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Poisson Lie symmetry and D-branes in WZW model on the Heisenberg Lie group H_4

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Abstract

We show that the WZW model on the Heisenberg Lie group H_4 has Poisson–Lie symmetry only when the dual Lie group is $A_2 \oplus 2A_1$. In this way, we construct the mutual T-dual sigma models on Drinfel'd double generated by the Heisenberg Lie group H_4 and its dual pair, $A_2 \oplus 2A_1$, as the target space in such a way that the original model is the same as the H_4 WZW model. Furthermore, we show that the dual model is conformal up to two-loop order. Finally, we discuss D-branes and the worldsheet boundary conditions defined by a gluing matrix on the H_4 WZW model. Using the duality map obtained from the canonical transformation description of the Poisson–Lie T-duality transformations for the gluing matrix which locally defines the properties of the D-brane, we find two different cases of the gluing matrices for the WZW model based on the Heisenberg Lie group H_4 and its dual model.

1. Introduction

The WZW models on non-semi-simple Lie groups [1–4] play an important role in string theory, since some of them provide exact string backgrounds that have a target space dimension equal to the integer and irrational Virasoro central charge of the affine non-semi-simple algebra [3,5]. The first of these models was based on the group E_2^c, a central extension of the two-

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dimensional Euclidean group; and the corresponding sigma model describes string propagation on a four-dimensional space–time in the background of a gravitational plane wave [1]. This construction was subsequently extended to other non-semi-simple Lie groups [3,4] in such a way that the WZW model on the Heisenberg group with arbitrary dimension was, for the first time, introduced by Kehagias and Meessen [4].

On the other hand, the T-duality is a very important symmetry of string theories, or more generally, two-dimensional sigma models [6] and the Poisson–Lie T-duality [7,8] is a generalization of Abelian and non-Abelian target space duality (T-duality). So far, there is one example for conformal sigma models related by Poisson–Lie T-duality [9] in a way that the duality relates the standard $SL(2,R)$ WZW model to a constrained sigma model defined on the $SL(2,R)$ group space. Moreover, we have recently shown that the WZW models on the Lie supergroups $GL(1|1)$ [10] and (C^3+A) [11] contain super Poisson–Lie symmetry such that in this process the dual Lie supergroups are the respective $B \oplus A \oplus A_{1,1|}.i$ and $C^3 \oplus A_{1,1|}.i$. In this paper we show that the WZW model on the Heisenberg Lie group H_4 has Poisson–Lie symmetry only when the dual Lie group is $A_2 \oplus 2A_1$. Furthermore, we show that the dual model is conformal up to two-loop order and in this manner we obtain the general form of the dilaton field of the dual model. We also study the worldsheet boundary conditions for our model and its dual.

The outline of the paper is as follows. In Section 2 we show that the WZW model on the Heisenberg Lie group H_4 has Poisson–Lie symmetry only when the dual Lie group is $A_2 \oplus 2A_1$. In Section 3, we first construct the Poisson–Lie T-dual sigma models on the Drinfel'd double $(H_4, A_2 \oplus 2A_1)$ in such a way that we show the original model is the same as the H_4 WZW model. Then, by calculation of the vanishing of the one-loop B-functions we obtain the general form of the dilaton field of the dual model and followed by, we show that the dual model is conformal up to two-loop order. Finally, in Section 4, we first review the worldsheet boundary conditions under the Poisson–Lie T-duality and reobtain, in general, the algebraic form of a duality map for the gluing matrix between both the original and dual models under the Poisson–Lie T-duality transformation. Then, we study the consequences of the duality transformation of the gluing matrix for the H_4 WZW model and its dual model. Some concluding remarks are given in the last section.

2. Poisson–Lie symmetry of the WZW model on the Heisenberg Lie group H_4

In this section, based on our previous works [10] and [11], we will describe a new example of a WZW model containing Poisson–Lie symmetry. The model is constructed on the Heisenberg Lie group H_4, a non-semi-simple Lie group of dimension four. As mentioned in introduction section, the WZW model based on the Heisenberg group H_4 was, for the first time, constructed by Kehagias and Meessen [4]. Here, we first obtain the WZW model on the Heisenberg group H_4 with a new background. Then, we will show that the model has Poisson–Lie symmetry. Before proceeding to construct model, let us first introduce the Lie algebra h_4 of the Lie group H_4 (the oscillator Lie algebra). The Lie algebra h_4 is generated by the generators $\{N, A_+, A_-, M\}$ with the following non-zero Lie brackets

$$[N, A_+] = A_+, \qquad [N, A_-] = -A_-, \qquad [A_-, A_+] = M. \tag{2.1}$$

One can show that the Lie algebra h_4 is isomorphic to the Drinfel'd double of a two-dimensional Lie bialgebra, i.e., $(\mathcal{A}_2, \mathcal{I}_2)$ [12,13] where $\mathcal{A}_2$ and $\mathcal{I}_2$ are two-dimensional non-Abelian and Abelian Lie algebras, respectively. The isomorphic transformation between the Lie algebras h_4 and $(\mathcal{A}_2, \mathcal{I}_2)$ is given by

$$N = T_1 + \alpha_0 T_4, \quad A_+ = \beta_0 T_2 - \alpha_0\beta_0 T_3, \quad A_- = \gamma_0 T_4, \quad M = -\beta_0\gamma_0 T_3,$$

where $\{T_1, \ldots, T_4\}$ are generators of the Lie algebra of the Drinfel'd double $(\mathcal{A}_2, \mathcal{I}_2)$ and $\alpha_0 \in \Re$; $\beta_0, \gamma_0 \in \Re - \{0\}$.

Let us now turn into the construction of our model. In general, given a Lie algebra with generators X_a and structure constants $f_{ab}{}^c$, to define a WZW model, one needs a non-degenerate ad-invariant symmetric bilinear form $\Omega_{ab} = <X_a\ ,\ X_b>$ on Lie algebra $\mathcal{G}$ such that it satisfies the following relation [1]

$$f_{ab}{}^d\, \Omega_{dc} + f_{ac}{}^d\, \Omega_{db} = 0. \tag{2.2}$$

Using the commutation relations (2.1), one can obtain the non-degenerate ad-invariant bilinear form Ω_{ab} on the Lie algebra h_4 as

$$\Omega_{ab} = \begin{pmatrix} 0 & 0 & 0 & -\kappa_0 \\ 0 & 0 & \kappa_0 & 0 \\ 0 & \kappa_0 & 0 & 0 \\ -\kappa_0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.3}$$

where κ_0 is a non-zero real constant. In general, we know that the WZW model based on a Lie group G is defined on a Riemannian surface Σ as a worldsheet by the following action [1]

$$S_{WZW}(g) = \frac{K}{4\pi}\int_\Sigma d\sigma^+ d\sigma^-\, L^a_+\, \Omega_{ab}\, L^b_- + \frac{K}{24\pi}\int_B d^3\sigma\ \varepsilon^{\gamma\alpha\beta} L^a_\gamma\, \Omega_{ad}\, L^b_\alpha\, f_{bc}{}^d\, L^c_\beta, \tag{2.4}$$

