Europ. J. Combinatorics (1999) **20**, 821–832
Article No. eujc.1999.0341
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Subgraphs with Restricted Degrees of their Vertices in Large Polyhedral Maps on Compact Two-manifolds

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Let $k \geq 2$, be an integer and $\mathbb{M}$ be a closed two-manifold with Euler characteristic $\chi(\mathbb{M}) \leq 0$. We prove that each polyhedral map G on $\mathbb{M}$, which has at least $(8k^2 + 6k - 6)|\chi(\mathbb{M})|$ vertices, contains a connected subgraph H of order k such that every vertex of this subgraph has, in G, the degree at most $4k + 4$. Moreover, we show that the bound $4k + 4$ is best possible. Fabrici and Jendrol' proved that for the sphere this bound is 10 if $k = 2$ and $4k + 3$ if $k \geq 3$. We also show that the same holds for the projective plane.

1. INTRODUCTION

This paper continues the investigations of [2] and [7]. Some of the definitions of [7] are repeated.

In this paper all manifolds are compact two-dimensional manifolds. If a graph G is embedded in a manifold $\mathbb{M}$ then the closure of the connected components of $\mathbb{M} - G$ are called *the faces* of G. If each face is a closed two-cell and each vertex has valence at least three then G is called a *map* in $\mathbb{M}$. If, in addition, no two faces have a multiply connected union then G is called a *polyhedral map* in $\mathbb{M}$. This condition on the union of two faces is equivalent to saying that any two faces that meet, meet on a single vertex or a single edge. When two faces in a map meet in one of these two ways we say that they *meet properly*.

In the following let $\mathbb{S}_g$ ($\mathbb{N}_q$) be an orientable (a non-orientable) surface of genus g (genus q, respectively). We say that H is a *subgraph* of a polyhedral map G if H is a subgraph of the underlying graph of the map G.

The degree of a face α of a polyhedral map is the number of edges incident to α. Vertices and faces of degree j are called j-valent vertices and j-valent faces, respectively. Let $v_i(G)$ and $p_j(G)$ denote the number of i-valent vertices and j-valent faces, respectively. For a polyhedral map G let $V(G)$, $E(G)$ and $F(G)$ be the vertex set, the edge set and the face set of G, respectively. The degree of a vertex A in G is denoted by $\deg_G(A)$ or $\deg(A)$ if G is known from the context. A path and a cycle on k vertices is defined to be the k-*path* and the k-*cycle*, respectively. A k-path passing through vertices $A_1, \dots, A_k$ is denoted by $[A_1, A_2, \dots, A_k]$ provided that $A_i, A_{i+1} \in E(G)$ for any $i = 1, 2, \dots, k-1$.

It is an old classical consequence of the famous Euler formula that each planar graph contains a vertex of degree at most 5. A beautiful theorem of Kotzig [11, 12] states that every three-connected planar graph contains an edge with degree-sum of its endvertices being at most 13. This result was further developed in various directions and served as a starting point for discovering many structural properties of embeddings of graphs, see e.g., [1, 4, 5, 7, 13].

Fabrici and Jendrol' [1] have proved that every three-connected planar graph G of maximum degree at least $5k$ contains a path P_k on k vertices such that each vertex of this path has, in G, a degree $\leq 5k$; the bound $5k$ being best possible. A slight modification of their proof provides the validity of this result also for every three-connected graph embedded in the projective plane. An analogous result has been found for two-manifolds other than the sphere and the projective plane, see [7].

More precisely, the following problem has been investigated.

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PROBLEM 1. For a given connected graph H let $\mathcal{G}(H, \mathbb{M})$ be the family of all polyhedral maps on a closed two-manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M})$ having a subgraph isomorphic with H. What is the minimum integer $\phi(H, \mathbb{M})$ such that every polyhedral map $G \in \mathcal{G}(H, \mathbb{M})$ contains a subgraph K isomorphic with H for which

$$\deg_G(A) \leq \phi(H, \mathbb{M}) \qquad \text{for every vertex } A \in V(K)?$$

(If such minimum does not exist we write $\phi(H, \mathbb{M}) = \infty$.)

The answer to this question for $\mathbb{S}_0$ and $\mathbb{N}_1$ is given in Theorem 1; the answer for each two-manifold other then $\mathbb{S}_0$ and $\mathbb{N}_1$ is given in Theorem 2.

THEOREM 1 (FABRICI AND JENDROL' [1]). *Let k be an integer, $k \geq 1$. Then*

$$\begin{aligned} \phi(P_k, \mathbb{S}_0) = \phi(P_k, \mathbb{N}_1) = 5k, &\qquad \text{for any } k \geq 1 \\ \phi(H, \mathbb{S}_0) = \phi(H, \mathbb{N}_1) = \infty, &\qquad \text{for any } H \neq P_k. \end{aligned}$$

THEOREM 2 (JENDROL' AND VOSS [7]). *Let k be an integer, $k \geq 1$, and $\mathbb{M}$ be a closed two-manifold with Euler characteristic $\chi(\mathbb{M}) \notin \{1, 2\}$. Then*

(i) $2\left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor \leq \phi(P_k, \mathbb{M}) \leq k \left\lfloor \frac{5+\sqrt{49-24\chi(\mathbb{M})}}{2} \right\rfloor$, *$k \geq 1$, and*
(ii) $\phi(H, \mathbb{M}) = \infty$, *for any $H \neq P_k$.*

Note that in Theorem 2 the upper bound is sharp for even k. The precise value of $\phi(P_k, \mathbb{M})$ for odd k, $k \geq 3$, has been studied in [10].

