A theoretical consideration of a free convective boundary layer on an isothermal horizontal conic

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Abstract

Approximate analytical solution of simplified Navier–Stokes and Fourier–Kirchhoff equations describing free convective heat transfer from isothermal surface of horizontal conic of the base angle \( \alpha \) has been presented. The solution is based on the typical for natural convection assumption that the normal to the surface component of velocity is negligibly small in comparison with the tangential one. The results obtained for boundary cases of conic under considerations are in good agreement with known solutions for a horizontal cylinder \( \alpha = \pi/2 \) and a vertical round plate \( \alpha = 0 \).

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1. Introduction

The results of theoretical and experimental study of free convective heat transfer from different configurations of heating objects are widely published and they are very useful to determine convective heat losses from apparatus, devices, pipes in industrial or energetic installations, electronic equipment, architectonic objects and so on by engineers and designers. Unfortunately the available data are not complete. There are many publications on flat (vertical, horizontal and inclined) surfaces as well as cylindrical and spherical ones. From the analysis of literature data it is obvious that for conical configurations of heating surface one can only find a few papers for example [1–7]. In the review Churchill’s paper [8], among about 120 results there are only four that

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are concerned with conical vertical surfaces. Since for horizontal conic we have found only one paper (Oosthuizen [7]), we decided to continue such investigations and maybe extend this subject. Moreover in [7] only experimental studies have been performed.

We started from a vertical cone [2]. In this paper we proposed a solution based on the cylindrical symmetry with respect to the gravity force direction. To build the solution we used our former results for inclined isothermal plate. We verified the solution experimentally for two fluids (air and water) and six conics of different base angle.

The presented paper is devoted to the theoretical solution of the problem of the description of a boundary layer near the isothermal surface of horizontal conic. The horizontal position of the cone does not allow to apply the above mentioned approach, because of symmetry break by the gravity field. The proposed physical model of the flow is based on boundary layer approximations made in the momentum and energy equations, which permit only convective tangential heat and momentum transfer. To apply such a model for the investigated geometry related to the gravity field direction (horizontal cone), we use two kinds of coordinates: cylindrical coordinate system coupled with the cone surface and the special local one. The choice of the local coordinate system is motivated by the physical model in which the coordinate curves are connected with stream lines. This statement is in good agreement with a picture of the pattern flow we observed directly [9]. The tangent vector of the curve is directed along the sum of the surface interaction force and the gravitation ones.

We consider an approximate analytical solution of simplified convective flow induced by an isothermal conical body with horizontal axis of symmetry. The choice of the coordinate system in the frame of the typical for laminar natural convection simplifications and for $Pr \approx 1$ allows to diminish the number of basic equations. After transformations the basic set of two equations for the thermal boundary layer results in the ordinary differential equation of second order with a singularity. As the method of solution we use power series expansion near the point of singularity of the basic equation for the boundary layer thickness as a function of local coordinates.

### 2. The physical model

The geometrical and physical variables for natural convection heat transfer from a horizontal isothermal conic with proposed notations have been shown in Fig. 1. The base angle $\alpha$ is a parameter of a conical surface which varied from $\alpha = \pi/2$—horizontal cylinder to $\alpha = 0$—round vertical plate.

The isothermal lateral conic surface is described by the equation

$$x^2 + y^2 - z^2 \cot^2(\alpha) = 0. \quad (1)$$

Each point on this surface $0 \leq z \leq H$ is described by $x, y, z$ that satisfy (1) or by $\rho = z \cot \alpha$ and $\epsilon$, where

$$x = \rho \sin(\epsilon), \quad y = \rho \cos(\epsilon). \quad (2)$$

At every arbitrary point $M$ of lateral conical surface $\Sigma$ one may distinguish two tangent vectors $\tau_\rho$ and $\tau_\epsilon$ and one normal $\tau$ to surface.
We have introduced coordinates $r, s$ that, from the point of gravity forces, look more natural for our problem. Decomposition of the gravity according to these coordinates gives two components of gravity force $g_r$ and $g_s$ that act in normal and in a tangent direction to the lateral surface of the conic.

