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Height counting of unlabeled interval and N-free posets

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Abstract

This paper enumerates according to height the classes of unlabeled N-free posets, interval orders, and posets that are both N-free and interval orders. The last two classes are enumerated according to height in terms of generating functions. We apply an algorithmic method for height counting of connected N-free posets. Numerical results for n-element posets of height k, $1 \leqslant k \leqslant n \leqslant 14$, are included.

MSC: 06A07; 05C30

Keywords: Counting; Enumeration; Interval order; N-free; Height

1. Introduction

The enumeration of various classes of posets is an interesting combinatorial problem for which several techniques have been applied. Numerous results ranging from exact and algorithmic counting to asymptotic estimates appear in the literature, see [1–5,10,12,13]. A related interesting problem is to count classes of posets according to the height of poset. Few studies are present in this field. For example, the generating function for height counting of unlabeled series–parallel posets have been derived in [6]. In the same paper, the authors gave a general technique for height counting of a class of posets closed with respect to series and parallel compositions provided that the height counting of irreducible posets in the class is known. This technique was applied in [11] to obtain the number of unlabeled prime, UPO, and general posets on n elements with height k, for $1 \leqslant k \leqslant n \leqslant 12$.

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In this paper, we consider the classes of N-free posets, interval orders, and posets that are N-free and interval orders at the same time. The last two classes are enumerated according to height in terms of generating functions. We apply an algorithmic method for height counting of connected N-free posets. A technique similar to that in [6] is derived. Then it is applied to get the results for N-free posets.

The paper consists of five sections. In Section 2, we give the basic definitions. Section 3 contains the height counting of unlabeled interval orders according to height. Sections 4 and 5 deal with unlabeled N-free interval posets and N-free posets, respectively. Finally, the appendix contains numerical results for posets in the above classes with n elements and height k, for $1 \leqslant k \leqslant n \leqslant 14$.

2. Basic definitions

Let $P=(V,<)$ be a poset where V is a finite non-empty set and $<$ is a partial order defined on V. A subset X of P is called a *chain* if for every $u,v \in X$ either $u<v$ or $v<u$. While X is called an *antichain* if for every $u,v \in X$ neither $u<v$ nor $v<u$. The *height* of an element $u \in P$, denoted by $h(u)$, is the maximum cardinality of a chain in P having u as its maximum element. The *height* of P is defined as $h(P)=\max\{h(u):\ u \in P\}$.

The poset P is said to be an *interval order* if each element $v \in P$ can be represented by an interval I_v of the real line such that $v<w$ if and only if I_v lies entirely to the left of I_w. It is known [7] that P is an interval order if and only if P does not contain two parallel edges, i.e., an induced subposet of four elements a,b,c,d with $a<c$ and $b<d$ (the only comparabilities), see Fig. 1(a). Another characterization of interval orders is as follows: For $u \in P$, let $D(u)=\{v \in P:\ v<u\}$ be the set of predecessors of u. Then P is an interval poset if and only if the sets of predecessors of the elements of P are linearly ordered by inclusion.

A poset P is called *N-free* if its directed covering graph has no induced subgraph isomorphic to the digraph $\boldsymbol{N}$ shown in Fig. 1(b).

There are two useful representations of an N-free poset, namely the block- and the matrix-representation, see [1,14]. Assume that P is N-free. By a *block* of P we mean a

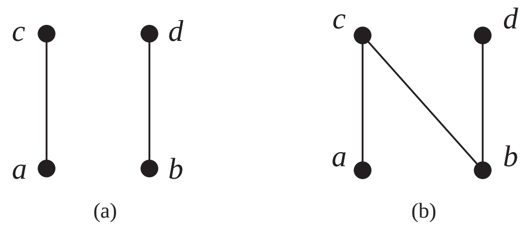
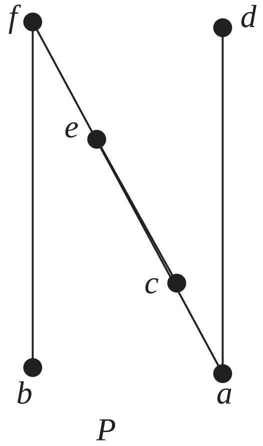


Fig. 1.



