. One has a natural action of the group $G := \mathrm{GL}(V)$ on the space

$$W := \mathrm{End}_{\mathbb{C}}(V) \times \mathrm{Hom}_{\mathbb{C}}(V_1, V)$$

given by $g \cdot (\varphi, \varphi_1) := (g \circ \varphi \circ g^{-1}, g \circ \varphi_1)$. The stability conditions (Definition 1) which could be introduced for this problem depend on a weight $\tau \in \mathcal{T}_{\mathrm{GL}(V)}$. Moreover, as pointed out, for instance, in [15] (for affine spaces) and in [16] (for representations of quivers), the set of weights has a GIT-fan structure. In the case of the group $\mathrm{GL}(V)$, this structure is a very simple one: there are essentially three different stability conditions, corresponding to the weights $\tau_0 := 0$, $\tau_1 := i\mathrm{tr}$, $\tau_{-1} := -i\mathrm{tr}$. We first claim that the semistable locus corresponding to the weight τ_{-1} is empty. Indeed, fix an arbitrary element $(\varphi, \varphi_1) \in W$. We consider the element of Hermitian type $s = -\mathrm{id}$ with associated filtration $\{0\} \subset V = \mathcal{F}_s^{\leqslant 0}$. Obviously, according to Examples 1 and 2, it holds $(\varphi, \varphi_1) \in W^{\leqslant 0}(s)$, but $\langle \tau, is \rangle = -\dim_{\mathbb{C}}(V) < 0$, that is (φ, φ_1) cannot be semistable. As a general rule, this proof can be adapted to show that any marked sink (or source) of a quiver factorization problem imposes some restrictions to the cone of effective weights (that is those weights for which the semistable locus is not empty). Thus, in this case, the cone of effective weights is given by the inequality $x \geqslant 0$. On the other hand, for $\tau_0 = 0$ the semistable locus is equal to the whole space, while the stable locus is empty (this statement can be proved using arguments similar to those above). This shows that τ_0 represents a 'wall' of the space of weights (i.e. the stable locus does not coincide with the semistable one).



We now aim to describe the set of (semi)stable points for the remaining character, τ_1. Put first $V_{\varphi,\varphi_1} := \sum_{0 \leqslant i \leqslant r-1} \mathrm{im}(\varphi^i \circ \varphi_1) \subseteq V$ and denote by $\mathcal{M}$ the set of pairs (φ, φ_1) such that $V_{\varphi,\varphi_1} = V$.

Theorem 1. *It holds* $\mathcal{M} = W^{s,\tau_1} = W^{ss,\tau_1}$.

Proof. Let (φ, φ_1) be an element of $\mathcal{M}$. Consider $s \in H(\mathrm{GL}(V))$ such that $(\varphi, \varphi_1) \in W^{\leqslant 0}(s)$. By Example 2 it follows that $\mathrm{im}(\varphi_1) \subseteq \mathcal{F}_s^{\leqslant 0}$ and by Example 1 the filtration $\mathcal{F}_s$ is φ-invariant, in particular $\varphi(\mathcal{F}_s^{\leqslant 0}) \subseteq \mathcal{F}_s^{\leqslant 0}$. We deduce that for any $0 \leqslant i \leqslant r-1$ it holds $\mathrm{im}(\varphi^i \circ \varphi_1) \subseteq \mathcal{F}_s^{\leqslant 0}$ and eventually $V_{\varphi,\varphi_1} \subseteq \mathcal{F}_s^{\leqslant 0}$. Since (φ, φ_1) is an element of $\mathcal{M}$, it results that $V_{\varphi,\varphi_1} = \mathcal{F}_s^{\leqslant 0} = V$. In particular all the eigenvalues of s are nonpositive and hence $\langle \tau_1, is \rangle = -\mathrm{tr}(s) \geqslant 0$. Moreover, if $s \neq 0$, then $\langle \tau_1, is \rangle > 0$. This proves that $\mathcal{M} \subseteq W^{s,\tau_1}$.