The unit vector $\hat{r}$, connected with $r$ and $s$, was calculated as follows

$$\hat{r} = \frac{\vec{r}}{\rho}, \quad \hat{\epsilon} = \frac{\vec{\epsilon}}{\epsilon},$$

where $\vec{r} = r(x, y, z) \in \Sigma$.

$$\tau_\rho = \frac{\partial \rho}{\partial \rho}, \quad \tau_\epsilon = \frac{\partial \rho}{\partial \epsilon}, \quad \tau_\rho = \frac{\partial y}{\partial \rho} = \cos(\epsilon), \quad \tau_\rho = \frac{\partial z}{\partial \rho} = \tan(\epsilon),$$

$$\tau_\rho = \frac{\partial x}{\partial \rho} = \rho \cos(\epsilon), \quad \tau_\epsilon = \frac{\partial y}{\partial \epsilon} = -\rho \sin(\epsilon), \quad \tau_\epsilon = \frac{\partial z}{\partial \epsilon} = 0.$$  \hspace{1cm} (3)

We have introduced coordinates $\sigma$, $\tau$ that, from the point of gravity forces, look more natural for our problem. Decomposition of the gravity according to these coordinates gives two components of gravity force $g_\sigma$ and $g_\tau$ that act in normal and in a tangent direction to the lateral surface of the conic.

The unit vector $\hat{\sigma}$, connected with $\tau$ and $\epsilon$, was calculated as follows

$$\hat{\sigma} = \frac{\vec{\sigma}}{\epsilon},$$

where the direction and parameter of the normal to the surface vector are

$$\vec{\sigma} = \vec{r}_\rho \times \vec{r}_\epsilon = \vec{i}(\rho \tan \alpha \sin \epsilon) + \vec{j}(\rho \tan \alpha \cos \epsilon) - \rho \vec{k}, \quad \vec{N} = \frac{\rho}{\cos \alpha},$$

$$N = \frac{\rho}{\cos \alpha},$$ \hspace{1cm} (4)
hence

\[ \vec{\sigma} = \vec{i} \sin \alpha \sin \epsilon + \vec{j} \sin \alpha \cos \epsilon - \vec{k} \cos \alpha. \]  

(7)

The choice of the coordinate system in Fig. 1 gives the vector of acceleration of gravity as \( \vec{g} = (-g, 0, 0) = -\vec{i}g \), which together with (7) gives the projection to the gravity field of the normal vector \( \sigma_n = (g, \vec{\sigma}) = g \sin \alpha \sin \epsilon \). The tangent unit vector can be calculated in a similar manner to \( \vec{\sigma} \):

\[ \vec{\tau} = \frac{-\vec{i}(1 - \sin^2 \alpha \sin^2 \epsilon) + \vec{j} \sin^2 \alpha \sin \epsilon \cos \epsilon - \vec{k} \cos \alpha \sin \epsilon \sin \epsilon}{\sqrt{1 - \sin^2 \alpha \sin^2 \epsilon}}. \]

(8)

Let the unit vector \( \vec{\tau} \) defines a curve \( S \) on the surface. At any point \( M_i \) the components of gravity \( g_\sigma \) and \( g_\tau \) are the normal and tangent ones to this curve \( S \) (Fig. 2). For the limiting cases (\( \alpha = \pi/2 \)—horizontal cylinder) or (\( \alpha = 0 \)—vertical round plate) the curves of longitudinal sections of the conic transfer into cross-sectional circles of the cylinder or into chords of the round vertical conic base.

The property of gravity and surface reaction fields caused that we decided to consider and solve the equations: Navier–Stokes, Fourier–Kirchhoff and continuity in these two characteristic directions \( \sigma \) and \( \tau \).

