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Compact Sobolev Imbeddings on Finite Measure Spaces

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We provide conditions on a finite measure μ on $\mathbb{R}^n$ which insure that the imbeddings $W^{k,p}(\mathbb{R}^n, d\mu) \hookrightarrow L^p(\mathbb{R}^n, d\mu)$ are compact, where $1 \leqslant p < \infty$ and k is a positive integer. The conditions involve uniform decay of the measure μ for large $|x|$ and are satisfied, for example, by $d\mu = e^{-|x|^\alpha}\, dx$, where $\alpha > 1$.

1. INTRODUCTION

For Gauss measure, $d\nu = (2\pi)^{-n/2}\, e^{-x^2/2}\, dx$ for $x \in \mathbb{R}^n$, L. Gross showed in [7] that the Sobolev space $W^{1,2}(\mathbb{R}^n, d\nu)$ is imbedded in the Orlicz space $L^2 \ln L(\mathbb{R}^n, d\nu)$. This is significant in light of the fact that, for a finite measure μ with an unbounded support set Γ contained in $\mathbb{R}^n$, there is no imbedding of the form $W^{k,p}(\Gamma, d\mu) \hookrightarrow L^q(\Gamma, d\mu)$ if q is greater than p. The imbedding $W^{1,2}(\mathbb{R}^n, d\mu) \hookrightarrow L^2 \ln L(\mathbb{R}^n, d\mu)$ has been shown to hold for finite measures other than Gauss measure (see, e.g., [4, 5, 8]). In each case, it is necessary that the measure μ decay in some uniform sense like Gauss measure. Stronger Sobolev–Orlicz space imbeddings have been exhibited for measures which satisfy more restrictive decay conditions (see [1, 11]).

There is hence a good indication that a relationship exists between decay properties of a finite measure μ on $\mathbb{R}^n$ and the existence of Sobolev–Orlicz space imbeddings of the form $W^{k,p}(\mathbb{R}^n, d\mu) \hookrightarrow L_X(\mathbb{R}^n, d\mu)$, where X is a Young's function. As a first step in establishing such a relationship, this article provides a link between decay properties of a measure μ and compactness of the imbeddings

$$W^{k,p}(\mathbb{R}^n, d\mu) \hookrightarrow L^p(\mathbb{R}^n, d\mu), \tag{1}$$

where k is a positive integer and $1 \leqslant p < \infty$.

As an indication of the importance of determining compactness of these imbeddings, suppose that $X(t)$ is a Young's function which increases essentially faster than t^p. We will see in Section 2 that the imbedding

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$W^{k,p}(\mathbb{R}^n, d\mu) \hookrightarrow L_X(\mathbb{R}^n, d\mu)$ does not exist unless the imbedding (1) is compact.

In Theorem 3.1, the main result of this paper, we provide conditions on a measure μ which imply compactness of the imbeddings (1). The conditions are satisfied, for example, by measures which decay in a uniform sense like $e^{-|x|^\alpha}$ for some $\alpha > 1$. Theorem 3.2 gives conditions which are necessary in order that the imbeddings (1) be compact. For example, if a measure decays as slowly as $e^{-\alpha|x|}\,dx$ for some $\alpha > 0$, then the imbeddings (1) are not compact.

The imbedding theory presented in this paper parallels a part of the classical imbedding theory for domains $\Gamma \subset \mathbb{R}^n$ equipped with Lebesgue measure. In many cases, we have been able to make use of this classical theory, and we frequently refer to the notable monograph on Sobolev spaces by R. A. Adams [2].

Finally, in Section 4, we show how our compact imbedding theory can be used to determine whether an elliptic differential operator on a finite measure space has a complete orthogonal system of eigenfunctions.

2. Preliminary Results

For $r > 0$, we let $B_r = \{x \in \mathbb{R}^n : |x| \leqslant r\}$ and $D_r = \overline{\mathbb{R}^n \setminus B_r} = \{x \in \mathbb{R}^n : |x| \geqslant r\}$. We suppose that Γ is an unbounded domain in $\mathbb{R}^n$ such that for all $r > 0$, $\Gamma \cap B_r$ and $\Gamma \cap D_r$ satisfy the cone property and the segment property (see, e.g., [2] for definitions of the cone and segment properties).