{a,b}	{c,d}	{e}	{f}	∅
∅	{a}	{c}	{b,e}	{d,f}

Fig. 2. An N-free poset P and its block-representation.

maximal complete bipartite graph in the directed covering graph of P. More precisely,a block of P is a pair (A,B), where $A,B \subseteq P$ are such that A is the set of all upper covers of every $v \in B$ and B is the set of all lower covers of every $u \in A$. By convention, $(\mathrm{Min}\, P, \emptyset)$ and $(\emptyset, \mathrm{Max}\, P)$ also are blocks where $\mathrm{Min}\, P$ and $\mathrm{Max}\, P$ are, respectively, the sets of minimal and maximal elements of P. The existence of blocks in an N-free poset is guaranteed by the well-known fact that for any two elements u, v of such a poset, the sets of lower covers (and the sets of upper covers) of u and v are either disjoint or identical. On the other hand, a poset which is not N-free might not contain proper blocks at all, e.g., the poset N in Fig. 1(b).

Let $(A_1,B_1),\dots,(A_k,B_k)$ be all the blocks of P. Then, the sets A_i's form a partition of P and so do the B_i's. We shall always assume that the blocks of P are ordered such that for any $v \in P$, if $v \in A_i$ and $v \in B_j$ then $i < j$. The *block-representation* of P is a $2 \times k$ matrix with the A_i's in its first row and the B_i's in the second row ordered as above. This is illustrated in Fig. 2. Clearly, every N-free poset has a block-representation that is unique up to a possible permutation of its columns.

The *matrix-representation* of P is the $k \times k$ matrix $M(P) = [m_{ij}]$, where $m_{ij} = |A_i \cap B_j|$. The matrix $M(P)$ is unique up to a possible permutation σ applied simultaneously to its rows and columns. The above prescribed order of the blocks implies that $m_{ij} = 0$ whenever $i \geqslant j$, thus $M(P)$ is a strictly upper triangular matrix.

In [1], it was proved that an N-free poset P is also an interval order if and only if $M(P)$ has no zeros on the super diagonal, i.e., $m_{i,i+1} \neq 0$ for each $i = 1,\dots,k-1$. In this case the matrix $M(P)$ is unique. Fig. 2 shows an N-free interval poset whose matrix-representation is given in Fig. 3.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

M (P)

Fig. 3. The matrix representation of P.

3. Height counting of interval orders

In [5], El-Zahar enumerated labeled and unlabeled interval orders. His method followed Hanlon's technique for counting labeled and unlabeled interval graphs [8]. Fortunately, the generating functions derived in [5] can easily be modified to account for height. We shall here introduce this modification for counting unlabeled interval orders according to height. Referring to [5], we recall and introduce the following definitions.

An interval poset P is called *reduced* if no two maximal elements of P have the same set of predecessors. A maximal element $v \in P$ is called a *chief* element if $h(v) = h(P)$, while if $h(v) < h(P)$ then v is called an *assistant* element. Let P be a reduced interval poset with height k and having n non-maximal elements, r assistant elements, and s chief elements. We give P the weight $y^n z^r w^s h^k$. We define the generating function

$$G(y,z,w,h) = \sum_{\substack{n,r \geqslant 0 \\ s,k \geqslant 1}} g_{nrsk}\, y^n z^r w^s h^k,$$

where g_{nrsk} denotes the number of unlabeled reduced interval posets with weight $y^n z^r w^s h^k$. Now, we describe how reduced interval posets are built from smaller ones. For each reduced interval poset P, let the *leader* element, $l(P)$, denote the unique maximal element $u \in P$ whose set of predecessors, $D(u)$, is maximum. In other words, $l(P)$ is the unique element of P which is larger than all its non-maximal elements. Obviously, $h(l(P)) = h(P)$. The predecessor of P is defined to be the reduced interval poset obtained from P by deleting $l(P)$ and identifying all pairs u_1, u_2 of maximal elements in $P - l(P)$ having $D(u_1) = D(u_2)$. Conversely, let Q be a reduced interval poset with $h(Q) = k$. To obtain all reduced interval posets P having Q as their predecessor, we proceed as follows:

1. We add a new element $l(P)$ larger than each non-maximal element of Q; $l(P)$ will have weight w.
2. All non-maximal elements of Q will remain non-maximal in P, i.e., keep their weights.
3. For an assistant element v of Q with weight z, there are three possibilities:
   (i) v remains assistant with weight z,

(ii) v splits into non-maximal elements $v_1,\dots,v_t$, $t \geqslant 1$, where $v_i < l(P)$ for each $1 \leqslant i \leqslant t$; thus the weight z of v is replaced by y^t,

(iii) v splits into $v_0, v_1,\dots,v_t$, $t \geqslant 1$, where v_0 remains assistant and $v_i < l(P)$ for $1 \leqslant i \leqslant t$; thus the weight z of v is replaced by zy^t.

To account for all possibilities, we replace the weight z of v by

$$z + \sum_{t\geqslant 1} y^t + z\sum_{t\geqslant 1} y^t = \frac{z+y}{1-y}.$$

4. For a chief element v of Q with weight w, there are two subcases.
   (a) $h(P) = h(Q)$. In this subcase, the only possibility for v is to remain a chief element without splitting; thus keeping its weight.
   (b) $h(P) = h(Q) + 1$. Here there are three possibilities:
      (i) v remains a maximal element. Since $h(v) < h(P)$, then v becomes an assistant element and its weight is replaced by z,
      (ii) v splits into non-maximal elements $v_1,\dots,v_t$ thus its weight w is replaced by y^t,
      (iii) v splits into $v_0, v_1,\dots,v_t$, $t \geqslant 1$, where v_0 is an assistant maximal element and $v_i < l(P)$ for each $1 \leqslant i \leqslant t$. Thus the weight w of v is replaced by zy^t.

   In conclusion, the weight w of v is not changed if $h(P) = h(Q)$ and is replaced by

$$z + \sum_{t\geqslant 1} y^t + z\sum_{t\geqslant 1} y^t = \frac{z+y}{1-y} \quad \text{if } h(P) = h(Q) + 1.$$

We note that $h(P)=h(Q)+1$ if and only if $l(P)$ covers at least either a chief element v of Q or a non-maximal element arising from the splitting of a chief element. Otherwise, $h(P) = h(Q)$.

Assume now that Q has weight $y^n z^r w^s h^k$. Let $E_1(y,z,w,h)$ and $E_2(y,z,w,h)$ denote, respectively, the weight enumerators of reduced interval posets having Q as their predecessor and of height, respectively, $k+1$ and k. The following two lemmas calculate these two enumerators.

Lemma 3.1. $E_1(y,z,w,h) = y^n((z+y)/(1-y))^r(((z+y)/(1-y))^s - z^s)wh^{k+1}$.

Proof. Let P be a reduced interval poset with height $k+1$ and having Q as its predecessor. Since $h(P) > h(Q)$, then the only chief element of P is $l(P)$. Furthermore, not all chief elements of Q remain maximal without splitting in P, since otherwise we would have $h(P)=h(Q)$. This explains the subtraction of the term z^s in the substitution for w^s. The substitutions for y^n and z^r follow respectively from (2) and (3) above. □

Lemma 3.2. $E_2(y,z,w,h) = y^n(((z+y)/(1-y))^r - z^r)w^{s+1}h^k$.