where the components of the left-invariant one-forms L^a_α's are defined via $g^{-1}\partial_\alpha g = L^a_\alpha\ X_a$, in which $g : \Sigma \to G$ is an element of Lie group G. Here B is a three-manifold bounded by worldsheet Σ and $\sigma^\pm = \frac{1}{\sqrt{2}}(\tau \pm \sigma)$ are the standard light-cone variables on the worldsheet. To calculate the L^a_α's we parameterize the corresponding Lie group H_4 with coordinates $x^\mu = \{x, y, u, v\}$ so that its elements can be written as

$$g = e^{vX_4}\, e^{uX_3}\, e^{xX_1}\, e^{yX_2}, \tag{2.5}$$

where we have introduced the new generators $\{X_1, X_2, X_3, X_4\}$ instead of $\{N, A_+, A_-, M\}$, respectively. We then obtain

$$L^1_\pm = \partial_\pm x, \quad L^2_\pm = y\partial_\pm x + \partial_\pm y, \quad L^3_\pm = e^x\ \partial_\pm u, \quad L^4_\pm = ye^x\ \partial_\pm u + \partial_\pm v. \tag{2.6}$$

Hence using relations (2.1), (2.3) and (2.6) and some algebraic calculations, the WZW action on the H_4 Lie group is worked out to be of the form

$$S_{WZW}(g) = \frac{\kappa_0 K}{4\pi}\int d\sigma^+ d\sigma^- \Big\{ - \partial_+ x\partial_- v - \partial_+ v\partial_- x + e^x\Big(\partial_+ y\partial_- u + \partial_+ u\partial_- y + y\partial_+ u\partial_- x - y\partial_+ x\partial_- u\Big)\Big\}. \tag{2.7}$$

As we know the non-linear sigma model for a bosonic string propagating in a d-dimensional space–time with the metric $g_{\mu\nu}$, the anti-symmetric tensor field $b_{\mu\nu}$ and the dilaton field Φ is given by[1]

[1] The dimensional coupling constant α' turns out to be the inverse string tension.

$$S = \frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-h} \left\{ \frac{1}{2}\left(h^{\alpha\beta} g_{\mu\nu} + \epsilon^{\alpha\beta} b_{\mu\nu}\right) \partial_\alpha x^\mu \partial_\beta x^\nu + \frac{1}{4}\alpha' \Phi(x^\mu)\ R^{(h)} \right\}, \tag{2.8}$$

where $h_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ are the worldsheet metric with $R^{(h)}$ the corresponding worldsheet curvature scalar and anti-symmetric tensor on the worldsheet, respectively, such that $h := \det h_{\alpha\beta}$ and the indices $\alpha, \beta = \tau, \sigma$. Note that here we consider $\Phi(x^\mu) = 0$. The model (2.8) is invariant under worldsheet reparametrization, therefore this symmetry allows us to switch to light-cone coordinates on the worldsheet; consequently, in the absence of the dilaton we have [7]

$$S = \frac{1}{2} \int d\sigma^+ d\sigma^- \left(g_{\mu\nu} + b_{\mu\nu}\right) \partial_+ x^\mu \partial_- x^\nu. \tag{2.9}$$

Here the space–time geometry is described by a Lorentz signature metric $g_{\mu\nu}$ and anti-symmetric tensor field $b_{\mu\nu}$, both of which may depend on the space–time coordinates x^μ. For the action (2.7), the corresponding space–time metric and the anti-symmetric tensor field are, respectively, given by

$$\begin{aligned} ds^2 &= -2dxdv + 2e^x\, dydu, \\ b &= -ye^x\, dx \wedge du. \end{aligned} \tag{2.10}$$

Thus, by identifying the action (2.7) with the sigma model of the form (2.9) we can read off the background matrix $\mathcal{E}_{\mu\nu} = g_{\mu\nu} + b_{\mu\nu}$ in the coordinate base $\{dx, dy, du, dv\}$ as

$$\mathcal{E}_{\mu\nu} = \begin{pmatrix} 0 & 0 & -ye^x & -1 \\ 0 & 0 & e^x & 0 \\ ye^x & e^x & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.11}$$

The sigma model (2.9) has Poisson–Lie symmetry with respect to the dual Lie group $\tilde{G}$ (with the same dimension G) when background matrix satisfies in the following relation [7,8]

$$\mathcal{L}_{V_a}(\mathcal{E}_{\mu\nu}) = \mathcal{E}_{\mu\rho}\ V_b{}^\rho\ \tilde{f}^{cb}{}_a\ V_c{}^\lambda\ \mathcal{E}_{\lambda\nu}, \tag{2.12}$$

where $\mathcal{L}_{V_a}$ stands for the Lie derivative corresponding to the left invariant vector field V_a and $\tilde{f}^{cb}{}_a$ are the structure constants of the dual Lie algebra $\tilde{\mathcal{G}}$. In the following we will show that the WZW model on the H_4 Lie group has Poisson–Lie symmetry. To this end, we need the left invariant vector fields on the H_4. Utilizing relation (2.6) in the equation $V_a{}^\mu\ L_\mu{}^b = \delta_a{}^b$ the V_a's are obtained to be

$$V_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \qquad V_2 = \frac{\partial}{\partial y}, \qquad V_3 = e^{-x}\ \frac{\partial}{\partial u} - y\frac{\partial}{\partial v}, \qquad V_4 = \frac{\partial}{\partial v}. \tag{2.13}$$

Now, by substituting relations (2.11) and (2.13) on the right hand side of (2.12) and, then, by direct calculation of Lie derivative of $\mathcal{E}_{\mu\nu}$ with respect to V_a, one can obtain the structure constants of the dual Lie algebra to the Lie algebra h_4 in such a way that only non-zero commutation relation of the dual pair is

$$[\tilde{X}^2, \tilde{X}^4] = \tilde{X}^2. \tag{2.14}$$

The Lie algebra deduced in this process is a four-dimensional decomposable Lie algebra. In the classification of four-dimensional Lie algebras [14] it has denoted by $\mathcal{A}_2 \oplus 2\mathcal{A}_1$ where $\mathcal{A}_1$ is one-dimensional Lie algebra. Note that both sets of generators (2.1) and (2.14) are maximally isotropic with respect to the non-degenerate invariant symmetric bilinear form defined by the brackets [8]

$$<X_a\ ,\ X_b> = <\tilde{X}^a\ ,\ \tilde{X}^b> = 0, \qquad <X_a\ ,\ \tilde{X}^b> = \delta_a{}^b.$$