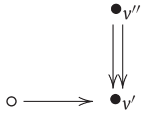
Let now (φ, φ_1) be a pair such that $V_{\varphi,\varphi_1} \neq V$. Consider s with eigenvalues 0 and 1 and whose associated filtration is $\{0\} \subset V_{\varphi,\varphi_1} \subset V$. Then one has $(\varphi, \varphi_1) \in W^{\leqslant 0}(s)$ and it holds

$$\langle \tau_1, is \rangle = -\mathrm{tr}(s) = -\dim_{\mathbb{C}}(V/V_{\varphi,\varphi_1}) < 0,$$

which shows that the given pair is not τ_1-semistable. Hence we proved that the semistable locus is included in $\mathcal{M}$. Since the stable locus is included in the semistable one, the required equalities follow. □

3.2. Enlargement procedure and relationship between stability conditions

The quiver Q contains a loop (particularly a closed oriented path). As pointed out in [7], one could apply to this quiver the 'enlargement' procedure, which means to consider the following augmented Kronecker quiver, denoted by $\widetilde{Q}$



and an appropriate quiver factorization problem. Actually, this construction translates in terms of quivers Helmke's construction [17,6]. More precisely, we take the group $\widetilde{G} := \mathrm{GL}(V) \times \mathrm{GL}(V)$, which acts on the space

$$\widetilde{W} := \mathrm{Hom}_{\mathbb{C}}(V, V)^2 \times \mathrm{Hom}_{\mathbb{C}}(V_1, V)$$

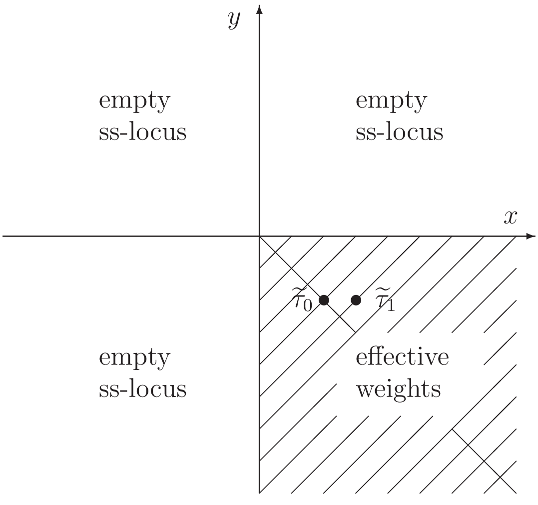
in a natural fashion

$$(g, h) \cdot (\psi, \varphi, \varphi_1) := (g \circ \psi \circ h^{-1}, g \circ \varphi \circ h^{-1}, g \circ \varphi_1).$$

Let us notice that, at set-theoretical level, one has a natural map which is compatible with the two group actions

$$\iota : W \to \widetilde{W}, \quad \iota(\varphi, \varphi_1) := (\mathrm{id}, \varphi, \varphi_1).$$

For this quiver factorization problem, the space of characters is isomorphic to $\mathbb{R}^2$. Since the vertex denoted by v' is a sink and the vertex denoted by v'' is a source, one can prove, using arguments similar to those mentioned above, that the cone of effective weights is given by the inequalities $x \geqslant 0$, $y \leqslant 0$. In this case, the wall structure is more complicated, and it is not our aim to describe here its complete structure. We will instead only prove that there exists a relationship between the stability conditions associated to the quiver Q and those corresponding to $\widetilde{Q}$.



We consider the weights $\widetilde{\tau}_0$ and $\widetilde{\tau}_1$ given by

$$\langle \widetilde{\tau}_\varepsilon, (\zeta', \zeta'') \rangle := \begin{cases} ir(\mathrm{tr}(\zeta') - \mathrm{tr}(\zeta'')), & \text{if } \varepsilon = 0 \\ i((r+1)\mathrm{tr}(\zeta') - r\mathrm{tr}(\zeta'')), & \text{if } \varepsilon = 1. \end{cases}$$

Then for any $s', s'' \in H(\mathrm{GL}(V))$ one has $\langle \widetilde{\tau}_0, i(s', s'') \rangle = -r\mathrm{tr}(s') + r\mathrm{tr}(s'')$ and $\langle \widetilde{\tau}_1, i(s', s'') \rangle = -(r+1)\mathrm{tr}(s') + r\mathrm{tr}(s'')$. We now claim that the following result holds.