Proof. Suppose that P is a reduced interval poset with height k and having Q as its predecessor. Since P is reduced, then not all assistant elements of Q remain assistant without splitting in P, since otherwise, $l(P)$ and $l(Q)$ would have the same set of predecessors. Therefore, the term z^r is replaced by $((z+y)/(1-y))^r - z^r$. The remaining terms are straightforward. □

As a result of the preceding two lemmas, $G(y,z,w,h)$ satisfies the following equation.

Theorem 3.3.

$$G(y,z,w,h) = wh + wh\left(G\left(y,\frac{z+y}{1-y},\frac{z+y}{1-y},h\right) - G\left(y,\frac{z+y}{1-y},z,h\right)\right)$$
$$+ w\left(G\left(y,\frac{z+y}{1-y},w,h\right) - G(y,z,w,h)\right). \quad (1)$$

Proof. The term wh on the right-hand side accounts for the single-element poset which has no predecessor. Every other reduced interval poset has a unique predecessor and therefore its weight appears exactly once in $\Sigma(E_1(y,z,w,h) + E_2(y,z,w,h))$, where the summation is taken for all $n,r \geqslant 0$ and $s,k \geqslant 1$. The required result now follows. □

Let $G^*(x,h) = \sum_{1 \leqslant k \leqslant n} g^*_{nk} x^n h^k$, where g^*_{nk} denotes the number of unlabeled interval posets having n elements and height k. Each interval poset P is obtained from a unique reduced interval poset P' by replacing some maximal elements of P by antichains and this operation does not increase its height. The generating function of all antichains is $x/(1-x)$.Therefore, we have

Theorem 3.4.

$$G^*(x,h) = G\left(x,\frac{x}{1-x},\frac{x}{1-x},h\right). \quad (2)$$

Eq. (1) can be used to recursively calculate the coefficients g_{nrsk} and from (2) we can then calculate g^*_{nk}. We obtain

$$G(y,z,w,h) = wh + ywh^2 + (yzw + y^2w)h^2 + y^2wh^3 + (y^2zw + y^3w + y^2w^2)h^2$$
$$+ (3y^3w + 2y^2zw)h^3 + y^3wh^4 + \cdots,$$

$$G^*(x,h) = xh + x^2(h+h^2) + x^3(h+3h^2+h^3) + x^4(h+7h^2+6h^3+h^4) + \cdots.$$

The values of g^*_{nk} for $1 \leqslant k \leqslant n \leqslant 14$ are included in Table 1 of the appendix.

4. Height counting of N-free interval posets

In this section, we consider unlabeled posets which are simultaneously N-free and interval order. This class of posets was first introduced in [1] in order to prove that almost all N-free posets are not series–parallel (posets obtained from the single-element poset by series and parallel compositions). As proved in [1], a poset is N-free interval order if and only if its matrix-representation has no zeros on the super diagonal. Therefore, there exists a one-to-one correspondence between N-free interval posets and their matrix-representations. Furthermore, an N-free interval poset is rigid, i.e., has no non-trivial automorphism, if and only if its matrix-representation has no element larger than one, see [1]. In Fig. 2, we give a rigid N-free interval poset whose matrix-representation is illustrated in Fig. 3.

To enumerate unlabeled rigid N-free interval posets according to height, we introduce the following generating function. Let $J(x,h)=\sum_{n\geqslant 1}\sum_{k=1}^{n} j_{nk}x^n h^k$, where j_{nk} denotes the number of unlabeled rigid n-element N-free interval posets with height k. The numbers j_{nk} are given by:

Lemma 4.1.

$$j_{nk}=\binom{k(k-1)/2}{n-k}. \tag{3}$$

Proof. Let P be a rigid N-free interval poset with n elements and $k+1$ blocks. Then, $M(P)=[m_{ij}]$ is a unique 0–1 matrix of order $k+1$ in which all the $(i,i+1)$ entries are 1's. This implies that $|A_i\cap B_{i+1}|=1$ for each $i=1,\dots,k$. Therefore, the ith block must precede the $(i+1)$th block, $i=1,\dots,k$, in any block representation of P. Let v_i be the unique element of $A_i\cap B_{i+1}$, $i=1,\dots,k$. Then $v_1<v_2<\cdots<v_k$ is a unique maximum chain of length $k-1$ in P and so the height of P equals k.