Nevertheless, the $(h_4\ ,\ \mathcal{A}_2 \oplus 2\mathcal{A}_1)$ as a Lie bialgebra satisfies mixed Jacobi identities. Having a Drinfel'd double which is simply a Lie group D, we can construct the Poisson–Lie symmetric sigma models on it. So we will, first, form the Drinfel'd double generated by the Lie algebra h_4 and its dual pair $\mathcal{A}_2 \oplus 2\mathcal{A}_1$. The Manin triple[2] $(\mathcal{D}, h_4, \mathcal{A}_2 \oplus 2\mathcal{A}_1)$ possesses eight generators $\{X_1, \ldots, X_4; \tilde{X}^1, \ldots, \tilde{X}^4\}$ so that they obey the following set of non-zero commutation relations

$$\begin{aligned}
&[X_1\ ,\ X_2] = X_2, && [X_1\ ,\ X_3] = -X_3, \quad [X_2\ ,\ X_3] = -X_4, \quad [\tilde{X}^2\ ,\ \tilde{X}^4] = \tilde{X}^2,\\
&[X_1\ ,\ \tilde{X}^2] = -\tilde{X}^2, && [X_1\ ,\ \tilde{X}^3] = \tilde{X}^3, \quad [X_3\ ,\ \tilde{X}^3] = -\tilde{X}^1, \quad [X_3\ ,\ \tilde{X}^4] = -\tilde{X}^2,\\
&[X_2\ ,\ \tilde{X}^2] = X_4 + \tilde{X}^1, && [X_2\ ,\ \tilde{X}^4] = -X_2 + \tilde{X}^3.
\end{aligned} \tag{2.15}$$

In the next section, we shall construct a pair of Poisson–Lie T-dual sigma models which is associated with the Drinfel'd double $(H_4\ ,\ A_2 \oplus 2A_1)$ and will show that the original sigma model on the H_4 is the same as the WZW model obtained in (2.7).

3. Poisson–Lie T-dual sigma models built on the Drinfel'd double $(H_4\ ,\ A_2 \oplus 2A_1)$

As mentioned above, having Drinfel'd doubles we can construct the Poisson–Lie T-dual sigma models on them. The construction of the models has been described in [7] and [8]. The models have target spaces as the Lie groups G and $\tilde{G}$ and are, respectively, given by the actions

$$S = \frac{1}{2}\int d\sigma^+\, d\sigma^-\ E^+_{ab}(g)\ R^a_+\ R^b_-, \tag{3.1}$$

$$\tilde{S} = \frac{1}{2}\int d\sigma^+\, d\sigma^-\ \tilde{E}^{+ab}(\tilde{g})\ (\tilde{R}_+)_a\ (\tilde{R}_-)_b, \tag{3.2}$$

where $R^a_\pm$ and $(\tilde{R}_\pm)_a$ are the components of the right-invariant one-forms on the Lie groups G and $\tilde{G}$, respectively, and are defined by

$$R^a_\pm := (\partial_\pm g\ g^{-1})^a = \partial_\pm x^\mu\ R_\mu{}^a, \qquad (\tilde{R}_\pm)_a := (\partial_\pm \tilde{g}\ \tilde{g}^{-1})_a = \partial_\pm \tilde{x}^\mu\ \tilde{R}_{\mu a}, \tag{3.3}$$

furthermore, the background fields $E^+(g)$ and $\tilde{E}^+(\tilde{g})$ are defined by

$$E^+(g) = \Big(\Pi(g) + (E^+_0)^{-1}(e)\Big)^{-1}, \quad \tilde{E}^+(\tilde{g}) = \Big(\tilde{\Pi}(\tilde{g}) + (\tilde{E}^+_0)^{-1}(\tilde{e})\Big)^{-1}, \tag{3.4}$$

in which $E^+_0(e)$ and $\tilde{E}^+_0(\tilde{e})$ are the sigma model constant matrices at the unit element of G and $\tilde{G}$, respectively, and are related to each other in the following way [8]

$$E^+_0(e)\tilde{E}^+_0(\tilde{e}) = \tilde{E}^+_0(\tilde{e})E^+_0(e) = \mathbb{1}. \tag{3.5}$$

Here $\Pi(g)$ is a bivector field on the Lie group manifold which gives a Poisson–Lie bracket on G and is defined as

[2] The Lie algebra $\mathcal{D}$ provided with non-degenerate ad-invariant symmetric bilinear form $< .\ ,\ . >$ will be called Drinfel'd double iff it can be decomposed into a pair of maximally isotropic sub-algebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ such that $\mathcal{D} = \mathcal{G} \oplus \tilde{\mathcal{G}}$. The triple $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$ is called Manin triple.

$$\Pi^{ij}(g) = b^{ik}(g)(a^{-1})_k{}^j(g), \tag{3.6}$$

where $a(g)$ and $b(g)$ are sub-matrices of the adjoint representation of G on the Lie algebra of the Drinfel'd double [8]. Analogously $\tilde{\Pi}(\tilde{g})$ is a Poisson–Lie bracket on the dual Lie group manifold $\tilde{G}$.

To construct the mutual T-dual sigma models with the Drinfel'd double (H_4 , $A_2 \oplus 2A_1$) whose Lie algebra defined in (2.15), we use the same parameterization (2.5) for both the original and dual models.[3] Using relation (2.5) and (3.3), one can calculate $R^a_\pm$'s as

$$\begin{aligned} R^1_\pm &= \partial_\pm x, & R^2_\pm &= e^x\, \partial_\pm y, \\ R^3_\pm &= u\, \partial_\pm x + \partial_\pm u, & R^4_\pm &= \partial_\pm v + u e^x\, \partial_\pm y, \end{aligned} \tag{3.7}$$

and considering relation (2.5) for the corresponding tilted symbols, we obtain

$$\begin{aligned} (\tilde{R}_\pm)_1 &= \partial_\pm \tilde{x}, & (\tilde{R}_\pm)_2 &= e^{-\tilde{v}}\, \partial_\pm \tilde{y}, \\ (\tilde{R}_\pm)_3 &= \partial_\pm \tilde{u}, & (\tilde{R}_\pm)_4 &= \partial_\pm \tilde{v}. \end{aligned} \tag{3.8}$$

Then, choosing the constant matrix at the unit element of H_4 as

$$E_0^+(e) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.9}$$

T-dual sigma models (3.1) and (3.2) are found to be given by

$$\begin{aligned} S = \frac{1}{2}\int d\sigma^+\, d\sigma^- \Big\{ &\partial_+ x \partial_- v + \partial_+ v \partial_- x - e^x \Big(\partial_+ y \partial_- u + \partial_+ u \partial_- y + y \partial_+ u \partial_- x \\ &- y \partial_+ x \partial_- u \Big) \Big\}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} \tilde{S} = \frac{1}{2}\int d\sigma^+\, d\sigma^- \Big\{ &\partial_+ \tilde{x} \partial_- \tilde{v} + \partial_+ \tilde{v} \partial_- \tilde{x} - \partial_+ \tilde{u} \partial_- \tilde{y} + \tilde{u} \partial_+ \tilde{v} \partial_- \tilde{y} + \tilde{y} \partial_+ \tilde{u} \partial_- \tilde{v} \\ &+ \frac{e^{-\tilde{v}}}{e^{-\tilde{v}} - 2} \Big(\partial_+ \tilde{y} \partial_- \tilde{u} + \tilde{u} \partial_+ \tilde{y} \partial_- \tilde{v} + \tilde{y} \partial_+ \tilde{v} \partial_- \tilde{u} + 2\tilde{y}\tilde{u} e^{\tilde{v}} \partial_+ \tilde{v} \partial_- \tilde{v} \Big) \Big\}. \end{aligned} \tag{3.11}$$

Now by rescaling κ_0 to $\frac{-2\pi}{K}$ in action (2.7) one can conclude that action (3.10) is nothing but the WZW action based on the Lie group H_4. Thus, we showed that the Poisson–Lie T-duality relates the H_4 WZW model to a sigma model defined on the dual Lie group of H_4, i.e., $A_2 \oplus 2A_1$. It is seen that in this case, the Poisson–Lie T-duality transforms rather extensive and complicated action (3.11) to much simpler form such as (3.10).