Let $d\mu = \rho(x)\,dx$ be a positive finite measure on Γ such that $\rho \in L^\infty(\Gamma)$ and ρ is locally bounded away from zero on $\bar{\Gamma}$, i.e., for any compact set $K \subset \bar{\Gamma}$, $\rho(x) \geqslant \varepsilon_K > 0$ a.e. on K. It will be convenient to consider μ as a measure on all of $\mathbb{R}^n$ by defining $\mu(U) = \mu(U \cap \Gamma)$ for all Borel subsets U of $\mathbb{R}^n$.

For any domain $\Omega \subset \mathbb{R}^n$, we denote by $C_0^\infty(\Omega)$ the set of all infinitely differentiable functions with compact support contained in Ω, and by $C_0^\infty(\bar{\Omega})$ the set of all functions $\psi\chi_\Omega$, where $\psi \in C_0^\infty(\mathbb{R}^n)$ and χ_Ω is the characteristic function of the set Ω.

For a non-negative integer k and for $1 \leqslant p < \infty$, the Sobolev space $W^{k,p}(\Omega, d\mu)$ is the set of all functions whose distributional derivatives of order less than or equal to k are in $L^p(\Omega, d\mu)$. $W^{k,p}(\Omega, d\mu)$ is a Banach space with respect to the norm

$$\|u\|_{k,p,\Omega} = \sum_{|\alpha| \leqslant k} (\|D^\alpha u\|_{p,\Omega}^p)^{1/p},$$

where $\|\cdot\|_{0,p,\Omega} = \|\cdot\|_{p,\Omega}$ is the usual L^p norm on Ω with respect to μ.

Remark. The fact that $\rho(x)$ is locally bounded away from zero on $\bar{\Gamma}$ guarantees that all functions in $L^p(\Gamma, d\mu)$ are locally integrable, and hence have distributional derivatives. If $\Omega \cap \Gamma$ satisfies the segment property, then $W^{k,p}(\Omega, d\mu)$ can equivalently be defined as the closure of $C_0^\infty(\bar{\Omega})$ with respect to the norm $\|\cdot\|_{k,p,\Omega}$ (this follows, for example, from Theorem 3.18 in [2]).

We state now two compactness results which will be used in the following section.

PROPOSITION 2.1. *Suppose that K is a subset of $L^q(\Gamma, d\mu)$, $1 \leqslant q < \infty$, such that*

(i) *for each $m \in \mathbb{Z}^+$, the set of restrictions of functions in K to $\Gamma \cap B_m$ is a precompact subset of $L^q(\Gamma \cap B_m, d\mu)$;*

(ii) *given $\varepsilon > 0$, there exists $m \in \mathbb{Z}^+$ such that*

$$\int_{\Gamma \cap D_m} |u|^q \, d\mu < \varepsilon \qquad \text{for all} \quad u \in K.$$

Then K is precompact in $L^q(\Gamma, d\mu)$.

The proof of this proposition is a simple modification of the proof of Theorem 2.22 in [2].

PROPOSITION 2.2. *For positive integer k and for $1 < p < \infty$, the imbedding $W^{k,p}(\Gamma, d\mu) \hookrightarrow L^1(\Gamma, d\mu)$ is compact.*

Proof. Since μ is a finite measure, the imbedding clearly exists. Let M be a bounded subset of $W^{k,p}(\Gamma, d\mu)$. It suffices to show that M is precompact in $L^1(\Gamma, d\mu)$.

M satisfies condition (i) of Proposition 2.1 with $q = 1$ since, on the bounded sets $\Gamma \cap B_m$, $d\mu$ is equivalent to Lebesgue measure dx, and since for bounded sets Ω satisfying the cone property, the imbedding $W^{k,p}(\Omega, dx) \hookrightarrow L^1(\Omega, dx)$ is compact as a result of the Rellich–Kondrachov theorem (see, e.g., Theorem 6.2 in [2]).