Theorem 2. *Take any $\varepsilon \in \{0, 1\}$. A pair $(\varphi, \varphi_1) \in W$ is τ_ε-(semi)stable if and only if $\iota(\varphi, \varphi_1) \in \widetilde{W}$ is $\widetilde{\tau}_\varepsilon$-(semi)stable.*

Proof. We first notice that for $\varepsilon \in \{0, 1\}$ and $s \in H(\mathrm{GL}(V))$ it holds: if $(\varphi, \varphi_1) \in W^{\leqslant 0}(s)$ and $\langle \tau_\varepsilon, is \rangle < (\leqslant) 0$, then $(\mathrm{id}, \varphi, \varphi_1) \in \widetilde{W}^{\leqslant 0}(s, s)$ and $\langle \widetilde{\tau}_\varepsilon, i(s, s) \rangle < (\leqslant) 0$. This means that if (φ, φ_1) is not τ_ε-(semi)stable, then $(\mathrm{id}, \varphi, \varphi_1)$ cannot be $\widetilde{\tau}_\varepsilon$-(semi)stable.

We now prove the converse assertion: if (φ, φ_1) is τ_ε-(semi)stable, then $(\mathrm{id}, \varphi, \varphi_1)$ is $\widetilde{\tau}_\varepsilon$-(semi)stable.

The case $\varepsilon = 0$. Since the stable locus for τ_0 is empty, we only have to prove that every element $(\mathrm{id}, \varphi, \varphi_1)$ is $\widetilde{\tau}_0$-semistable. Indeed, if (s', s'') is a pair such that $(\mathrm{id}, \varphi, \varphi_1) \in \widetilde{W}^{\leqslant 0}(s', s'')$, then, by Remark 3, we deduce that for any $i = 1, \ldots, r$ one has $\lambda'(i) \leqslant \lambda''(i)$ and hence

$$\langle \widetilde{\tau}_0, i(s', s'') \rangle = -r(\mathrm{tr}(s') - \mathrm{tr}(s'')) = r \sum_{i=1}^{r} (-\lambda'(i) + \lambda''(i)) \geqslant 0.$$

The case $\varepsilon = 1$. We will prove that if $(\mathrm{id}, \varphi, \varphi_1)$ is not $\widetilde{\tau}_1$-stable, then (φ, φ_1) is not τ_1-semistable. Let $(s', s'') \neq (0, 0)$ be a pair for which it holds $(\mathrm{id}, \varphi, \varphi_1) \in \widetilde{W}^{\leqslant 0}(s', s'')$ and such that $\langle \widetilde{\tau}_1, i(s', s'') \rangle \leqslant 0$. The latter inequality means that $(r+1)\mathrm{tr}(s') - r\mathrm{tr}(s'') \geqslant 0$ and we can hence apply Lemma 2. We therefore find q such that $\mathcal{F}_{s'}^{\leqslant 0} = V'_q$, respectively, t such that $q \leqslant j(t) < a$ and such that $d''_t = d'_{j(t)}$.

We already showed in the proof of Lemma 1 that $V''_t \subseteq V'_{j(t)}$ and since their dimensions are equal ($d''_t = d'_{j(t)}$), it follows that these subspaces of V coincide. Moreover, since $\varphi \in \mathrm{Hom}_{\mathbb{C}}(V,V)^{\leqslant 0}(s', s'')$, we can use the relation (4), deducing

$$\varphi(V'_{j(t)}) = \varphi(V''_t) = \varphi\left(\bigoplus_{\lambda''_i \leqslant \lambda''_t} V(\lambda''_i)\right) \subseteq \bigoplus_{\lambda'_i \leqslant \lambda''_t} V(\lambda'_i) = \bigoplus_{\lambda'_i \leqslant \lambda'_{j(t)}} V(\lambda'_i) = V'_{j(t)}$$

and it holds

$$\mathrm{im}(\varphi_1) \subseteq \mathcal{F}_{s'}^{\leqslant 0} = V'_q \subseteq V'_{j(t)}.$$