3.1. Conformal invariance of the dual sigma model up to the two-loop B-functions

Consistency of the string theory requires that the action (2.8) defines a conformally invariant quantum field theory, and the conditions for conformal invariance can be interpreted as effective field equations for $g_{\mu\nu}$, $b_{\mu\nu}$ and Φ of the string effective action [15,16]. The conditions for conformal invariance of the sigma model to the order α'^2 (at the two-loop level) have been

[3] The parameters $\tilde{x}$, $\tilde{y}$, $\tilde{u}$ and $\tilde{v}$ are applied for the dual model.

obtained in [17] (see also Ref. [18]). It has shown that these conditions can be derived from any one of a family of space–time effective actions.

When the β-functions are trivial, i.e., they vanish up to the ambiguities inherent in their definition, then the theory is rigid scale invariant, i.e., the integrated trace anomaly vanishes. The local scale or conformal invariance needed here requires that the trace anomaly vanishes locally, which requires the vanishing of certain *B-functions* [17]. The vanishing of two-loop B-function gives us the conformal invariance conditions of the sigma model (2.8) to the order α'^2 [17]. In the following, we use these conditions to show that the dual sigma model (3.11) is conformally invariant up to two-loop order.

3.1.1. Conformal invariance of the dual sigma model up to the one-loop B-functions

The conditions for conformal invariance to hold in two-dimensional sigma model (2.8) in the lowest non-trivial approximation are the vanishing of the one-loop B-functions. The one-loop B-functions are given by [17]

$$\begin{aligned}
B^{g}_{\mu\nu} &= -\alpha'\Big[R_{\mu\nu} - (H^2)_{\mu\nu} + \nabla_\mu\nabla_\nu\Phi\Big] + \mathcal{O}(\alpha'^2),\\
B^{b}_{\mu\nu} &= -\alpha'\Big[-\nabla^\lambda H_{\lambda\mu\nu} + H_{\mu\nu}{}^{\lambda}\nabla_\lambda\Phi\Big] + \mathcal{O}(\alpha'^2),\\
B^{\Phi} &= -\alpha'\Big[-\frac{1}{2}\nabla^2\Phi + \frac{1}{2}(\nabla\Phi)^2 - \frac{1}{3}H^2\Big] + \mathcal{O}(\alpha'^2),
\end{aligned} \tag{3.12}$$

where $R_{\mu\nu}$ is the Ricci tensor of the metric $g_{\mu\nu}$,

$$H_{\mu\nu\rho} = \frac{1}{2}\big(\partial_\mu b_{\nu\rho} + \partial_\nu b_{\rho\mu} + \partial_\rho b_{\mu\nu}\big), \tag{3.13}$$

is the torsion of the anti-symmetric field $b_{\mu\nu}$, $(H^2)_{\mu\nu} = H_{\mu\rho\sigma}H^{\rho\sigma}{}_{\nu}$ and $H^2 = H_{\mu\nu\rho}H^{\mu\nu\rho}$.

Notice that the original model (the model described by action (3.10)) as a WZW model should be conformally invariant. One can find that the only non-zero components of $R_{\mu\nu}$ and H are $R_{xx} = -\frac{1}{2}$ and $H_{xyu} = -\frac{e^x}{2}$, respectively. Thus, it is straightforward to get $R = 0 = H^2$ and verify that the only non-zero component of $(H^2)_{\mu\nu}$ is $2H_{xyu}H^{yu}{}_{x} = -\frac{1}{2}$. Consequently, the metric of this model is flat in the sense that its scalar curvature vanishes. Employing the above results in the vanishing of equations (3.12) we conclude that dilaton is constant. Nevertheless, by solving the vanishing of equations (3.12) one can also find a non-constant dilaton as

$$\Phi(x^\mu) = \Phi_0 + \mathcal{C}x, \tag{3.14}$$

where Φ_0 and $\mathcal{C}$ are integration constants.

Let us now turn into the dual model. With regard to action (3.11) the line element of the dual model is

$$d\tilde{s}^2 = 2d\tilde{x}d\tilde{v} + \frac{2}{e^{-\tilde{v}} - 2}\Big[d\tilde{y}d\tilde{u} + \tilde{u}(e^{-\tilde{v}} - 1)\, d\tilde{y}d\tilde{v} + \tilde{y}(e^{-\tilde{v}} - 1)\, d\tilde{u}d\tilde{v} + \tilde{y}\tilde{u}\, d\tilde{v}^2\Big]. \tag{3.15}$$

Analogously, for the dual model we find that the only non-zero component of $\tilde{R}_{\mu\nu}$ is $\tilde{R}_{\tilde{v}\tilde{v}} = -\frac{1}{2(e^{-\tilde{v}}-2)^2}\Big(e^{-2\tilde{v}} - 4e^{-\tilde{v}} + 16\Big)$; as $\tilde{g}^{\tilde{v}\tilde{v}} = 0$, $\tilde{R} = 0$. Therefore, the metric of dual model is also flat in the sense that its scalar curvature vanishes. As shown in the above, the dual metric has an apparent singularity. This singularity is the coordinate singularity in the metric. In general relativity [19], to investigate the types of singularities one has to study the invariant characteristics of space–time. To detect the singularities it is sufficient to study only three of them, the Ricci scalar R, $R_{\mu\nu}R^{\mu\nu}$ and the so-called Kretschmann scalar $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$. For the line element (3.15), $\tilde{R}$, $\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}$ and the Kretschmann scalar vanish. Therefore, the singular point is not an essential point, that is, it can be removed by a coordinate transformation.