We show then that M satisfies condition (ii) of Proposition 2.1 with $q = 1$. Let $\varepsilon > 0$. There exists $R < \infty$ such that $\|u\|_{p,\Gamma} \leqslant R$ for all $u \in M$ since M is bounded in $W^{k,p}(\Gamma, d\mu)$. Choosing m so that $R(\mu(D_m))^{p/(p-1)} < \varepsilon$, we have

$$\int_{D_m} |u| \, d\mu \leqslant \left(\int_{D_m} |u|^p \, d\mu \right)^{1/p} \left(\int_{D_m} 1 \, d\mu \right)^{p/(p-1)} < \varepsilon$$

for all $u \in M$. Hence Proposition 2.1 is applicable and Proposition 2.2 follows.

Next, we present two results concerning Orlicz space imbeddings. We recall that, given a domain $\Omega \subset \mathbb{R}^n$ and a Young's function $X(t)$, we can define an Orlicz space

$$L_X(\Omega, d\mu) = \left\{ u(x) : \int_\Omega X(|u(x)|)\, d\mu(x) < \infty \right\}.$$

$L_X(\Omega, d\mu)$ is a Banach space with respect to the norm

$$\|u\|_{X,\Omega} = \inf \left\{ r > 0 : \int_\Omega X\left(\frac{|u(x)|}{r}\right) d\mu(x) \leqslant 1 \right\}.$$

For example, if $X(t) = t^p$ for $1 \leqslant p < \infty$, then $L_X(\Omega, d\mu)$ is simply $L^p(\Omega, d\mu)$.

Suppose that $X(t)$ and $Y(t)$ are Young's functions. If there exist constants R and t_0 such that $X(t) \leqslant Y(Rt)$ for all $t \geqslant t_0$, then we say that Y dominates X at infinity. If in addition X dominates Y at infinity, then we say that X and Y are equivalent at infinity. If Y dominates X at infinity but X and Y are not equivalent at infinity, then we say that X increases essentially more slowly than Y, or that Y increases essentially faster than X.

PROPOSITION 2.3. *If X and Y are Young's functions and Y dominates X at infinitely, then the imbedding $L_Y(\Omega, d\mu) \hookrightarrow L_X(\Omega, d\mu)$ exists and is continuous. If X and Y are equivalent at infinity, then the norms $\|\cdot\|_{X,\Omega}$ and $\|\cdot\|_{Y,\Omega}$ are equivalent.*

The proof of this proposition is a routine calculation which depends on the fact that μ is a finite measure (see, e.g., Theorem 8.12 in [2]).

PROPOSITION 2.4. *Suppose that X and Y are Young's functions and X increases essentially more slowly than Y. If for a positive integer k and for $1 < p < \infty$ the imbedding $W^{k,p}(\Gamma, d\mu) \hookrightarrow L_Y(\Gamma, d\mu)$ is continuous, then the imbedding $W^{k,p}(\Gamma, d\mu) \hookrightarrow L_X(\Gamma, d\mu)$ is compact.*

The proof of this proposition can be obtained by making minor modifications in the proofs of Theorems 8.22 and 8.23 in [2]. The result depends upon the fact that μ is a finite measure and on the fact that the imbedding $W^{k,p}(\Gamma, d\mu) \hookrightarrow L^1(\Gamma, d\mu)$ is compact.

3. COMPACT IMBEDDING THEOREMS

We begin by introducing a method for gauging the rate of decay of a measure for large $|x|$. We assume that Γ and μ are as in the previous section.

For $\Omega \subset \mathbb{R}^n$, we denote by $\mathscr{B}(\Omega)$ the set of Borel sets contained in Ω. For $U \subset \mathbb{R}^n$ and $s > 0$, we define $T_s(U)$ by

$$T_s(U) = \left\{ x + s\frac{x}{|x|} : x \in U, x \neq 0 \right\}.$$

For $r, s \geqslant 0$, we define $\gamma(r, s)$ by

$$\gamma(r, s) = \sup_{U \in \mathscr{B}(D_{r-s})} \frac{\mu(T_s(U))}{\mu(U)}.$$

We now state our main result.

THEOREM 3.1. *Suppose that*

(i) $\lim_{r \to \infty} \gamma(r, 1) = 0$;

(ii) $\lim_{r \to \infty} \left(\int_0^1 \gamma(r, s)\, ds \right) = 0$.

Then for positive integer k and for $1 \leqslant p < \infty$, the imbedding $W^{k,p}(\Gamma, d\mu) \hookrightarrow L^p(\Gamma, d\mu)$ is compact.