If $\mathbb{M}$ is the torus $\mathbb{S}_1$ or Klein's bottle $\mathbb{N}_2$ then Theorem 2 implies

$$12\left\lfloor \frac{k}{2} \right\rfloor \leq \phi(P_k, \mathbb{S}_1), \phi(P_k, \mathbb{N}_2) \leq 6k.$$

We proved that these bounds are also valid for polyhedral maps on two-manifolds $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M}) < 0$, if these maps have enough vertices.

Thus the following problem has been investigated.

PROBLEM 2. Let $N \geq 1$ be an integer. For a given connected graph H let $\mathcal{G}_N(H, \mathbb{M})$ be the family of all polyhedral maps on a closed two-manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M})$ having a subgraph isomorphic with H and having $\geq N$ vertices. What is the minimum integer $\phi_N(H, \mathbb{M})$ such that every polyhedral map $G \in \mathcal{G}_N(H)$ contains a subgraph K isomorphic with H for which

$$\deg_G(A) \leq \phi_N(H, \mathbb{M}) \qquad \text{for every vertex } A \in V(K)?$$

Obviously, $\phi_1(H, \mathbb{M}) = \phi(H, \mathbb{M})$.

Let N_k denote the largest number of vertices in a connected graph with maximum degree $\leq 6k$ containing no path with k vertices. Obviously, $N_k \leq (6k)^{k/2+2}$.

Let $\mathbb{M}$ be a compact two-manifold of characteristic $\chi(\mathbb{M}) \leq 0$, $R_1 := (14(k-1)N_k + 6)|\chi(\mathbb{M})|$ and $R_2 := 30\,000(|\chi(\mathbb{M})| + 1)^3(N_k + 3(|\chi(\mathbb{M})| + 1))$. For large polyhedral maps on $\mathbb{M}$ we proved:

THEOREM 3 (JENDROL' AND VOSS [8, 9]). *For any compact two-manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M}) \leq 0$ and any integer $k \geq 1$*

$$\phi_N(P_k, \mathbb{M}) = \begin{cases} 6k, & \text{if } k = 1 \text{ or } k \text{ is even and } N > R_1, \\ 6k-2, & \text{if } k \geq 3 \text{ is odd and } N > R_2. \end{cases}$$

Each polyhedral map G on a closed two-manifold $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M}) \leq 0$ with vertex number $v(G) > |\chi(\mathbb{M})|$ contains a vertex of degree at most 6. If G has enough vertices of degree > 6 then G contains even a vertex of degree at most 5. So we introduced the sum $\sum_{j>6k}(j-6k)v_j$ and investigated the following problem.

PROBLEM 3. Let L be an integer. For a given connected graph H let $\mathcal{G}^L(H,\mathbb{M})$ be the family of all polyhedral maps on a closed two-manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M})$ which have a subgraph isomorphic with H and for which $\sum_{j>6k}(j-6k)v_j \geq L$. What is the minimum integer $\phi^L(H,\mathbb{M})$ such that each polyhedral map $G \in \mathcal{G}^L(H)$ contains a subgraph K isomorphic with H for which

$$\deg_G(A) \leq \phi^L(H,\mathbb{M}) \qquad \text{for every vertex } A \in V(K)?$$

We have proved:

THEOREM 4 (JENDROL' AND VOSS [8]). *For any closed two-manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M}) \leq 0$, any integer $k \geq 1$, and any integer $L > 6k|\chi(\mathbb{M})|$, there is:*

(i) $\phi^L(P_k,\mathbb{M}) = 5k$,
(ii) $\phi^L(H,\mathbb{M}) = \infty$ for any $H \neq P_k$.

Fabrici and Jendrol' [2] have proved that every three-connected planar graph G of order at least k contains a subgraph on k vertices such that each vertex of this subgraph has, in G, a degree $\leq 4k+3$, for $k \geq 3$. More precisely, for the sphere the following problem has been investigated.

PROBLEM 4. Let N, k be positive integers with $N \geq k$. Let $\mathcal{H}_N(k,\mathbb{M})$ be the family of all polyhedral maps of order $N \geq k$ on a compact two-manifold $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M})$. What is the minimum integer $\tau_N(k,\mathbb{M})$ such that every graph $G \in \mathcal{H}_N(k,\mathbb{M})$ contains a connected subgraph H of order k such that

$$\deg_G(A) \leq \tau_N(k,\mathbb{M})$$

holds for every vertex $A \in V(H)$?

Let $\tau(k,\mathbb{M}) := \tau_k(k,\mathbb{M})$. For the sphere Euler's formula gives $\tau(1,\mathbb{S}_0) = 5$. Kotzig's result [11, 12] yields $\tau(2,\mathbb{S}_0) = 10$.

THEOREM 5 (FABRICI AND JENDROL' [2]). *Let k be an integer, $k \geq 1$. Then:*

(i) $\tau(1,\mathbb{S}_0) = 5$,
(ii) $\tau(2,\mathbb{S}_0) = 10$,
(iii) $\tau(k,\mathbb{S}_0) = 4k+3$ for any $k \geq 3$.

In this paper we shall prove Theorems 6–8.

Since each connected subgraph with one, two or three vertices contains a path with one, two or three vertices, respectively, the Theorem 1 [1] implies: $\tau(1,\mathbb{N}_1) = 5$, $\tau(2,\mathbb{N}_1) = 10$, and $\tau(3,\mathbb{N}_1) = 15$.