By considering anti-symmetric tensor field $\tilde{b}_{\mu\nu}$ of the action (3.11), one quickly finds that the only non-zero component of $\tilde{H}$ is $\tilde{H}_{\tilde{y}\tilde{u}\tilde{v}} = -\frac{e^{-\tilde{v}}-4}{2(e^{-\tilde{v}}-2)^2}$. Consequently, the only non-zero component of $(\tilde{H}^2)_{\mu\nu}$ is $2\tilde{H}_{\tilde{v}\tilde{y}\tilde{u}}\tilde{H}^{\tilde{y}\tilde{u}}{}_{\tilde{v}} = -\frac{1}{2}\Big(\frac{e^{-\tilde{v}}-4}{e^{-\tilde{v}}-2}\Big)^2$ and $\tilde{H}^2 = 0$, too. Inserting the above results in equations (3.12), the conformal invariance conditions up to one-loop (the vanishing of the one-loop B-functions) are satisfied with the dilaton field

$$\tilde{\Phi}(\tilde{x}^{\mu}) = \tilde{\Phi}_0 + \tilde{\Phi}_1\tilde{v} + \ln(\frac{e^{-\tilde{v}}}{e^{-\tilde{v}}-2}), \tag{3.16}$$

where $\tilde{\Phi}_0$ and $\tilde{\Phi}_1$ are the constants of integration.

At the end of this subsection we check an interesting result. The dilaton fields (3.14) with $\mathcal{C} = 0$ and (3.16) with $\tilde{\Phi}_0 = \Phi_0$, $\tilde{\Phi}_1 = 0$ satisfy the following transformations[4]

$$\begin{aligned}
\Phi(x^{\mu}) &= \Phi_0 + \ln\big(\det E^{+}(g)\big) - \ln\big(\det E_0^{+}(e)\big),\\
\tilde{\Phi}(\tilde{x}^{\mu}) &= \Phi_0 + \ln\big(\det \tilde{E}^{+}(\tilde{g})\big).
\end{aligned} \tag{3.17}$$

The above transformations have been obtained by quantum considerations based on a regularization of a functional determinant in a path integral formulation of Poisson–Lie duality [20] (see also Ref. [21]).

3.1.2. *Conformal invariance of the dual sigma model up to the two-loop B-functions*

The two-loop B-functions found by Hull and Townsend [17] (see also Ref. [18]) are given by

$$\begin{aligned}
B^{g}_{\mu\nu} &= -\alpha'\Big[R_{\mu\nu} - H_{\mu\rho\sigma}H^{\rho\sigma}{}_{\nu} + \nabla_{\mu}\nabla_{\nu}\Phi\Big] - \frac{1}{2}\alpha'^2\Big[R_{\mu\rho\sigma\lambda}R_{\nu}{}^{\rho\sigma\lambda} + 2R_{\mu\rho\sigma\nu}(H^2)^{\rho\sigma}\\
&\quad + 2R_{\rho\sigma\lambda(\mu}H_{\nu)}{}^{\lambda\delta}H^{\rho\sigma}{}_{\delta} + \frac{1}{3}(\nabla_{\mu}H_{\rho\sigma\lambda})(\nabla_{\nu}H^{\rho\sigma\lambda}) - (\nabla_{\lambda}H_{\rho\sigma\mu})(\nabla^{\lambda}H^{\rho\sigma}{}_{\nu})\\
&\quad + 2H_{\mu\rho\sigma}H_{\nu\lambda\delta}H^{\eta\delta\sigma}H_{\eta}{}^{\lambda\rho} - 2H_{\mu\rho}{}^{\sigma}H_{\nu\sigma\lambda}(H^2)^{\lambda\rho}\Big] + \mathcal{O}(\alpha'^3),\\
B^{b}_{\mu\nu} &= -\alpha'\Big[-\nabla^{\lambda}H_{\lambda\mu\nu} + \nabla^{\lambda}\Phi' H_{\mu\nu\lambda}\Big] - \frac{1}{2}\alpha'^2\Big[2\nabla^{\lambda}H^{\rho\sigma}{}_{[\nu}R_{\mu]\lambda\rho\sigma} + 2(\nabla_{\lambda}H_{\rho\mu\nu})(H^2)^{\lambda\rho}\\
&\quad - 4(\nabla^{\lambda}H^{\rho\sigma}{}_{[\nu})H_{\mu]\rho\delta}H_{\lambda\sigma}{}^{\delta}\Big] + \mathcal{O}(\alpha'^3),\\
B^{\Phi} &= -\alpha'\Big[-\frac{1}{2}\nabla^2\Phi' + \frac{1}{2}(\nabla\Phi')^2 - \frac{1}{3}H_{\mu\nu\rho}H^{\mu\nu\rho}\Big] - \frac{1}{2}\alpha'^2\Big[\frac{1}{4}R_{\mu\rho\sigma\lambda}R^{\mu\rho\sigma\lambda}\\
&\quad - \frac{1}{3}(\nabla_{\lambda}H_{\mu\nu\rho})(\nabla^{\lambda}H^{\mu\nu\rho}) - \frac{1}{2}H^{\mu\nu}{}_{\lambda}H^{\rho\sigma\lambda}R_{\rho\sigma\mu\nu} - R_{\mu\nu}(H^2)^{\mu\nu}\\
&\quad + \frac{3}{2}(H^2)_{\mu\nu}(H^2)^{\mu\nu} + \frac{5}{6}H_{\mu\nu\rho}H^{\mu}{}_{\sigma\lambda}H^{\nu\sigma}_{\ \ \delta}H^{\rho\lambda\delta}\Big] + \mathcal{O}(\alpha'^3),
\end{aligned} \tag{3.18}$$

[4] For our example, the sigma model constant matrix $E_0^{+}(e)$ has given by (3.9). The background fields $E^{+}(g)$ and $\tilde{E}^{+}(\tilde{g})$ have explicitly written in (4.28).

where $\Phi' = \Phi + \alpha' q H^2$, $(H^2)^{\mu\nu} = H^{\mu\rho\sigma} H_{\rho\sigma}{}^{\nu}$ and $R_{\mu\rho\sigma\lambda}$ is the Riemann tensor field. We note that round brackets denote the symmetric part on the indicated indices whereas square brackets denote the anti-symmetric part. For the line element (3.15) of the dual model one finds that the only non-zero components of $\tilde{R}_{\mu\nu\rho\sigma}$ and $\tilde{R}^{\mu\nu\rho\sigma}$ are $\tilde{R}_{\tilde{y}\tilde{v}\tilde{u}\tilde{v}} = -\frac{e^{-2\tilde{v}}-4e^{-\tilde{v}}+16}{4(e^{-\tilde{v}}-2)^3}$ and $\tilde{R}^{\tilde{x}\tilde{y}\tilde{x}\tilde{u}} = -\frac{e^{-2\tilde{v}}-4e^{-\tilde{v}}+16}{4(e^{-\tilde{v}}-2)}$, respectively. Furthermore, we find that the only non-zero components of $(\tilde{H}^2)^{\mu\nu}$ and $\nabla_\lambda \tilde{H}_{\mu\nu\rho}$ are $2\tilde{H}^{\tilde{x}\tilde{y}\tilde{u}} \tilde{H}_{\tilde{y}\tilde{u}}{}^{\tilde{x}} = -\frac{1}{2}\left(\frac{e^{-\tilde{v}}-4}{e^{-\tilde{v}}-2}\right)^2$ and $\nabla_{\tilde{v}} \tilde{H}_{\tilde{y}\tilde{u}\tilde{v}} = \frac{e^{-\tilde{v}}}{(e^{-\tilde{v}}-2)^3}$. We must also note that since $\tilde{H}^2 = 0$, hence, $\tilde{\Phi}' = \tilde{\Phi}$. Putting all these together into equations (3.18) we conclude that the vanishing of the two-loop B-functions satisfy. Therefore the dual model is conformally invariant up to two-loop order.