Proof. We first note that, given the definition of $\gamma(r, s)$,

$$\int_{D_r} \left| u\left(x - \frac{tx}{|x|}\right) \right| d\mu(x) \leqslant \gamma(r, t) \int_{D_{r-t}} |u(x)|\, d\mu(x) \tag{2}$$

for any measurable function $u(x)$, and for $0 \leqslant t \leqslant r$.

Consider a function $u \in C^1(\bar{\Gamma}) \cap L^\infty(\Gamma) \cap W^{1,1}(\Gamma, d\mu)$. We have for $x \in \mathbb{R}^n$ such that $|x| > 1$,

$$u\left(x - \frac{x}{|x|}\right) = u(x) + \int_0^1 \frac{d}{dt} u\left(x - \frac{tx}{|x|}\right) dt. \tag{3}$$

Also,

$$\left| \frac{d}{dt} u\left(x - \frac{tx}{|x|}\right) \right| \leqslant \left| \frac{\partial u}{\partial x_1}\left(x - \frac{tx}{|x|}\right) \right| + \cdots + \left| \frac{\partial u}{\partial x_n}\left(x - \frac{tx}{|x|}\right) \right|. \tag{4}$$

It follows from (2) and (4) that

$$\int_{D_r} \left| u\left(x - \frac{x}{|x|}\right) \right| d\mu \leqslant \gamma(r, 1) \int_{D_{r-1}} |u(x)|\, d\mu \tag{5}$$

and that

$$\int_{D_r} \left| \int_0^1 \frac{d}{dt} u\left(x - \frac{tx}{|x|}\right) dt \right| d\mu(x)$$
$$\leqslant \int_0^1 \int_{D_r} \left(\left| \frac{\partial u}{\partial x_1}\left(x - \frac{tx}{|x|}\right) \right| + \cdots + \left| \frac{\partial u}{\partial x_n}\left(x - \frac{tx}{|x|}\right) \right| \right) d\mu(x)\, dt$$
$$\leqslant \int_0^1 \gamma(r, t) \int_{D_{r-t}} \left(\left| \frac{\partial u}{\partial x_1}(x) \right| + \cdots + \left| \frac{\partial u}{\partial x_n}(x) \right| \right) d\mu(x)\, dt. \tag{6}$$

Letting $\delta(r) = \max\{\gamma(r, 1), \int_0^1 \gamma(r, s)\, ds\}$, it follows from (3), (5), and (6) that

$$\int_{D_r} |u(x)|\, d\mu \leqslant \int_{D_r} \left| u\left(x - \frac{x}{|x|}\right) \right| d\mu + \int_{D_r} \left| \int_0^1 \frac{d}{dt} u\left(x - \frac{tx}{|x|}\right) dt \right| d\mu(x)$$
$$\leqslant \delta(r)\, \|u\|_{1,1,\Gamma}. \tag{7}$$

This inequality can clearly be extended to hold for all $u \in W^{1,1}(\Gamma, d\mu)$.

Next, consider $u \in W^{1,p}(\Gamma, d\mu)$, u real valued, where $1 \leqslant p < \infty$. The first-order distributional derivatives of $|u|^p$ are locally integrable and are given, for $j = 1,\ldots, n$, by

$$\begin{aligned} (|u|^p)_{x_j} &= p\,|u|^{p-1} u_{x_j} && \text{if} \quad u(x) > 0 \\ &= 0 && \text{if} \quad u(x) = 0 \\ &= -p\,|u|^{p-1} u_{x_j} && \text{if} \quad u(x) < 0 \end{aligned}$$

for almost all $x \in \Gamma$. This computation requires an application of the chain rule useing a characterization of distributional derivatives for absolute values given, for example, in Section 7.4 of [6].