In this notation and after typical for natural convection assumptions according to Squire [10] and Eckert [11] (see also [12]):

- fluid is incompressible and its flow is laminar,
- inertia forces are negligible small in comparison with viscosity forces,
- thicknesses of the thermal and hydraulic boundary layers are the same (\( Pr \approx 1 \)), the Navier–Stokes equations may be written,
- physical properties of the fluid (the mass density \( \rho_f \), kinematic viscosity \( \nu \) and volumetric expansion \( \beta \)) in boundary layer and in the undisturbed region (index \( \infty \)) are constant.
tangent to the heated surface component of the velocity inside the boundary layer is significantly larger than normal one \( W_s \). From this assumption two marginal regions are excluded: first where the boundary layer arises \( \epsilon = \epsilon_m \) and the second where it is transferred into the free buoyant plum \( \epsilon = \epsilon_m \).

- temperature of the lateral conical surface \( T_w \) is constant,
- the temperature distribution inside the boundary layer can be used as a solution of Fourier–Kirchhoff equation [10,11] that depends on only the variable of normal distance from the heating surface

\[
\Theta = \frac{T - T_\infty}{T_w - T_\infty} = \left(1 - \frac{\sigma}{\delta}\right)^2 \quad \text{or} \quad T - T_\infty = \Delta T \left(1 - \frac{\sigma}{\delta}\right)^2.
\]  

The basic Navier–Stokes equations with such assumptions about the natural convective fluid dynamics are as follows

\[
\frac{\partial^2 W_s}{\partial \sigma^2} - g_s \beta (T - T_\infty) \frac{1}{\rho_f} \frac{\partial p}{\partial \tau} = 0, 
\]

\[
-g_s \beta (T - T_\infty) \frac{1}{\rho_f} \frac{\partial p}{\partial \sigma} = 0. 
\]

From Eqs. (6) and (8) we can evaluate the normal and tangent components of gravity as

\[
g_\sigma = \bar{\sigma} \cdot \vec{g} = -g \sin \alpha \sin \epsilon, 
\]

\[
g_\tau = \bar{\tau} \cdot \vec{g} = g \sqrt{1 - \sin^2 \alpha \sin^2 \epsilon}. 
\]

Plugging (12), (13) and (9) into (10) and (11) gives

\[
\frac{\partial^2 W_s}{\partial \sigma^2} - g_\beta \Delta T \left(1 - \frac{\sigma}{\delta}\right)^2 \sqrt{1 - \sin^2 \alpha \sin^2 \epsilon} \frac{1}{\rho_f} \frac{\partial p}{\partial \tau} = 0, 
\]

\[
-g_\beta \Delta T \sin \alpha \sin \epsilon \left(1 - \frac{\sigma}{\delta}\right)^2 = \frac{1}{\rho_f} \frac{\partial p}{\partial \sigma}. 
\]

3. Boundary conditions and solution

Integration of Eq. (15) for the boundary condition \( \sigma = \delta, p = p_\infty(\sigma \geq \delta) \) gives a formula for the pressure distribution in a boundary layer directed tangent to the heating surface.

\[
p = -p_\infty(\sigma \geq \delta) - \rho_f g_\beta \Delta T \sin \alpha \sin \epsilon \left(\sigma \frac{\sigma^2}{\delta} + \frac{\sigma^3}{3\delta^2} - \frac{\delta}{3}\right). 
\]  

Pressure \( p_\infty(\sigma \geq \delta) \) represents the excess of pressure over the hydrostatic pressure, existing on the border of the boundary layer on the following level, which is a function of fluid layer thickness over the heating surface. Because our considerations are concerned with unlimited space the
thickness of this layer is much greater than the thickness of the boundary layer \( \delta \) so the value of the pressure \( p_{\infty(\sigma \gg \delta)} \) is constant.