THEOREM 6. *Let k be an integer, $k \geq 1$. Then for the projective plane holds:*

(i) $\tau(1,\mathbb{N}_1) = 5$,
(ii) $\tau(2,\mathbb{N}_1) = 10$,

(iii) $\tau(k, \mathbb{N}_1) = 4k + 3$ for any $k \geq 3$.

Theorem 3 [8, 9] implies $\tau_N(1, \mathbb{M}) = 6$ for $N > 6|\chi(\mathbb{M})|$; $\tau_N(2, \mathbb{M}) = 12$ and $\tau_N(3, \mathbb{M}) = 16$ for large N.

THEOREM 7. *For any closed two-manifold* $\mathbb{M}$ *with Euler characteristic* $\chi(\mathbb{M}) \leq 0$, *any integer* $k \geq 1$, *and any integer* $N > (8k^2 + 6k - 6)|\chi(\mathbb{M})|$ *it holds:*

(i) $\tau_N(1, \mathbb{M}) = 6$, *and*
(ii) $\tau_N(k, \mathbb{M}) = 4k + 4$ *for any* $k \geq 2$.

Moreover, $\tau_N(1, \mathbb{M}) = 6$ *for* $N > 6|\chi(\mathbb{M})|$.

PROBLEM 5. Let L, k be positive integers. Let $\mathcal{H}^L(k, \mathbb{M})$ be the family of all polyhedral maps of order at least k on a compact two-manifold $\mathbb{M}$ of Euler characteristic $\chi(\mathbb{M})$ for which $\sum_{j>4k+4}(j - (4k + 4))v_j \geq L$. What is the minimum integer $\tau^L(k, \mathbb{M})$ such that each polyhedral map $G \in \mathcal{H}^L(k, \mathbb{M})$ contains a connected subgraph H of order k such that

$$\deg_G(A) \leq \tau^L(k, \mathbb{M})$$

holds for every vertex $A \in V(H)$?

Theorem 4 of [8] implies $\tau^L(1, \mathbb{M}) = 5$, $\tau^L(2, \mathbb{M}) = 10$, and $\tau^L(3, \mathbb{M}) = 15$ for all $L > 6|\chi(\mathbb{M})|$. Here we prove:

THEOREM 8. *For any closed two-manifold* $\mathbb{M}$ *with Euler characteristic* $\chi(\mathbb{M}) \leq 0$, *any integer* $k \geq 1$, *and any integer* $L > (4k + 4)|\chi(\mathbb{M})|$ *it holds:*

(i) $\tau^L(1, \mathbb{M}) = 5$,
(ii) $\tau^L(2, \mathbb{M}) = 10$,
(iii) $\tau^L(k, \mathbb{M}) = 4k + 3$ for any $k \geq 3$.

2. PROOF OF THEOREMS 6–8; UPPER BOUNDS

In this section we shall prove Theorems 6(iii), 7(ii) and 8(iii). Let a polyhedral map G on a compact two-manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M})$ be a counterexample to Theorem 6(iii), or 7(ii), or 8(iii) on a minimum number of vertices, say n, and a maximum number of edges, say m, among all counterexamples on n vertices.

(A) If G is a counterexample of order $\geq k$ to Theorem 6(iii) then $\mathbb{M} = \mathbb{N}_1$ is the projective plane and each connected subgraph of G of order k contains a vertex of degree $\geq 4k + 4$, $k \geq 3$.

(B) If G is a counterexample of order $\geq k$ to Theorem 7(ii) then $\chi(\mathbb{M}) \leq 0$, the map G has an order $\geq N > (8k^2 + 6k - 8)|\chi(\mathbb{M})|$ and each connected subgraph of G of order k has a vertex of degree $\geq 4k + 5$, $k \geq 2$.

(C) If G is a counterexample to Theorem 8(iii) then $\chi(\mathbb{M}) \leq 0$, the map G satisfies the inequality $\sum_{j>4k+4}(j - (4k + 4))v_j \geq L > (4k + 4)|\chi(\mathbb{M})|$ and each connected subgraph of G of order k has a vertex of degree $\geq 4k + 4$, $k \geq 3$.

In the cases (A) and (C) a vertex A is a *minor* vertex if $\deg_G(A) \leq 4k + 3$ and is a *major* vertex if $\deg_G(A) \geq 4k + 4$, $k \geq 3$. In the case (B) a vertex A is a *minor* vertex if $\deg_G(A) \leq 4k + 4$ and is a *major* vertex if $\deg_G(A) \geq 4k + 5$, $k \geq 2$.

Now we shall investigate properties of G. The first property is easy to see.

PROPERTY 1. G is a polyhedral map on $\mathbb{M}$ such that each connected subgraph of order k in it contains a major vertex.

PROPERTY 2. G is a triangulation.

PROOF. Let G contain an r-face α, $r \geq 4$. If α is incident with a major vertex A we insert a diagonal AB into α where B is a vertex incident with α and not adjacent with A. Because the diagonal AB cannot create a minor subgraph of order at least k, we get a counterexample with $m+1$ edges, a contradiction. If α is incident only with minor vertices, all these vertices belong to the same minor component and we can again add a diagonal into α without loss of Property 1; a contradiction to the maximality of m, the number of edges of G. □

Let $H = H(G)$ and $H' = H'(G)$ be the subgraph of G induced on all major or minor vertices of G, respectively. Note that each component K of H' contains at most $k-1$ vertices. Our aim is to transform the triangulation G on a semitriangulation G^* with the same set of major vertices so that the degrees of the major vertices do not decrease, and the major vertices of G^* induce a semitriangulation H^* of $\mathbb{M}$. (A *semitriangulation* of a surface $\mathbb{M}$ is an embedding of a pseudograph (i.e., multiple edges and loops are allowed) in $\mathbb{M}$ in such a way that all faces of this embedding are triangles.) In such a way we obtain an upper bound for the degree sum $\sum_{A\in V(H)} \deg_G(A) \leq \sum_{A\in V(H)} \deg_{G^*}(A)$.