The main point is that we found a *proper* φ-invariant subspace V' which includes $\mathrm{im}(\varphi_1)$ (namely $V'_{j(t)}$). Consider now s with eigenvalues 0 and 1 and with associated filtration $\mathcal{F}_s$:

$$\{0\} \subset V' \subset V.$$

The conditions above mean that $(\varphi, \varphi_1) \in W^{\leqslant 0}(s)$. On the other hand, one has

$$\langle \tau_1, is\rangle = -\mathrm{tr}(s) = -\dim_{\mathbb{C}}(V/V') < 0$$

and we conclude that (φ, φ_1) is not τ_1-semistable. □

3.3. Augmented Kronecker quivers and generalized linear systems

We remain in the framework of the quiver factorization problem associated to $\widetilde{Q}$, stated in Section 3.2, and we aim to explicitly describe the set of (semi)stable points corresponding to the character $\widetilde{\tau}_1$.

We begin by fixing some notation and terminology. Following [18], a triple $(\psi, \varphi, \varphi_1)$ will be called *admissible* if $\det(\lambda\psi + \mu\varphi) \not\equiv 0$. We notice that, if $\xi = (\psi, \varphi, \varphi_1)$ is admissible, then one can find $(\psi', \varphi', \varphi_1)$ lying in the same $\widetilde{G}$-orbit as ξ and such that, for suitable $\lambda_0, \mu_0 \in \mathbb{C}$, one has $\lambda_0\psi' + \mu_0\varphi' = \mathrm{id}$. Moreover, ψ' and φ' commute [19, Proposition 2.1].

Following [19], we say that a triple $(\psi, \varphi, \varphi_1)$ is *reachable* if $V_{\psi,\varphi,\varphi_1} = V$, where the subspace $V_{\psi,\varphi,\varphi_1}$ is defined by $V_{\psi,\varphi,\varphi_1} = \sum_{0\leqslant p,q\leqslant r-1} \mathrm{im}(\psi^p \circ \varphi^q \circ \varphi_1)$. Notice that, under the assumption of admissibility, the reachability condition is equivalent to the *controllability* condition used in [5,6], namely that $\mathrm{im}(\lambda\psi + \mu\varphi) + \mathrm{im}(\varphi_1) = V$ for any $(\lambda, \mu) \neq (0, 0)$ (see [19, Theorem 4.1]).

Consider now the following set

$$\widetilde{\mathcal{M}} = \{(\psi, \varphi, \varphi_1) \in \widetilde{W} \mid (\psi, \varphi, \varphi_1) \text{ admissible and reachable}\}.$$

Theorem 3. *It holds* $\widetilde{\mathcal{M}} = \widetilde{W}^{s,\widetilde{\tau}_1} = \widetilde{W}^{ss,\widetilde{\tau}_1}$.

Proof. We first take $(\psi, \varphi, \varphi_1) \in \widetilde{\mathcal{M}}$ and we aim to show that it is $\widetilde{\tau}_1$-stable. We now use the fact that stability is a property of the orbits (in the quiver problem considered in this Theorem, this fact follows at once from Remark 2). By using the admissibility condition, we may hence assume, without loss of generality, that $\lambda_0\psi + \mu_0\varphi = \mathrm{id}$, for λ_0, μ_0 suitable chosen. Suppose now that the stability condition is not fulfilled. This means that we can find $(s', s'') \neq (0, 0)$ with $(\psi, \varphi, \varphi_1) \in \widetilde{W}^{\leqslant 0}(s', s'')$ and such that $\langle \widetilde{\tau}_1, i(s', s'')\rangle \leqslant 0$. We first notice that $\mathrm{id} \in \mathrm{Hom}_{\mathbb{C}}(V,V)^{\leqslant 0}(s', s'')$, particularly Lemmas 1 and 2 can be applied. As in the proof of Theorem 2, we can construct a proper subspace $V' \subset V$ which is both ψ and φ-invariant and such that $\mathrm{im}(\varphi_1) \subseteq V'$. This would imply that $V_{\psi,\varphi,\varphi_1} \subseteq V'$ (one tacitly uses the fact that ψ and φ commute), that is the triple $(\psi, \varphi, \varphi_1)$ is not reachable and this yields a contradiction. We conclude that $\widetilde{\mathcal{M}} \subseteq \widetilde{W}^{s,\widetilde{\tau}_1}$.