4. Worldsheet boundary conditions under the Poisson–Lie T-duality

In this section we generally consider the worldsheet boundary conditions and their transformation under the Poisson–Lie T-duality. Then, we study the worldsheet boundary conditions specially for the H_4 WZW model. Consider a d-dimensional target space with Dp-branes, i.e., there are $d-(p+1)$ Dirichlet directions along which the field x^i $(i = p+1, \ldots, d-1)$ is frozen. At any given point on a Dp-brane we can choose local coordinates such that x^i are the directions normal to the brane and x^m $(m = 0, \ldots, p)$ as label Neumann directions are coordinates on the brane. Such a coordinate system is called adapted to the brane [22]. With this choice the Dirichlet condition takes the familiar form

$$\partial_\tau x^i = 0, \qquad i = p+1, \ldots, d-1. \tag{4.1}$$

The worldsheet boundary is by definition confined to a D-brane. Since the boundary relates left-moving fields $\partial_+ x^\mu$ to the right-moving fields $\partial_- x^\mu$, one can make a general ansatz for this relation. The goal is then to find the restrictions on this ansatz arising from varying the action (2.8). The most general local boundary condition may be expressed as [23]

$$\partial_- x^\rho = \mathcal{R}^\rho{}_\nu(x^\mu)\, \partial_+ x^\nu, \tag{4.2}$$

where $\mathcal{R}^\rho{}_\nu$ is a locally defined object which is called the *gluing matrix*. This matrix encodes the information about the Neumann and Dirichlet directions in its the eigenvalues and eigenvectors. We assume that $\mathcal{R}^\rho{}_\nu$ is in the form of a 2×2 block matrix as

$$\mathcal{R}^\rho{}_\nu(x^\mu) = \left(\begin{array}{c|c} \mathcal{R}^m{}_n & 0 \\ \hline 0 & \mathcal{R}^i{}_j \end{array}\right), \tag{4.3}$$

where the submatrices $\mathcal{R}^m{}_n$ and $\mathcal{R}^i{}_j$ are Neumann–Neumann and Dirichlet–Dirichlet parts, respectively. By going to adapted coordinates at a point and by using equations (4.1) and (4.2) we get $\mathcal{R}^i{}_j = -\delta^i{}_j$. The Neumann condition $\mathcal{R}^m{}_n$ still remains very general.

The boundary conditions mentioned above preserve conformal invariance at the boundary. We know that each symmetry corresponds to a conserved current, obtained by varying the action with respect to the appropriate field. For the case of conformal invariance, the corresponding current is the energy–momentum tensor and is derived by varying the action (2.8) with respect to the metric $h_{\alpha\beta}$. Its components in lightcone coordinates are [22]

$$T_{\pm\pm} = \partial_\pm x^\mu\, g_{\mu\nu}\, \partial_\pm x^\nu. \tag{4.4}$$

The T_{++} component is called the left-moving current, whereas T_{--} is referred to as right-moving current. In the conformally invariant case, energy–momentum conservation requires that the T_{++} and T_{--} components depend only on σ^+ and σ^-, respectively. To ensure conformal symmetry on the boundary, we need to impose boundary conditions on the currents (4.4). In general, one can find the boundary condition for a given current by using its associated charge. Applied to the energy–momentum tensor, the result is

$$T_{++} - T_{--} = 0. \tag{4.5}$$

Now, using the equations (4.2), (4.4) and (4.5) we find

$$\mathcal{R}^\rho{}_\mu\, g_{\rho\sigma}\, \mathcal{R}^\sigma{}_\nu = g_{\mu\nu}. \tag{4.6}$$

This condition states that the gluing matrix $\mathcal{R}^\mu{}_\nu$ preserves the metric $g_{\mu\nu}$.

In the next, we investigate structures on D-branes. We begin by defining a Dirichlet projector $\mathcal{Q}^\mu{}_\nu$ on the worldsheet boundary, which projects vectors onto the space normal to the brane. These vectors (Dirichlet vectors) are eigenvectors of $\mathcal{R}^\mu{}_\nu$ with eigenvalue -1. Hence we can use it to write the Dirichlet condition (4.1) on the desired covariant form

$$\mathcal{Q}^\mu{}_\nu\, \partial_\tau x^\nu = 0. \tag{4.7}$$

By contracting equations (4.2) and (4.7), we then obtain

$$\mathcal{Q}^\mu{}_\rho\, \mathcal{R}^\rho{}_\nu = \mathcal{R}^\mu{}_\rho\, \mathcal{Q}^\rho{}_\nu = -\mathcal{Q}^\mu{}_\nu. \tag{4.8}$$

Similarly, we may define a Neumann projector $\mathcal{N}^\mu{}_\nu$ which projects vectors onto the tangent space of the brane (vectors tangent to the brane are eigenvectors of $\mathcal{R}^\mu{}_\nu$ with eigenvalue 1) and is defined as complementary to $\mathcal{Q}^\mu{}_\nu$, i.e.,

$$\mathcal{N}^\mu{}_\nu = \delta^\mu{}_\nu - \mathcal{Q}^\mu{}_\nu, \qquad \mathcal{N}^\mu{}_\rho\, \mathcal{Q}^\rho{}_\nu = 0. \tag{4.9}$$

The Neumann projector satisfies the following conditions [24,25]

$$\mathcal{N}^\rho{}_\mu\, \mathcal{E}_{\sigma\rho}\, \mathcal{N}^\sigma{}_\nu - \mathcal{N}^\rho{}_\mu\, \mathcal{E}_{\rho\sigma}\, \mathcal{N}^\sigma{}_\lambda\, \mathcal{R}^\lambda{}_\nu = 0, \tag{4.10}$$

$$\mathcal{N}^\mu{}_\rho\, g_{\mu\nu}\, \mathcal{Q}^\nu{}_\sigma = 0, \tag{4.11}$$

$$\mathcal{N}^\mu{}_\gamma\, \mathcal{N}^\rho{}_\nu\, \mathcal{N}^\delta{}_{[\mu,\rho]} = 0. \tag{4.12}$$

The condition (4.10) is a condition on the Neumann–Neumann part of $\mathcal{R}^\mu{}_\nu$. In fact it states the definition of the b-field. In adapted coordinates, for a spacefilling brane (along the Neumann directions) equation (4.10) implies, schematically,[5] $\mathcal{R} = \mathcal{E}^{-1}\, \mathcal{E}^T$. The condition (4.11) implies the diagonalization of the metric with respect to the D-brane and the latter condition is the integrability condition for $\mathcal{N}^\mu{}_\nu$ [25].