If H has at most two vertices then the three-connectedness of G implies that H' has precisely one component and G has at most $(k-1)+2 = k+1$ vertices. Hence each vertex has a degree $\leq k$, contradicting (A), (B), and (C).

Next let H have at least three vertices.

Let K be a component of H' which is joined by edges with the vertices $A_1, A_2, \ldots, A_s$, $s \geq 3$, which are major vertices of G. (Note $s \geq 3$ because G is a polyhedral map, i.e., G is three-connected.)

Case 1. Let $s \geq 4$. In the subgraph induced by $K \cup \{A_1, A_2, \ldots, A_s\}$ there exists obviously a tree D such that $A_1, A_2, \ldots, A_s$ are the only vertices of degree 1. This tree is embedded in $\mathbb{M}$. We form a (small) two-cell F on $\mathbb{M}$ along D, completely containing D and w.l.o.g. having the endvertices $A_1, A_2, \ldots, A_s$ on its boundary in this cyclic order.

We delete K together with all incident edges. Thus from D only the (major) vertices $A_1, A_2, \ldots, A_s$ remain. We form in F a *simple* s-cycle $[A_1, A_2, \ldots, A_s]$ by introducing new edges A_iA_{i+1}, $i = 1, 2, \ldots, s$, with $A_{s+1} \equiv A_1$. If the old graph already contains an A_iA_{i+1}-edge bounding a two-cell together with the new A_iA_{i+1}-edge, then delete the new A_iA_{i+1}-edge. So if a multiple edge occurs, no two edges of them bound a two-cell face of the graph induced by the major vertices. (Note: since G is three-connected and by the construction each component of the graph induced by the minor vertices is joined to at least three major vertices.) Next the interior of F is triangulated by the $s-3$ new edges A_1A_i, $i = 3, 4, \ldots, s-1$ (Figure 1).

The result of our construction is that locally a component K of H' is replaced by a triangulated two-cell face F of H. Thus H is locally triangulated. Globally, i.e., outside F, the embedding has no loops and no two-cell faces of size 2. But non-two-cell faces of size ≥ 2 may occur. The next step of our construction is to insert minor vertices into the triangles of F so that the degree requirements at $A_1, A_2, \ldots, A_s$ are again satisfied.

Since G is a polyhedral map each vertex of K is joined with A_i by at most one edge, and K is joined by at most $k-1$ edges with A_i, $i = 1, 2, \ldots, s$. We introduce in each triangle of F a path of length $k-1$ (the length of a path is its number of vertices). In the triangle $[A_1, A_2, A_3]$

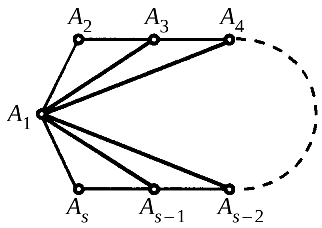

FIGURE 1.

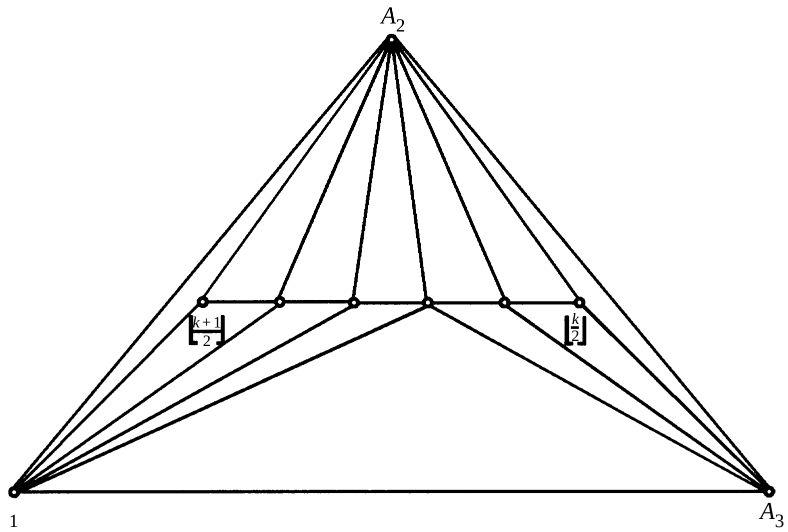

FIGURE 2.

we join all $k-1$ vertices of this path with A_2, $\lfloor \frac{k+1}{2} \rfloor$ vertices of it with A_1 and $\lfloor \frac{k}{2} \rfloor$ vertices with A_3 (for $k-1=6$ see Figure 2).