It follows, using Hölders inequality, that

$$\int_\Gamma |(|u|^p)_{x_j}|\, d\mu \leqslant p \left(\int_\Gamma |u|^p\, d\mu \right)^{(p-1)/p} \left(\int_\Gamma |u_{x_j}|^p\, d\mu \right)^{1/p}$$
$$\leqslant p\, \|u\|_{1,p,\Gamma}^p$$

so that $|u|^p \in W^{1,1}(\Gamma, d\mu)$ with $\||u|^p\|_{1,1,\Gamma} \leqslant K\, \|u\|_{1,p,\Gamma}^p$, K constant. Hence by (7), we have

$$\int_{D_r} |u|^p\, d\mu \leqslant \delta(r) K\, \|u\|_{1,p,\Gamma}^p. \tag{8}$$

This inequality clearly holds for complex valued $u \in W^{1,p}(\Gamma, d\mu)$ if we adjust the constant K.

Suppose now that M is a bounded set in $W^{1,p}(\Gamma, d\mu)$. We show that M satisfies conditions (i) and (ii) of Proposition 2.1 with $q = p$. Condition (i) holds as in the proof of Proposition 2.2 since the Rellich–Kondrachov theorem asserts that, for bounded domains Ω satisfying the cone property, the imbedding $W^{1,p}(\Omega, dx) \hookrightarrow L^p(\Omega, dx)$ is compact.

To verify condition (ii) of Proposition 2.1, let $\varepsilon > 0$. There exists a positive integer m such that $\delta(m)KR < \varepsilon$ since $\lim_{r\to\infty} \delta(r) = 0$, and where R is the $W^{1,p}$ bound for the set M. It then follows from (8) that for all $u \in M$,

$$\int_{D_m} |u|^p \, d\mu < \varepsilon.$$

Hence M is precompact in $L^p(\Gamma, d\mu)$ and the theorem follows for $k = 1$.

For an integer $k > 1$, the theorem follows since the imbedding $W^{k,p}(\Gamma, d\mu) \hookrightarrow W^{1,p}(\Gamma, d\mu)$ is continuous and the imbedding $W^{1,p}(\Gamma, d\mu) \hookrightarrow L^p(\Gamma, d\mu)$ is kcompact. Hence the proof of Theorem 3.1 is complete.

Remark. Theorem 6.47 in [2] states that the imbeddings $W^{k,p}(\Omega, dx) \hookrightarrow L^p(\Omega, dx)$ are compact if Ω is a domain in $\mathbb{R}^n$ whose Lebesgue measure decays for large $|x|$ in a uniform sense like the μ measure of Γ is required to decay in Theorem 3.1. The idea for, and many of the details of our proof of Theorem 3.1 were derived from the proof of Theorem 6.47 in [2].

EXAMPLE. Suppose that $\Gamma = \mathbb{R}^n$ and that $d\mu = e^{-|x|^\alpha}$, where $\alpha > 1$. Suppose that $0 < s \leqslant 1 < r < \infty$ and that $U \subset D_{r-s}$. Then

$$\begin{aligned}
\mu(T_s(U)) &= \int_{T_s(U)} e^{-|x|^\alpha} \, dx = \int_U \left(\frac{|x| + s}{|x|} \right)^{n-1} e^{-(|x|+s)^\alpha} \, dx \\
&\leqslant 2^{n-1} \int_U e^{-|x|^\alpha} e^{-\alpha s |x|^{\alpha-1}} \, dx \\
&\leqslant 2^{n-1} e^{-\alpha s (r-s)^{\alpha-1}} \mu(U).
\end{aligned}$$

The second equality is a result of the change of variables $x \to x + s(x/|x|)$; the first inequality follows from the fact that $(|x| + s)^\alpha \geqslant |x|^\alpha + \alpha s\, |x|^{\alpha-1}$ for $|x| \geqslant r > s$.

Hence $\gamma(r, s) \leqslant (\text{const})\, e^{-\alpha s (r-s)^{\alpha-1}}$ and the conditions of Theorem 3.1 are satisfied. Thus for $d\mu = e^{-|x|^\alpha} dx$ where $\alpha > 1$, the imbeddings $W^{k,p}(R^n, d\mu) \hookrightarrow L^p(\mathbb{R}^n, d\mu)$ are compact.

Our next theorem gives in some sense a lower bound on the rate of decay which allows compactness of the imbeddings $W^{k,p} \to L^p$.