Before starting the main result of this section, let us first to write down the boundary conditions (4.2), (4.6), (4.8) and (4.10)–(4.12) in the Lie algebra frame (related to the model on the Lie group). To this end, we use relations $\Omega_{ab} = (R^{-1})^\mu{}_a\, g_{\mu\nu}\, (R^{-1})^\nu{}_b$, $\mathcal{R}^a{}_b = R_\mu{}^a\, \mathcal{R}^\mu{}_\nu\, (R^{-1})^\nu{}_b$ and similarly for $\mathcal{Q}^\mu{}_\nu$ and $\mathcal{N}^\mu{}_\nu$. Then, we get

$$R^a_- = \mathcal{R}^a{}_b\, R^b_+, \tag{4.13}$$

$$\mathcal{R}^c{}_a\, \Omega_{cd}\, \mathcal{R}^d{}_b = \Omega_{ab}, \tag{4.14}$$

[5] The superscript “T” means transposition of the matrix.

$$\mathcal{Q}^a{}_b\, \mathcal{R}^b{}_c = \mathcal{R}^a{}_b\, \mathcal{Q}^b{}_c = -\mathcal{Q}^a{}_c, \tag{4.15}$$

$$\mathcal{N}^d{}_a\, E^+_{cd}\, \mathcal{N}^c{}_b - \mathcal{N}^d{}_a\, E^+_{dc}\, \mathcal{N}^c{}_e\, \mathcal{R}^e{}_b = 0, \tag{4.16}$$

$$\mathcal{N}^c{}_a\, \Omega_{cd}\, \mathcal{Q}^d{}_b = 0, \tag{4.17}$$

$$\mathcal{N}^c{}_a\, \mathcal{N}^e{}_b\, \mathcal{N}^d{}_{[c,e]} = 0. \tag{4.18}$$

From the relation (4.13) it is seen that the object $\mathcal{R}^a{}_b$ as a gluing map between currents at the worldsheet boundary maps R^a_+ to R^a_-, which are elements of the Lie algebra. As explained above in adapted coordinates, the Dirichlet–Dirichlet block of $\mathcal{R}$ is $-\delta^i{}_j$. To obtain the non-zero Neumann–Neumann block of $\mathcal{R}$ one must use the condition (4.16). Then $\mathcal{R}$ takes, schematically, the following form [23]

$$\mathcal{R} = \left(\begin{array}{c|c} (\mathcal{N}^T E^+ \mathcal{N})^{-1}\, (\mathcal{N}^T E^+ \mathcal{N})^T & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right). \tag{4.19}$$

To continue, we obtain the transformation of the gluing matrix which defines how the Poisson–Lie T-duality acts on the sigma model boundary conditions. In this way, we use the canonical transformation of the Poisson–Lie T-duality transformations found by Sfetsos [26] and [27]. The canonical transformation between the pairs of variables $(R^a_\sigma\ ,\ P_a)$ and $\big((\tilde{R}_\sigma)_a\ ,\ \tilde{P}^a)$ is given by [27]

$$R^a_\sigma = (\delta^a{}_b - \Pi^{ac}\tilde{\Pi}_{cb})\tilde{P}^b - \Pi^{ab}(\tilde{R}_\sigma)_b, \tag{4.20}$$

$$P_a = \tilde{\Pi}_{ab}\tilde{P}^b + (\tilde{R}_\sigma)_a, \tag{4.21}$$

where

$$R^a_\sigma = \frac{1}{2}(R^a_+ - R^a_-), \tag{4.22}$$

$$P_a = (R^{-1})^\mu{}_a\, P_\mu = \frac{1}{2}(E^+_{ba} R^b_+ + E^+_{ab} R^b_-). \tag{4.23}$$

Now, one can use equations (3.4), (4.22) and (4.23) to write the canonical transformations (4.20) and (4.21) as a transformation from $R_\pm$ to $\tilde{R}_\pm$. The resulting map is

$$(\tilde{R}_+)_a = (\tilde{E}^{+^{-1}})_{ba}\, (E_0^{+^{-1}})^{cb}\, E^+_{dc}\, R^d_+, \tag{4.24}$$

$$(\tilde{R}_-)_a = -(\tilde{E}^{+^{-1}})_{ab}\, (E_0^{+^{-1}})^{bc}\, E^+_{cd}\, R^d_-. \tag{4.25}$$

Ultimately, by substituting equation (4.13) into (4.25) and then by using (4.24) we obtain

$$(\tilde{R}_-)_a = \tilde{\mathcal{R}}_a{}^b\, (\tilde{R}_+)_b, \tag{4.26}$$

in which [23]

$$\tilde{\mathcal{R}}_a{}^b = -(\tilde{E}^{+\,-1})_{ac}\, (E_0^{+\,-1})^{cd}\, E^+_{de}\, \mathcal{R}^e{}_f\, (E^{+\,-1})^{hf}\, E^+_{0\,gh}\, \tilde{E}^{+\,bg}, \tag{4.27}$$

is the duality transformation of the gluing matrix. Note that one can immediately get $\det\tilde{\mathcal{R}} = \det(-\mathcal{R})$. We use relation (4.27) for analyzing the dual branes of the H_4 WZW model in the following subsection.

4.1. Example

In this subsection we study the worldsheet boundary conditions and D-branes in conformal Poisson–Lie symmetric sigma models generated by the Heisenberg Lie group H_4 and its dual pair, i.e., the Lie group $A_2 \oplus 2A_1$. In this example we analyze the consequences of the gluing matrix duality transformation (4.27). The conformal Poisson–Lie T-dual sigma models built by the Drinfel'd double $\big(H_4\ ,\ A_2 \oplus 2A_1\big)$ have been given by relation (3.10) and (3.11). For these models the background fields $E^+_{ab}(g)$ and $\tilde{E}^{+ab}(\tilde{g})$ can be obtained from equations (3.4)–(3.6) and (3.9) as follows:

$$E^+_{ab}(g) = \begin{pmatrix} 0 & 0 & ye^x & 1 \\ 0 & 0 & -1 & 0 \\ -ye^x & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{E}^{+ab}(\tilde{g}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{e^{-\tilde{v}}-2} & \frac{\tilde{u}}{e^{-\tilde{v}}-2} \\ 0 & -e^{\tilde{v}} & 0 & \tilde{y} \\ 1 & \tilde{u}\, e^{\tilde{v}} & \frac{\tilde{y}e^{-\tilde{v}}}{e^{-\tilde{v}}-2} & \frac{2\tilde{y}\tilde{u}}{e^{-\tilde{v}}-2} \end{pmatrix}. \tag{4.28}$$

Thus, inserting $E^+_{ab}(g)$ and $\tilde{E}^{+ab}(\tilde{g})$ from (4.28) together with the $E^+_0(e)$ of relation (3.9) into equation (4.27) one can get the dual gluing matrix $\tilde{\mathcal{R}}$ for any given original matrix $\mathcal{R}$. From the form of the matrix $E^+_{ab}(g)$ (relation (4.28)) it is seen that 1×1, 2×2 and 3×3 submatrices of $E^+_{ab}(g)$ are not invertible. Therefore according to relation (4.19) the brane does not include one, two and or three Neumann directions. Consequently, we can have two different types of D-branes: $D(-1)$ and $D3$; that is, all directions are either Dirichlet or Neumann. In the following we compute the dual gluing matrix for each of these cases.