In $[A_1, A_i, A_{i+1}]$, $i = 3, 4, \ldots, s-1$, we join all $k-1$ vertices of the internal path with A_{i+1}, $\lfloor \frac{k+1}{2} \rfloor$ with A_i and $\lfloor \frac{k}{2} \rfloor$ with A_1. Thus A_i is joint with the inserted paths by at least $k-1$ edges. The vertex A_i has been adjacent to at most $k-1$ vertices of K. Hence the degree of A_i (and, consequently, of all major vertices) do not decrease. The interior of the two-cell face F has been triangulated so that the newly introduced paths are new components of H', i.e., all newly inserted vertices are minor vertices. The degree of no major vertex has been decreased, i.e., the set of major vertices remains unchanged (only new edges are added joining major vertices). Outside F, the embedding has no loops and no two-cell faces of size 2.

Case 2. Let $s = 3$. The number of edges joining K with A_i is denoted k_i, $i = 1, 2, 3$. Since G is polyhedral, $k_i \leq k-1$.

Case 2.1. Let $k_1 + k_2 + k_3 \leq 2k-1$. Enlarge k_1, k_2, k_3 so that $k_1 + k_2 + k_3 = 2k-1$ and $k_i \leq k-1$ for $i = 1, 2, 3$. As in Case 1, in $G - K$ a triangle $[A_1, A_2, A_3]$ is formed, and a tree with at most one vertex of degree 3 is placed so that k_i vertices of that tree can be joined with the vertex A_i, $i = 1, 2, 3$. We arrive at the same conclusion as in Case 1, where F is a triangle

Case 2.2. Let $k_1 + k_2 + k_3 > 2k - 1$. Then at least two vertices of K, say P_1 and P_2, are adjacent to all three vertices A_1, A_2, A_3. The star S_i formed by the vertex set $\{P_i, A_1, A_2, A_3\}$ and edge set $\{P_i A_1, P_i A_2, P_i A_3\}$ is embedded in $\mathbb{M}$. With the help of S_1 and S_2 two triangles F_1 and F_2 with vertex set $\{A_1, A_2, A_3\}$ are formed as before. In both F_1 and F_2 a path of length $k - 1$ is placed. In F_1 all $k - 1$ vertices of this path are joined with A_1 and $\lfloor \frac{k+1}{2} \rfloor$ are joined with A_2, and $\lfloor \frac{k}{2} \rfloor$ are joined with A_3. In F_2, all $k - 1$ vertices of this path are joined with A_2 and $\lfloor \frac{k+1}{2} \rfloor$ are joined with A_3 and $\lfloor \frac{k}{2} \rfloor$ are joined with A_1. So A_i, $i = 1, 2, 3$, is joined by at least $k - 1$ edges with vertices of $S_1 \cup S_2$. We arrive at the same conclusion as in Case 1, where F consists of two triangles F_1 and F_2.

The obtained embedding is denoted by $\widetilde{G}$, the subgraph of $\widetilde{G}$ induced by the major vertices is $\widetilde{H}$, where $V(\widetilde{H}) = V(H)$. Thus each component K of H' is replaced by some trees of order $k - 1$ each lying in a triangle of $\widetilde{H}$. By our construction the degrees of the vertices of $V(\widetilde{H}) = V(H^*)$ did not decrease. Perhaps $\widetilde{G}$ is no longer a two-cell embedding. We add successively a maximum number of edges so that the number of faces remains unchanged. (Here it is permitted to add loops and multiple edges). Thus a two-cell embedding is obtained. Each face with more than three vertices is triangulated. In each triangle bounded by three major vertices a path of length $k - 1$ is placed and joined with the vertices of the triangle as before. Thus a semitriangulation G^* of $\mathbb{M}$ is obtained, where the subgraph H^* induced by the major vertices is also a semitriangulation of $\mathbb{M}$. The semitriangulation H^* satisfies the equation

$$2e(H^*) = 3f(H^*),$$

and Euler's formula

$$n(H^*) - e(H^*) + f(H^*) = \chi(\mathbb{M}).$$

Hence

$$f(H^*) = 2(n(H^*) - \chi(\mathbb{M})), \tag{1}$$

and

$$e(H^*) = 3(n(H^*) - \chi(\mathbb{M})). \tag{2}$$

With (1) and (2) we have

$$\begin{aligned}\sum_{A \in V(H)} \deg_G(A) &\le \sum_{A \in V(H^*)} \deg_{G^*}(A) \\ &\le \sum_{A \in V(H^*)} \deg_{H^*}(A) + f(H^*)(2k - 1) \\ &= 2e(H^*) + (2k - 1) f(H^*) \\ \sum_{A \in V(H)} \deg_G(A) &\le (4k + 4)(n(H^*) - \chi(\mathbb{M})). \end{aligned} \tag{3}$$

If $\mathbb{M} = \mathbb{N}_1$ is the projective plane then $\chi(\mathbb{N}_1) = 1$ and (3) implies the existence of a major vertex B of degree $\deg_G(B) \le 4k + 3$. This contradiction completes the proof of Theorem 6(iii). (With $\chi(\mathbb{S}_0) = 2$ we also obtain a new proof of Theorem 5(iii).)

Next Theorem 8(iii) can be proved in the following way. The lower bound $L > (4k + 4)|\chi(\mathbb{M})|$, $\chi(\mathbb{M}) \le 0$, of Theorem 8(iii) implies

$$\sum_{A \in V(H)} (\deg_G(A) - (4k + 4)) = \sum_{j > 4k+4} (j - (4k + 4))v_j > (4k + 4)|\chi(\mathbb{M})|,$$

and

$$\sum_{A \in V(H)} \deg_G(A) > (4k + 4)(n(H^*) + |\chi(\mathbb{M})|),$$

contradicting (3). This contradiction completes the proof of Theorem 8(iii).

Next the proof of Theorem 7(ii) is continued.