THEOREM 3.2. *Suppose that the imbedding $W^{k,p}(\Gamma, d\mu) \hookrightarrow L^p(\Gamma, d\mu)$ is compact. Then for all $\varepsilon, \delta > 0$, there exists R such that $r > R$ implies $\mu(D_r) \leqslant \delta\mu\{x: r - \varepsilon \leqslant |x| \leqslant r\}$.*

Proof. Suppose, to the contrary, that there exist $\varepsilon, \delta > 0$ and a sequence of integers $\{r_m\}$ such that $\mu(D_{r_m}) > \delta\mu\{x: r_m - \varepsilon \leqslant |x| \leqslant r\}$. We assume for convenience that $\varepsilon < 1$.

We choose spherically symmetric functions $\varphi_m(x) \in C^\infty(\mathbb{R}^n)$ such that $\varphi_m \equiv 1$ on D_{r_m}, $\varphi_m \equiv 0$ on $B_{r_m - \epsilon}$, $0 \leqslant \varphi_m \leqslant 1$ on $\mathbb{R}^n$, and such that

$$\sup_{|\alpha| \leqslant k} \sup_{m \in \mathbb{Z}^+} |D^\alpha \varphi_m| \leqslant M < \infty.$$

We assume that $M \geqslant 1$.

We define $\psi_m = C_m \varphi_m$, where $C_m = 1/(\mu(D_m))^{1/p}$. Then

$$\begin{aligned} \|\psi_m\|_{k,p}^p &= C_m^p \sum_{|\alpha| \leqslant k} \int_\Gamma |D^\alpha \varphi_m|^p \, d\mu \\ &\leqslant KC_m^p(\mu(D_{r_m}) + \mu\{r_m - \varepsilon \leqslant |x| \leqslant r_m\}) \\ &\leqslant KC_m^p(1 + 1/\delta)\,\mu(D_m) = K(1 + 1/\delta), \end{aligned}$$

where the constant K depends only on k and M. Hence $\{\psi_m\}$ is a bounded sequence in $W^{k,p}(\Gamma, d\mu)$.

If a subsequence of $\{\psi_m\}$ is convergent, it can clearly only converge to $\psi \equiv 0$. But this is not possible since

$$\|\psi_m\|_p^p \geqslant C_m^p \int_{D_m} |\varphi_m|^p \, d\mu = 1$$

for all m. Hence the theorem follows by contradiction.

COROLLARY 3.3. *Suppose that the imbedding $W^{k,p}(\Gamma, d\mu) \hookrightarrow L^p(\Gamma, d\mu)$ is compact. Then for all $\lambda > 0$, $\lim_{m \to \infty} e^{\lambda m}\mu(D_m) = 0$.*

Proof. Fix λ and set $\delta = e^{-(\lambda+1)}$, $\varepsilon = 1$. It follows from Theorem 3.2 that there exists R such that $r \geqslant R$ implies $\mu(D_{r+1}) \leqslant \delta\mu(D_r)$. We assume for convenience that R is an integer. For $m \in \mathbb{Z}^+$, we have

$$\begin{aligned} e^{\lambda(R+m)}\mu(D_{R+m}) &\leqslant e^{\lambda R} e^{\lambda m} \delta^m \mu(D_R) \\ &= e^{\lambda R} e^{-m} \mu(D_R) \to 0 \quad \text{as} \quad m \to \infty. \end{aligned}$$

The corollary follows.

Remark. Theorem 3.2 and Corollary 3.3 parallel Theorem 6.40 and Corollary 6.41 in [2] which exhibit a relationship between compactness of imbeddings $W^{k,p}(\Omega, dx) \hookrightarrow L^p(\Omega, dx)$ and the decay of the Lebesgue measure of Ω for large $|x|$.

We give next an example of a measure which satisfies the necessary conditions set forth in Theorem 3.2 but for which the imbeddings $W^{k,p} \hookrightarrow L^p$ are not compact. This example should give some indication of the importance of requiring uniform decay in the hypotheses of Theorem 3.1.