Differentiating Eq. (16) with respect to \( \tau \) gives

\[
\frac{\partial p}{\partial \tau} = \frac{\partial p}{\partial \epsilon} \frac{\partial \epsilon}{\partial \tau},
\]

where

\[
\frac{\partial p}{\partial \epsilon} = -g \beta T \sin \alpha \left[ \cos \epsilon \left( \sigma - \frac{\sigma^2}{\delta} + \frac{\sigma^3}{3 \delta^2} - \frac{\delta}{3} \right) + \sin \epsilon \left( \frac{\sigma^2}{\delta^2} - \frac{2 \sigma^3}{3 \delta^3} - \frac{1}{3} \right) \frac{d \delta}{d \epsilon} \right].
\]

(18)

Calculation of the derivative \( \partial \epsilon / \partial \tau \) is more complicated and needs some explanations shown in Appendix A. The result of the calculation is (A.6)

\[
\frac{d \epsilon}{d \tau} = \frac{(\cos \epsilon)^{\cos^2 \varphi + 1}}{\rho_0 \sqrt{1 - \sin^2 \epsilon \sin^2 \varphi}}.
\]

(19)

Substitution of (19) and (18) into (17) gives

\[
\frac{\partial p}{\partial \tau} = -\rho \gamma g \beta T \sin \alpha \frac{(\cos \epsilon)^{\cos^2 \varphi + 1}}{\rho_0 \sqrt{1 - \sin^2 \epsilon \sin^2 \varphi}} \left[ \cos \epsilon \left( \sigma - \frac{\sigma^2}{\delta} + \frac{\sigma^3}{3 \delta^2} - \frac{\delta}{3} \right) + \sin \epsilon \left( \frac{\sigma^2}{\delta^2} - \frac{2 \sigma^3}{3 \delta^3} - \frac{1}{3} \right) \frac{d \delta}{d \epsilon} \right].
\]

(20)

Plugging of the equality (20) into Eq. (14) leads to

\[
\sqrt{\frac{\partial^2 W_{\tau}}{\partial \sigma^2}} + \rho \gamma g \beta T \left\{ - \left( 1 - \frac{\sigma}{\delta} \right)^2 \sqrt{1 - \sin^2 \alpha \sin \epsilon} + \frac{\sin \alpha (\cos \epsilon)^{\cos^2 \varphi + 1}}{\rho_0 \sqrt{1 - \sin^2 (\epsilon) \sin^2 (\varphi)}} \frac{\cos \epsilon}{\cos \epsilon} \right. \\
\times \left( \sigma - \frac{\sigma^2}{\delta} + \frac{\sigma^3}{3 \delta^2} - \frac{\delta}{3} \right) + \sin \epsilon \left( \frac{\sigma^2}{\delta^2} - \frac{2 \sigma^3}{3 \delta^3} - \frac{1}{3} \right) \frac{d \delta}{d \epsilon} \right\} = 0.
\]

(21)

For the boundary condition (for \( \sigma = 0 \) and \( \delta, W_{\tau} = 0 \)) a double integration of Eq. (21) allows the evaluation of the formula of local and next mean velocity in boundary layer

\[
W_{\tau} = \frac{1}{\delta} \int_0^\delta W_{\tau} d\sigma = \frac{g \beta T \delta^2 (\cos \epsilon)^{\cos^2 \varphi + 1}}{\sqrt{1 - \sin^2 \epsilon \sin^2 \varphi}} \left( - \frac{1 - \sin^2 \epsilon \sin^2 \alpha}{4 \cos (\epsilon)^{\cos^2 \varphi + 1}} \right) \left( \frac{\sin \alpha \cos \epsilon}{180 \rho_0} \right) \left( \frac{\sin \alpha \sin \epsilon}{72 \rho_0} \right) \frac{d \delta}{d \epsilon}.
\]

(22)

Taking into account the choice of the unit vector direction the change in mass flow intensity is

\[
dm = -d(\rho \gamma A W_{\tau}),
\]

(23)

where \( A \) is the cross-section area of the boundary layer (see Fig. 3). The amount of heat necessary to create this change in mass flux is

\[
dQ = \Delta \Delta m = -\rho \gamma c_p (T - T_{\infty}) d(A W_{\tau}).
\]