Case (1): In this case all directions are Dirichlet; that is, $\mathcal{Q}^a{}_b = \delta^a{}_b$ and $\mathcal{N}^a{}_b = 0$. Since the numbers of Neumann directions are $p+1$, so we have, in this case, a $D(-1)$-brane. From relation (4.19) the corresponding gluing matrix is given by

$$\mathcal{R}^a{}_b = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.29}$$

Then, equation (4.27) yields the dual gluing matrix

$$\tilde{\mathcal{R}}_a{}^b = \begin{pmatrix} 1 & \frac{2(\tilde{u}+ye^x)}{e^{-\tilde{v}}-2} & 2\tilde{y} & \frac{4\tilde{y}\tilde{u}+2y\tilde{y}e^{x-\tilde{v}}(3-e^{-\tilde{v}})}{e^{-\tilde{v}}-2} \\ 0 & \frac{-e^{-\tilde{v}}}{e^{-\tilde{v}}-2} & 0 & \frac{-2\tilde{y}e^{-\tilde{v}}}{e^{-\tilde{v}}-2} \\ 0 & 0 & 2e^{\tilde{v}}-1 & 2(ye^{x-\tilde{v}} - \tilde{u}e^{\tilde{v}} - 2ye^x) \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.30}$$

It has determinant $\det\tilde{\mathcal{R}} = \det(-\mathcal{R}) = 1$, so the dual brane may include the following directions:

(1.i) Four Dirichlet directions, i.e., the dual brane is also a $D(-1)$-brane. The dual $D(-1)$-brane is nontrivially embedded in the dual manifold, and the embedding can be found by diagonalizing $\tilde{\mathcal{R}}$. Then, in this case, the only solution is $\tilde{\mathcal{R}} = -\mathbb{1}$, which happens only for backgrounds $E^+(g)$ and $\tilde{E}^+(\tilde{g})$ such that $\widetilde{E}^+(\widetilde{E}^{+T})^{-1} = -(E^+_0)^{-1}E^+(E^{+T})^{-1}E^{+T}_0$.

(1.ii) Two Dirichlet directions and two Neumann directions, i.e., it is a $D1$-brane.

(1.iii) Zero Dirichlet directions. This is a $D3$-brane whose embedding in $\tilde{G}$ is given by $\tilde{\mathcal{R}}$. Since it is spacefilling it should satisfy the dual version of equation (4.16), $\tilde{\mathcal{R}} = \tilde{E}^{+\,-1}\tilde{E}^{+\,T}$. Thus, relation (4.27) reduces to $\mathbb{1} = (E_0^+)^{-1}E^+(E^{+\,T})^{-1}E_0^{+\,T}$, implying $\Pi\big(E_0^+ + E_0^{+\,T}\big) = 0$, and hence since the condition $E_0^+ + E_0^{+\,T} = 0$ is equal to a vanishing metric we find $\det\Pi = 0$ [23].

Case (2): In this case we have a spacefilling brane, i.e., a $D3$-brane. The corresponding gluing matrix according to equation (4.16) is given by

$$\mathcal{R} = E^{+\,-1}E^{+\,T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2ye^x & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2ye^x & 1 \end{pmatrix}, \tag{4.31}$$

and from (4.27) the dual gluing matrix reads

$$\tilde{\mathcal{R}} = \begin{pmatrix} -1 & \frac{-2\tilde{u}}{e^{-\tilde{v}}-2} & -2\tilde{y} & \frac{-4\tilde{y}\tilde{u}}{e^{-\tilde{v}}-2} \\ 0 & \frac{e^{-\tilde{v}}}{e^{-\tilde{v}}-2} & 0 & \frac{2\tilde{y}e^{-\tilde{v}}}{e^{-\tilde{v}}-2} \\ 0 & 0 & 1-2e^{\tilde{v}} & 2\tilde{u}e^{\tilde{v}} \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.32}$$

Its determinant is $\det\tilde{\mathcal{R}} = 1$, so the dual brane may include the following directions:

(2.i) Four Dirichlet directions. In this case we obtain exactly the reverse situation of case (1.iii), that is, we have $\tilde{\mathcal{R}} = -1$ and the $D3$-brane is dual to a $D(-1)$-brane provided the $\det\tilde{\Pi}$ on $\tilde{G}$ vanishes.

(2.ii) Two Dirichlet directions and two Neumann directions, i.e., a $D1$-brane.

We note that since $\det\tilde{\mathcal{R}} = 1$, the dual brane may include zero Dirichlet directions, i.e., a $D3$-brane. But, on the other hand, since it is spacefilling it should satisfy the dual version of (4.16), $\tilde{\mathcal{R}} = \tilde{E}^{+\,-1}\tilde{E}^{+\,T}$. In this situation, equation (4.27) would require $E_0^+ + E_0^{+\,T} = 0$ and hence a vanishing metric. Thus, we conclude that $D3$-branes are dual either to $D1$-branes or $D(-1)$-branes provided by $\det\tilde{\Pi} = 0$, but that $D3$-branes are never dual to $D3$-branes.

5. Concluding remarks

In the present work, first we have constructed a WZW model based on the Heisenberg Lie group H_4 by choosing a convenient parametrization of the group. The most interesting feature of our results is the existence of the Poisson–Lie symmetry in the H_4 WZW model. We have shown that the Poisson–Lie T-duality relates the H_4 WZW model to a sigma model defined on the dual Lie group $A_2 \oplus 2A_1$. We moreover explicitly worked out the dual model is conformal up to two-loop order and in this manner we have obtained the general form of the dilaton field of the dual model. We have obtained the gluing matrices for the H_4 WZW model and its dual model by using the duality map of the gluing matrix obtained by the canonical transformation description of the Poisson–Lie T-duality transformations. We have shown that there are two different cases of the worldsheet boundary conditions for the H_4 WZW model; all directions are either Dirichlet ($D(-1)$-brane) or Neumann ($D3$-brane). Case (1) refers to a $D(-1)$-brane; in this case the dual brane includes four Dirichlet directions ($D(-1)$-brane) or two Dirichlet directions ($D1$-brane), and/or zero Dirichlet directions ($D3$-brane). In case (2) we have a spacefilling brane, i.e., a $D3$-brane; in this case we have shown that $D3$-branes are dual either to $D1$-branes or $D(-1)$-branes provided by $\det \tilde{\Pi} = 0$, but that $D3$-branes are never dual to $D3$-branes.

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