EXAMPLE. We define $d\mu = \rho(x)\,dx$ on $\mathbb{R}^1$ by setting $\rho(x) = e^{-n^2}$ for $n \leqslant x < n+1$, $n \in \mathbb{Z}^+$, and $\rho(x) = 2e^{-x^2} - \rho(-x)$ for $x < 0$. There exist functions $\varphi_n \in C_0^\infty(n, n+1)$ such that $\varphi_n(x) = 1$ for $x \in [n - 1/3, n + 2/3]$ and $0 \leqslant \varphi_n(x) \leqslant 1$ for all x, and such that

$$\sup_{0 \leqslant j \leqslant k} \sup_{n \in \mathbb{Z}^+} \sup_{x \in \mathbb{R}} \left| \frac{d^j \varphi_n}{dx} (x) \right| \leqslant M < \infty.$$

It is a simple calculation that $\{e^{n^2/p}\varphi_n(x)\}$ is a bounded set in $W^{k,p}(\mathbb{R}, d\mu)$ which has no subsequences convergent in $L^p(\mathbb{R}, d\mu)$. This example parallels Example 6.44 in [2].

We end this section by introducing a notion of an outward derivative for a measure μ and indicating how such a derivative might be related to the existence of Sobolev–Orlicz imbeddings $W^{k,p}(\Gamma, d\mu) \hookrightarrow L_X(\Gamma, d\mu)$, where X is a Young's function. We remark that if $X(t)$ is a Young's function which increases essentially faster than t^q for some $q > p$, then there can be no imbedding $W^{k,p}(\Gamma, d\mu) \hookrightarrow L_X(\Gamma, d\mu)$ if the support of the measure μ is unbounded. The proof is essentially the same as the proof of Theorem 5.30 in [2].

We define our outward derivative, which we denote by μ', by setting

$$\mu'(U) = \lim_{s \to 0^+} \frac{\mu(T_s(U)) - \mu(U)}{s}.$$

Under fairly mild conditions on μ, this derivative will be a measure on Γ. We can develop a general theory for outward derivatives of measures similar to the theory for directional derivatives of measures given in [3].

Given a measure μ whose outward derivative μ' is again a measure, it is fairly easy to see that conditions (i) and (ii) of Theorem 3.1 will both be satisfied if we require that

$$\zeta_\mu(r) = \sup_{U \in D_r} \mu'(U) \to -\infty \qquad \text{as} \quad r \to \infty.$$

It is likely the case that there is a relationship between the rate at which $\zeta_\mu(r)$ approaches $-\infty$ and the best possible Sobolev–Orlicz imbedding $W^{k,p}(\Gamma, d\mu) \hookrightarrow L_X(\Gamma, d\mu)$.

Upon examining properties of the outward derivatives of measures studied in [4, 5, 7, and 8], one is led naturally to the following

Conjecture. Suppose that the outward derivative of a measure μ exists and that $\zeta_\mu(r) < -r$ for all r sufficiently large. Then $W^{1,2}(\Gamma, d\mu)$ is imbedded in $L^2 \ln L(\Gamma, d\mu)$.

4. AN APPLICATION IN ELLIPTIC DIFFERENTIAL EQUATIONS

Consider a domain Γ and a measure $d\mu = \rho(x)\,dx$ as described in Section 2. We say that an $n \times n$ matrix valued function $a(x) = (a_{ij}(x))$ is ρ-strongly elliptic on Γ if there exist positive constants λ and Λ such that for any n dimensional vector ξ,

$$\lambda\rho(x)\,|\xi|^2 \leqslant \xi \cdot a(x)\bar{\xi} = \sum_{i,j=1}^{n} \xi_i \bar{\xi}_j\, a_{ij}(x) \leqslant \Lambda\rho(x)\,|\xi|^2 \tag{9}$$

for almost all $x \in \Gamma$. We assume for convenience that $\lambda \leqslant 1$ and $\Lambda \geqslant 1$.

For such a function $a(x)$, the sesquilinear form

$$h(u, v) = \int_\Gamma \nabla u \cdot (a\overline{\nabla v})\,dx = \sum_{i,j=1}^{n} \int_\Gamma u_{x_i}\overline{v_{x_j}} a_{ij}\,dx$$

with form domain $W^{1,2}(\Gamma, d\mu)$ is closed and positive. Hence, it has associated with it a positive self-adjoint operator A which is given by

$$Au = -\frac{1}{\rho}\nabla \cdot (a\nabla u) = -\frac{1}{\rho}\sum_{i,j=1}^{n} (a_{ij}u_{x_i})_{x_j}.$$

If $a(x)$ and $\rho(x)$ are C^1 on $\bar{\Gamma}$, then the domain of A consists of those functions in $W^{2,2}(\Gamma, d\mu)$ which satisfy zero Neumann boundary conditions, i.e., whose normal derivatives equal zero on the boundary of Γ (see [9] for a more complete discussion). A is a positive unbounded self-adjoint operator on $\mathscr{H} = L^2(\Gamma, d\mu)$.