(24)
Substitution of the mean value of the temperature

\[
\left( T - T_\infty \right) = \frac{1}{\delta} \int_0^\delta \Delta T \left( 1 - \frac{\sigma}{\delta} \right)^2 \, d\sigma = \frac{\Delta T}{3},
\]  

(25)
gives

\[
dQ = -\frac{1}{3} \rho_f c_p \Delta T \, d(AW_e).
\]  

(26)

The heat flux described by Eq. (26) may be compared with the heat flux determined by Newton’s equation (27):

\[
dQ = \alpha \Delta T \, dA_k = -\lambda \left( \frac{\partial \Theta}{\partial \sigma} \right)_{\sigma=0} \Delta T \, dA_k,
\]  

where \(dA_k\) is the control surface of the conic (see Fig. 3).

From simplifying assumption of the temperature profile inside boundary layer (9), the dimensionless temperature gradient on the heated surface may be evaluated as

\[
\left( \frac{\partial \Theta}{\partial \sigma} \right)_{\sigma=0} = -\frac{2}{\delta}.
\]  

(28)

Plugging the boundary condition (28) into Eq. (27) and equating the result with Eq. (26), one obtains dependence (29)

\[
\frac{1}{6\lambda} \rho_f c_p \delta \, d(AW_e) = -dA_k.
\]  

(29)
Derivation of formulas for the cross-sectional area and the control surface $A$ and $dA_k$ is shown in Fig. 3. As one can see in this figure, the differential width $d\zeta$ of the control surfaces $A$ and $dA_k$ is the module of the scalar product

$$d\zeta = |[\vec{\sigma} \times \vec{\tau}]d\vec{r}|,$$

where the vector product of normal (7) and tangent (8) to the surface vectors is

$$[\vec{\sigma} \times \vec{\tau}] = \left(0, \frac{-\cos \alpha}{\sqrt{1 - \sin^2 \alpha \sin^2 \epsilon}}, \frac{-\sin \alpha \cos \epsilon}{\sqrt{1 - \sin^2 \alpha \sin^2 \epsilon}} \right).$$  \hspace{1cm} (31)

Differentiation of the three dimensional coordinate vector $\vec{r} = r(x, y, z)$ (3) with the account of relations (2) and Eq. (A.3) leads to a relation for the differential form of this vector

$$d\vec{r} = d\rho_0 (\cos \epsilon)^{-\cos^2 \alpha}(\sin \epsilon, \cos \epsilon, \tan \alpha).$$  \hspace{1cm} (32)

Plugging Eqs. (31) and (32) into (30) allowed to found the width (33) and next the relations for the cross-sectional area $A$ as well as control surface $dA_k$

$$d\zeta = \frac{-(\cos \epsilon)^{1-\cos^2 \alpha} d\rho_0}{\cos \alpha \sqrt{1 - \sin^2 \alpha \sin^2 \epsilon}},$$

$$A = d\zeta \delta = \frac{-(\cos \epsilon)^{1-\cos^2 \alpha} d\rho_0 \delta}{\cos \alpha \sqrt{1 - \sin^2 \alpha \sin^2 \epsilon}},$$

$$dA_k = d\zeta \, d\tau = \frac{-(\cos \epsilon)^{-2\cos^2 \alpha} \rho_0 \, d\epsilon \, d\rho_0}{\cos \alpha}.$$  \hspace{1cm} (35)