Now the matrix-representations of P, which is of order $k+1$, has zero's on and below its diagonal, its super diagonal consists of exactly k one's and the remaining elements are either 0 or 1. Note that there are exactly n non-zero entries since the poset has n elements. Thus there are $\binom{k(k-1)/2}{n-k}$ ways to choose the non-zero elements above the super diagonal. Therefore, $j_{nk}=\binom{k(k-1)/2}{n-k}$ which completes the proof of the lemma. □

Let $J^*(x,h)=\sum_{n\geqslant 1}\sum_{k=1}^{n} j^*_{nk}x^n h^k$ where j^*_{nk} denotes the number of unlabeled n-element N-free interval posets with height k. An N-free interval poset is obtained from a rigid one by substituting antichains for some of its elements. Therefore, we have

Theorem 4.2.

$$J^*(x,h)=J\left(\frac{x}{1-x},h\right). \tag{4}$$

Eqs. (3) and (4) can be used to recursively calculate j^*_{nk}. The numerical results for $1\leqslant k\leqslant n\leqslant 14$ appear in Table 2 of the appendix.

5. Height counting of N-free posets

The enumeration of N-free posets according to height is achieved through an algorithmic method. We designed an algorithm to generate matrix-representations of n-element N-free posets in a certain order and count the corresponding posets up to isomorphism according to height. To reduce the running time, we counted only those matrices that represent connected N-free posets. We shall not give the details of the algorithm here but the result is that we get v_{nk}, the number of connected n-element unlabeled N-free posets with height k. To get the number, f_{nk}, of unlabeled N-free posets having n elements and height k, we derive the relations between v_{nk} and f_{nk}. Actually, these relations are similar to those given in [6]. Define the generating functions:

$$V(x,h)=\sum_{n=1}^{\infty}\sum_{k=1}^{n} v_{nk}x^n h^k=\sum_{k=1}^{\infty} V_k(x)h^k$$

and

$$F(x,h)=\sum_{n=1}^{\infty}\sum_{k=1}^{n} f_{nk}x^n h^k=\sum_{k=1}^{\infty} F_k(x)h^k.$$

Let e_{nk} denote the number of n-element unlabeled N-free posets with the property that each component of which has height k. Define the generating function:

$$E(x,h)=\sum_{n=1}^{\infty}\sum_{k=1}^{n} e_{nk}x^n h^k=\sum_{k=1}^{\infty} E_k(x)h^k.$$

As a direct consequence of Riddell's Theorem [9, p. 90], we get

Lemma 5.1. $1+E_k(x)=\exp\sum_{i=1}^{\infty}(V_k(x^i))/i$.

The following theorem calculates $F_k(x)$ in terms of $E_k(x)$.

Theorem 5.2. $F_k(x)=(1+\sum_{j=1}^{k-1}F_j(x))E_k(x)$.

Proof. Let P be an N-free poset of height k. Then P can be uniquely written as the parallel composition $P=Q\cup P_k$ where P_k is an N-free poset each component of which has height k and Q is a (possibly empty) N-free poset of height j for some $1\leqslant j<k$. Now, the term $1+\sum_{j=1}^{k-1}F_j(x)$ counts the posets Q (including the empty one) in this representation, while the posets P_k are counted by $E_k(x)$. This implies the required result. □

Finally, we outline the procedure for calculating the numbers f_{nk}, $k\leqslant n$. As we mentioned earlier, a computer program were used to calculate the coefficients v_{nk}. We then apply Lemma 5.1 to obtain e_{nk}, $k\leqslant n$. Using Theorem 5.2, we can recursively calculate f_{nk}. The numerical results for $v_{nk}$and f_{nk}, $1\leqslant k\leqslant n\leqslant 14$ are given respectively in Tables 3 and 4 in the following appendix.