For a bounded domain Γ, it is a classical result that $\mathscr{H}$ has a complete orthonormal system of eigenvectors $\{\eta_i\}_{i=1}^{\infty}$ of A with corresponding eigenvalues $\{\tau_i\}_{i=1}^{\infty}$ such that $\tau_i \to \infty$ as $i \to \infty$. For an unbounded domain Γ, we have

THEOREM 4.1. *Suppose that Γ, μ, and A are as defined above. Then the following are equivalent.*

(i) *The imbedding $W^{1,2}(\Gamma, d\mu) \hookrightarrow L^2(\Gamma, d\mu)$ is compact.*

(ii) *$\mathscr{H}$ has a complete orthonormal system of eigenvectors $\{\eta_i\}_{i=1}^{\infty}$ of A with corresponding eigenvalues $\{\tau_i\}_{i=1}^{\infty}$ such that $\tau_i \to \infty$ as $i \to \infty$.*

Remark. Recall that for a bounded domain Γ, the imbedding $W^{1,2}(\Gamma, d\mu) \to L^2(\Gamma, d\mu)$ is compact.

Proof of Theorem 4.1. Theorem XIII.64 in [10] asserts that condition (ii) holds if and only if the following condition holds.

(i′) For all $b \geqslant 0$, the set of all ψ in the form domain of A such that $\|\psi\|_{2,\Gamma} \leqslant 1$ and $\langle (A+I)\psi, \psi\rangle \leqslant b$ is compact in $\mathscr{H}$,

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathscr{H}$. But since h is the form associated with A, the form domain of A is $W^{1,2}(\Gamma, d\mu)$ and for all ψ in the form domain, we have

$$\begin{aligned}\langle (A+I)\psi, \psi\rangle &= h(\psi, \psi) + \langle \psi, \psi\rangle \\ &= \int_\Gamma \nabla\psi \cdot (a\overline{\nabla\psi})\, dx + \int_\Gamma |\psi|^2\, d\mu.\end{aligned}$$

It follows from (9) that (since $d\mu = \rho(x)\, dx$)

$$\begin{aligned}\lambda \|\psi\|^2_{1,2,\Gamma} &= \lambda \left(\int_\Gamma |\nabla\psi|^2\, d\mu + \int_\Gamma |\psi|^2\, d\mu\right) \\ &\leqslant \langle (A+I)\psi, \psi\rangle \leqslant \Lambda \|\psi\|^2_{1,2,\Gamma}.\end{aligned}$$

Hence conditions (i) and (i′) are equivalent, and the theorem follows.

Remark. Suppose that we define the form h to have as its domain $W_0^{1,2}(\Gamma, d\mu)$ instead of $W^{1,2}(\Gamma, d\mu)$, where $W_0^{1,2}(\Gamma, d\mu)$ is the closure of $C_0^\infty(\Gamma)$ in the norm $\|\cdot\|_{1,2,\Gamma}$. Then h will be closed and positive and will have an associated operator, which we again denote by A, and which is given by the same formula as before. But in this case, the domain of A is $W_0^{2,2}(\Gamma, d\mu)$, the closure of $C_0^\infty(\Gamma)$ in the norm $\|\cdot\|_{2,2,\Gamma}$. And condition (ii) will hold for A if and only if the imbedding $W_0^{1,2}(\Gamma, d\mu) \hookrightarrow L^2(\Gamma, d\mu)$ is compact.

EXAMPLE. For either choice of form domain, if A generates a hypercontractive semigroup of operators (see, e.g., [8] for definition), then the imbedding $W^{1,2}(\Gamma, d\mu) \hookrightarrow L^2 \ln L(\Gamma, d\mu)$ is continuous so that the imbedding $W^{1,2}(\Gamma, d\mu) \hookrightarrow L^2(\Gamma, d\mu)$ is compact. Hence condition (ii) of Theorem 4.1 holds.

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