We will now prove that, if $(\psi, \varphi, \varphi_1)$ is $\widetilde{\tau}_1$-semistable, then it is an element of $\widetilde{\mathcal{M}}$. Suppose, on the contrary, that this is not the case. We distinguish two situations.

If $(\psi, \varphi, \varphi_1)$ would not be admissible, then the relation $\det(\lambda\psi + \mu\varphi) \equiv 0$ would be verified. This means that the matrix pencil (M_ψ, M_φ) obtained by fixing some bases of V is singular. Using the canonical form of a singular pencil [20, p. 37], one proves that there exist subspaces W', W'' with $\dim_{\mathbb{C}}(W') < \dim_{\mathbb{C}}(W'')$ and such that $(\lambda\psi + \mu\varphi)(W'') \subseteq W'$ for all scalars λ, μ. Construct now s'' with associated filtration $\{0\} \subset W'' \subseteq V$ and with eigenvalues -1; 0, respectively, s' with associated filtration $\{0\} \subseteq W' \subset V$ and with the same eigenvalues. Denote by $\delta = \dim_{\mathbb{C}}(W')$, $d = \dim_{\mathbb{C}}(W'')$; one has $0 \leqslant \delta \leqslant d-1 < r$ and obviously $\mathrm{tr}(s') = -\delta$, $\mathrm{tr}(s'') = -d$. It is easy to check that ψ, φ satisfy the condition in Example 4. Since obviously $\mathrm{im}(\varphi_1) \subseteq \mathcal{F}_{s'}^{\leqslant 0}$, it follows that $(\psi, \varphi, \varphi_1) \in \widetilde{W}^{\leqslant 0}(s', s'')$. On the other hand, we have the following relations:

$$\langle \tilde{\tau}_1, i(s', s'') \rangle = -(r+1)\mathrm{tr}(s') + r\mathrm{tr}(s'') = (r+1)\delta - rd = r(\delta - d) + \delta \leqslant -r + \delta < 0$$

and this shows that $(\psi, \varphi, \varphi_1)$ cannot be $\tilde{\tau}_1$-semistable.

Let us now suppose that $(\psi, \varphi, \varphi_1)$ is not reachable, but it is admissible. Again by using the fact that semistability is a property of the orbits we may assume, without loss of generality, that $\lambda_0\psi + \mu_0\varphi = \mathrm{id}$; particularly ψ and φ commute. Consider now $s' = s''$ with associated filtration $\{0\} \subseteq V_{\psi,\varphi,\varphi_1} \subset V$ and with eigenvalues 0 and 1. Then obviously $\mathrm{im}(\varphi_1) \subseteq \mathcal{F}_{s'}^{\leqslant 0}$. Moreover, one can check that ψ and φ verify the conditions of Example 4 (the fact that they commute is crucial). On the other hand, we have

$$\langle \tilde{\tau}_1, i(s', s'') \rangle = -(r+1)\mathrm{tr}(s') + r\mathrm{tr}(s'') = -\dim_{\mathbb{C}}(V/V_{\psi,\varphi,\varphi_1}) < 0$$

and this shows that $(\psi, \varphi, \varphi_1)$ is not $\tilde{\tau}_1$-semistable. We conclude that the semistable locus is contained in $\widetilde{\mathcal{M}}$.

Finally, since the stable locus is always included in the semistable one, we get the desired statement. □

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