Let H' and $H^{*'}$ denote the subgraph of G or G^* induced by the minor vertices of G or G^*, respectively. By the construction described above G, H, and H' are transformed into G^*, H^*, and $H^{*'}$, respectively, such that $n(G^*) \geq n(G)$, $n(H^{*'}) \geq n(H')$, and $n(H^*) = n(H)$, where even $V(H^*) = V(H)$. Since $H^{*'}$ has at most $f(H^*)$ components and each component has $\leq k-1$ vertices, (1) implies that the number $n(H^{*'})$ of vertices of $H^{*'}$ is

$$n(H^{*'}) \leq (k-1) f(H^*) = (2k-2)(n(H^*) + |\chi(\mathbb{M})|), \tag{4}$$

and the number $n(H^*)$ of the vertices of the triangulation H^* is

$$n(H^*) = n(G^*) - n(H^{*'}) \geq n(G^*) - (2k-2)(n(H^*) + |\chi(\mathbb{M})|).$$

Consequently,

$$n(H^*) \geq \frac{1}{2k-1}(n(G^*) - (2k-2)|\chi(\mathbb{M})|). \tag{5}$$

The lower bound $n(G^*) \geq n(G) \geq N > (8k^2 + 6k - 6)|\chi(\mathbb{M})|$ of the hypothesis implies

$$n(H^*) > (4k+4)|\chi(\mathbb{M})|. \tag{6}$$

Equations (3) and (6) imply: there is a vertex $B \in V(H)$ such that its degree

$$\begin{aligned} \deg_G(B) &\leq \frac{(4k+4)(n(H^*) + |\chi(\mathbb{M})|}{n(H^*)} \\ &= 4k + 4 + \frac{(4k+4)|\chi(\mathbb{M})|}{n(H^*)} \\ &< 4k + 4 + \frac{(4k+4)|\chi(\mathbb{M})|}{(4k+4)|\chi(\mathbb{M})|} = 4k + 5. \end{aligned} \tag{7}$$

Therefore, the degree of the major vertex B in G is $\leq 4k+4$. This contradiction proves the validity of Theorem 7(ii).

3. Proof of Theorem 7(ii)—Lower Bound

The main goal of this part is to prove that

$$\tau_N(k, \mathbb{M}) \geq 4k + 4, k \geq 2, \chi(\mathbb{M}) \leq 0,$$

that is to construct a large polyhedral map G on a compact two-manifold $\mathbb{M}$ with Euler characteristic $\chi(\mathbb{M}) \leq 0$ so that each connected subgraph of order k in it contains a vertex of degree at least $4k+4$. This construction is very similar to our construction presented in Section 4 of [9].

Let $P_n \times P_m$ be the Cartesian product of two paths of length n and m with vertex set $\{(i, j)|i, j \in \mathbb{Z}, 1 \leq i \leq n, 1 \leq j \leq m\}$ and edge set $\{((i, j), (i, j+1))|1 \leq i \leq n, 1 \leq j \leq m-1\} \cup \{((i, j), (i+1, j)|1 \leq i \leq n-1, 1 \leq j \leq n\}$. Add the edge set $\{((i, j), (i+1, j-1)|1 \leq i \leq n-1, 2 \leq j \leq m\}$ (see Figure 3).

Identifying opposite sides of the rectangle results in a toroidal map T_n, and reversing one side of this rectangle and then identifying opposite sides of this rectangle results in a map Q_n on the Klein bottle, respectively. Into each triangle D of the obtained triangulation we insert

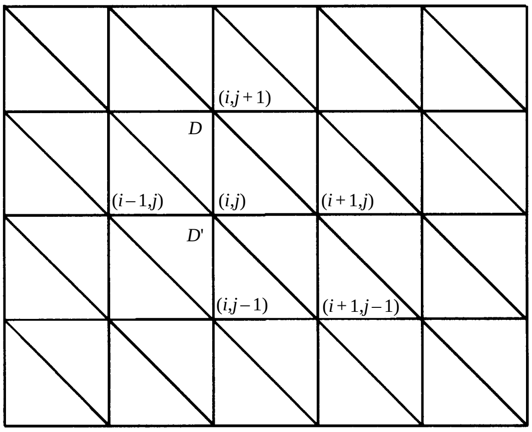

FIGURE 3.

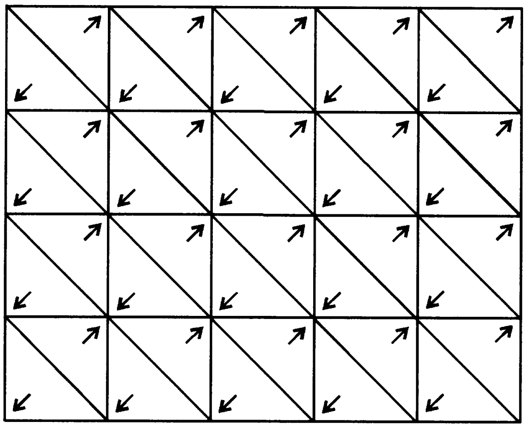

FIGURE 4.

a generalized three-star $S^{(k-1)}$ consisting of a central vertex Z and three paths starting in Z, one of length $\lfloor \frac{k+1}{3} \rfloor$, the second of length $\lfloor \frac{k+2}{3} \rfloor$, and the third of length $\lfloor \frac{k+3}{3} \rfloor$.