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Appendix

Tables 1–4 give the number of n-element unlabeled posets of height $k(1 \leqslant k \leqslant n \leqslant 14)$, which are, respectively, interval order, N-free interval order, connected N-free, and N-free.

Table 1
The number of n-element unlabeled interval orders of height k, $1 \leqslant k \leqslant n \leqslant 14$

	n													
k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	3	7	15	31	63	127	255	511	1023	2047	4095	8191
3			1	6	26	100	366	1317	4743	17275	64029	242371	938741	3723210
4				1	10	69	412	2305	12551	67933	370168	2046980	11546918	66665327
5					1	15	150	1270	9920	74525	551232	4072130	30322587	228997375
6						1	21	286	3236	33301	325860	3109628	29395997	278111527
7							1	28	497	7210	93926	1151416	13644127	158939927
8								1	36	806	14540	232891	3477454	49791316
9									1	45	1239	27147	522840	9308502
10										1	55	1825	47665	1084540
11											1	66	2596	79596
12												1	78	3587
13													1	91
14														1
Total	1	2	5	15	53	217	1014	5335	31240	201608	1422074	10886503	89903100	796713191

Table 2
The number of unlabeled n-element N-free interval posets of height k, $1 \leqslant k \leqslant n \leqslant 14$

	n													
k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	3	6	10	15	21	28	36	45	55	66	78	91
3			1	6	21	56	126	252	462	792	1287	2002	3003	4368
4				1	10	55	220	715	2002	5005	11440	24310	48620	92378
5					1	15	120	680	3060	11628	38760	116280	319770	817190
6						1	21	231	1771	10626	53130	230230	888030	3108105
7							1	28	406	4060	31465	201376	1107568	5379616
8								1	36	666	8436	82251	658008	4496388
9									1	45	1035	16215	194580	1906884
10										1	55	1540	29260	424270
11											1	66	2211	50116
12												1	78	3081
13													1	91
14														1
Total	1	2	5	14	43	143	510	1936	7775	32869	145665	674338	3251208	16282580

Table 3
The number of n-element unlabeled connected N-free posets of height k, $1 \leqslant k \leqslant n \leqslant 14$

	n													
k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1													
2		1	2	3	4	5	6	7	8	9	10	11	12	13
3			1	5	17	45	115	278	679	1666	4167	10591	27452	72301
4				1	9	50	218	851	3161	11507	41837	153158	567427	2131683
5					1	14	114	709	3818	19042	91383	431375	2029244	9583860
6						1	20	224	1867	13113	83222	497913	2883579	16436980
7							1	27	398	4276	37898	297293	2157924	14923081
8								1	35	657	8845	96614	918526	7952292
9									1	44	1025	16913	223496	2536157
10										1	54	1529	30369	478118
11											1	65	2199	51787
12												1	77	3068
13													1	90
14														1
Total	1	1	3	9	31	115	474	2097	9967	50315	268442	1505463	8840306	54169431

Table 4
The number of n-element unlabeled N-free posets of height k, $1 \leqslant k \leqslant n \leqslant 14$

	n													
k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	3	7	13	25	43	76	128	216	354	583	937	1505
3			1	6	24	77	228	644	1776	4854	13184	35819	97408	265845
4				1	10	61	291	1229	4872	18711	70858	267337	1010627	3842536
5					1	15	130	856	4840	25107	124167	599133	2860982	13639325
6						1	21	246	2136	15543	101538	621216	3656776	21077891
7							1	28	427	4733	43120	346187	2559866	17954298
8								1	36	694	9577	106963	1036689	9120021
9									1	45	1071	18031	242694	2799313
10										1	55	1585	32011	511830
11											1	66	2266	54121
12												1	78	3147
13													1	91
14														1
Total	1	2	5	15	49	180	715	3081	14217	69905	363926	1996922	11500336	69269925

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