Let the paths be denoted by p_1, p_2, and p_3 so that p_1, p_2, p_3 are in this anticlockwise cycle order in D and p_1 and p_2 have the same length. If $D = ((i, j), (i+1, j), (i, j+1))$ then (i, j) is joined to all vertices of p_1 and p_2, $(i+1, j)$ is joined to all vertices of p_2 and p_3 and $(i, j+1)$ is joined to all vertices of p_3 and p_1. We do the same in $D' = ((i, j), (i-1, j), (i, j-1))$.

The situation is presented in Figure 4, where in each triangle Δ an arrow indicates which vertex of Δ is joined with two paths of equal length.

The introduced trees have $\lfloor \frac{k+1}{3} \rfloor + \lfloor \frac{k+2}{3} \rfloor + \lfloor \frac{k+3}{3} \rfloor - 2 = k - 1$ vertices, and the degree of the vertex (i, j) is

$$\begin{aligned} \deg(i, j) = 6 + 2&\left(\left\lfloor \frac{k+1}{3} \right\rfloor + \left\lfloor \frac{k+2}{3} \right\rfloor - 1 \right) \\ +2&\left(\left\lfloor \frac{k+2}{3} \right\rfloor + \left\lfloor \frac{k+3}{3} \right\rfloor - 1 \right) \end{aligned}$$

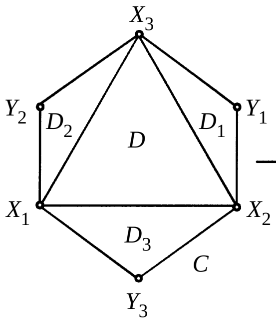

FIGURE 5.

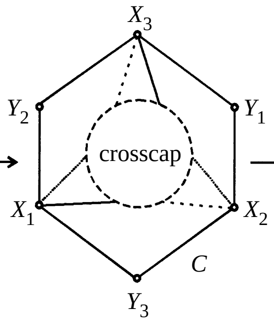

FIGURE 6.

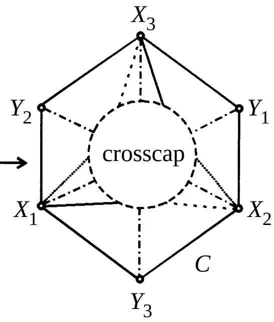

FIGURE 7.

$$+2\left(\left\lfloor\frac{k+3}{3}\right\rfloor+\left\lfloor\frac{k+1}{3}\right\rfloor-1\right)$$
$$=4\left(\left\lfloor\frac{k+1}{3}\right\rfloor+\left\lfloor\frac{k+2}{3}\right\rfloor+\left\lfloor\frac{k+3}{3}\right\rfloor\right)=4k+4.$$

Thus the construction results in a polyhedral triangulation T_n^* of the torus and in a polyhedral triangulation Q_n^* of the Klein bottle, both satisfying the degree requirements.

The required polyhedral map on an orientable compact two-manifold $\mathbb{S}_g$ of genus $g \geq 2$ will be constructed from the toroidal triangulation T_n^* with the underlying triangulation T_n. We choose $2g-2$ triangles of T_n so that any two of them have a distance ≥ 2 in T_n (this is possible if n is large enough). In T_n^* from each of these triangles Δ we delete the interior part so that the bounding three-cycle of Δ bounds now a hole of the torus. We join repeatedly two holes of T_n^* by a handle, and $g-1$ handles are added to the torus in this way.

The handles are triangulated in the following way: if $[X_1X_2X_3]$ and $[Y_1Y_2Y_3]$ are the bounding cycles of some handle which are around the handle in the same cyclic order then add the cycle $[X_1Y_1X_2Y_2X_3Y_3]$. In each of the new triangles a generalized three-star $S^{(k-1)}$ is placed in the same manner as before. The obtained polyhedral triangulation of $\mathbb{S}_g$ fulfils also the degree requirements.

The required polyhedral map on an unorientable compact two-manifold $\mathbb{N}_q$ of genus $q \geq 3$ will be constructed from the triangulation Q_n^* of the Klein bottle with underlying triangulation Q_n. We choose $q-2$ triangles of Q_n so that any two of them have a distance ≥ 4 in Q_n.

Let D be one of these triangles with bounding cycle $[X_1X_2X_3]$ and D_1, D_2, D_3 the three neighbouring triangles in Q_n with bounding cyles $[Y_1X_3X_2]$, $[Y_2X_1X_3]$, and $[Y_3X_2X_1]$ (see Figure 5–7). In Q_n^* we delete the inserted trees of D, D_1, D_2, D_3 and the separating edges X_1X_2, X_2X_3 and X_3X_1. A greater face F with bounding six-cycle $C=[X_1Y_3X_2Y_1X_3Y_2]$ is obtained (for the notation see Figure 6).

In F a crosscap is placed and the edges X_1X_2, X_2X_3, and X_3X_1 are again added so that the interior of C is subdivided into three quadrangles (see Figure 6). These quadrangles are subdivided by the edges X_iY_i, $i=1,2,3$ (see Figure 7). Finally, in each of the new triangles a generalized three-star $S^{(k-1)}$ is placed. The obtained polyhedral triangulation of $\mathbb{N}_q$ fulfils the degree requirements.

4. PROOF OF THEOREMS 6(iii) AND 8(iii)—LOWER BOUNDS

Let $\mathbb{M}$ be a compact two-manifold with Euler characteristic $\chi(\mathbb{M})$. Firstly we construct a polyhedral graph of the plane with degree sum $\sum_{j>4k+4}(j-(4k+4))v_j \geq L > (4k+$

$4)|\chi(\mathbb{M})|$, $k \geq 3$, so that each subgraph of order k contains a vertex of degree at least $4k + 3$, $k \geq 3$. This gives again good examples for the lower bound in Theorem 5(iii). Our method used here is very similar to the one used in [2]. We start with the Cartesian product $P_{n+1} \times P_m$ with $n > \frac{L}{4} + 2k + 2$ and $m \geq 8$ as described in Section 3. The opposite 'vertical sides' are identified, i.e., the two paths $(1, 1), (1, 2), \ldots, (1, m)$ and $(n+1, 1), (n+1, 2), \ldots, (n+1, m)$ are identified in the given order. A plane polyhedral graph is obtained which can be embedded in a closed finite cylinder so that the top face F_1 and the bottom face F_2 are the only faces of degree $n \geq \frac{L}{4} + 2k + 2$, all other faces are triangles.

In each of these triangles a generalized three-star with paths of lengths $\lfloor \frac{k+1}{3} \rfloor$, $\lfloor \frac{k+2}{3} \rfloor$ and $\lfloor \frac{k+3}{3} \rfloor$ is introduced as described in Section 3. In the resulting polyhedral plane graph the vertices (i, j) have degree $4k + 4$ for $1 \leq i \leq n$ and $2 \leq j \leq m - 1$, and the vertices $(i, 1)$ and (i, m) have degree $2k + 3$ for $1 \leq i \leq n$. Next the degrees of the vertices of degree $2k + 3$ bounding the top face F_1 and the bottom F_2 are increased by 1 and the degrees of their inner neighbours of degree $4k + 4$ are decreased by 1. Then these vertices have degrees $2k + 4$ and $4k + 3$, respectively.

However, we will describe this process in more detail. If $k \equiv 0(3)$ then the number $\lfloor \frac{2k}{3} \rfloor - 1$ of edges joining the inner tree D of the triangle $\Delta((i, m), (i+1, m-1), (i+1, m))$, $1 \leq i \leq n$, with the vertex $(i + 1, m)$ is smaller than the number $\lfloor \frac{2k}{3} \rfloor$ of edges joining this tree with each of the two other vertices. Rearrange the edges so that D is now joined with $(i + 1, m - 1)$ by $\lfloor \frac{2k}{3} \rfloor - 1$ edges and with the other two vertices of Δ by $\lfloor \frac{2k}{3} \rfloor$ edges (i.e., now the arrow of this triangle points to $(i + 1, m - 1)$). Do the same for the triangles $\Delta((i, 2), (i, 1), (i + 1, 1))$.

If $k \equiv 1(3)$ then the number $\lfloor \frac{2k}{3} \rfloor + 1$ of edges joining the inner tree D of the triangle $\Delta((i, m), (i, m - 1), (i + 1, m - 1))$, $1 \leq i \leq n$, with the vertex $(i, m - 1)$ is larger than the number $\lfloor \frac{2k}{3} \rfloor$ of edge joining D with each of the other vertices. Rearrange the edges so that D is now joined with (i, m) by $\lfloor \frac{2k}{3} \rfloor + 1$ edges and with the other two vertices of Δ by $\lfloor \frac{2k}{3} \rfloor$ edges (i.e., now the arrow of this triangle points to (i, m)). Do the same for the triangle $\Delta((i, 2), (i, 1), (i + 1, 1))$.

If $k \equiv 2(3)$ then each vertex of the triangle $\Delta((i, m), (i, m - 1), (i + 1, m - 1))$, $1 \leq i \leq n$; is joint with D by the same number of edges $\lfloor \frac{2k}{3} \rfloor$. Then replace D by a tree D^* so that D^* is joined with (i, m), $(i, m - 1)$, and $(i + 1, m - 1)$ by $\lfloor \frac{2k}{3} \rfloor + 1$, $\lfloor \frac{2k}{3} \rfloor$, and $\lfloor \frac{2k}{3} \rfloor - 1$ edges.

Thus, the vertices bounding the top face F_1 and the bottom face F_2 of the cylinder have degree $2k + 4$. In order to complete our construction we put into F_i a new vertex X_i and join X_i with all bounding vertices of F_i, $i = 1, 2$. In each new triangle Δ a path p of length $k - 1$ is introduced. One endvertex of p is joined with each of the two remaining vertices of Δ. In the obtained triangulation the vertices bounding F_i have degree $2k+4+1+2(k-2) = 4k+3$ and X_i has degree $\deg X_i \geq n \geq \frac{L}{2} + 2k + 2$. Thus examples of the lower bound in Theorem 4(ii) are obtained.

Next the wanted polyhedral maps of $\mathbb{M}$ will be constructed. If $\mathbb{M}$ is an orientable two-manifold $\mathbb{S}_g$ of genus g then g handles have to be added.

If $\mathbb{M}$ is a nonorientable two-manifold $\mathbb{N}_q$ of genus q then q crosscaps have to be added. This is accomplished in the same way as in Section 3; the required triangles are chosen only among the triangles between the cycles $(1, 4), (2, 4), \ldots, (n, 4)$ and $(1, 5), (2, 5), \ldots, (n, 5)$. This implies that all triangles involved into this construction are incident only with vertices of degree $4k+4$. Hence adding g handles or q crosscap causes, according to Section 3, no problems.

ACKNOWLEDGEMENT

Support from the Slovak VEGA grant 1/4377/97 is acknowledged.

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Received 10 April 1998 and accepted in revised form 21 July 1999

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