Substituting Eqs. (22), (34) and (35) into Eq. (29) leads to the nonlinear differential equation

$$K\delta \left(X_1 \delta^3 + X_2 \frac{\delta^4}{\rho_0} + X_3 \frac{\delta^3}{\rho_0} \frac{d\delta}{d\epsilon}\right) = X_4 \rho_0 \, d\epsilon,$$  \hspace{1cm} (36)

where

$$K = \frac{RaR}{240R^3} = \frac{\rho_f c_p \, g \beta \Delta T}{240 \lambda}, \quad Ra_R = \frac{g \beta \Delta T R^3}{va},$$  \hspace{1cm} (37)

$$X_1 = -(\cos \epsilon)^{1-\cos^2 \alpha},$$  \hspace{1cm} (38)

$$X_2 = \frac{2}{9} \left(\cos \epsilon\right)^{3+\cos^2 \alpha} \frac{\sin \alpha}{(1 - \sin^2 \alpha \sin^2 \epsilon)},$$  \hspace{1cm} (39)

$$X_3 = \frac{5}{9} \left(\cos \epsilon\right)^{2+\cos^2 \alpha} \frac{\sin \alpha \sin \epsilon}{(1 - \sin^2 \alpha \sin^2 \epsilon)},$$  \hspace{1cm} (40)

$$X_4 = (\cos \epsilon)^{-2\cos^2 \alpha}.$$  \hspace{1cm} (41)
Differentiation in the relation (36) yields the second order equation

\[ X_3 (\delta \delta'' + 3 \delta'^2) + (4X_2 + X_3') \delta \delta' + X_2' \delta^2 + 3X_1 \rho_0 \delta' + X_1' \rho_0 \delta = \frac{\rho_0^2 X_4}{K \delta^3}. \]  

(42)

Let us introduce the new variables

\[ y(\epsilon) = \delta K^{1/3}, \quad r = \rho_0 K^{1/3}, \]  

(43)

so Eq. (42) has the form

\[
y^4(\epsilon) E \frac{\partial^2 y(\epsilon)}{\partial \epsilon^2} + 3y^3(\epsilon) E \frac{\partial y(\epsilon)}{\partial \epsilon} + y^2(\epsilon) \frac{\partial y(\epsilon)}{\partial \epsilon} G + y^4(\epsilon) H + y^4(\epsilon) F = r^2 (1 - \sin^2 \alpha \sin^2 \epsilon) \cos^{-2} \cos^2 \epsilon, \]

(44)

where the coefficients are defined by

\[ E = X_3 (1 - \sin^2 \alpha \sin^2 \epsilon) = \frac{5}{9} \cos^2 + \cos^2 \epsilon \sin \alpha \sin \epsilon, \]

(45)

\[ G = [y(4X_2 + X_3') + 3X_1 r](1 - \sin^2 \alpha \sin^2 \epsilon) \]

\[ = 3(\cos^1 - \cos^2 \epsilon \sin^2 \epsilon) r (\cos^2(\epsilon + \cos^2 \alpha) - \cos^2 \epsilon \cos^2 \alpha) + \frac{8}{9}(\cos^3 + \cos^2 \epsilon \sin \alpha) y(\epsilon), \]

(46)

\[ H = X_2'(1 - \sin^2 \alpha \sin^2 \epsilon) \]

\[ = \frac{2}{9} \sin \epsilon \sin \alpha \]

\[ \sin^2 \alpha \sin^2 \epsilon - 1 \cos^2 + \cos^2 \epsilon \epsilon (\sin^2 \epsilon \cos^4 \alpha + 3 \cos^2 \alpha + \cos^2 \epsilon), \]

(47)

\[ F = X_1' r(1 - \sin^2 \alpha \sin^2 \epsilon) = \frac{r \sin^2 \alpha (1 - \sin^2 \alpha \sin^2 \epsilon)}{\cos^2 + \cos^2 \epsilon} \sin \epsilon. \]

(48)

A search for explicit analytical solutions seems to be rather difficult. Below we consider an asymptotic solution as a power series in the vicinity of the point \( \epsilon = 0 \). This point is the singularity point of the equation: the coefficient by the second derivative is equal to zero when \( \epsilon = 0 \). The formal Taylor series expansion is

\[ y(\epsilon) = \sum_{i=0}^{\infty} c_i \epsilon^i = y(0) + g \epsilon + f \epsilon^2 / 2 + \cdots. \]

(49)

The coefficients of the expansion we determine directly from the differential equation (44) in the point \( \epsilon = 0 \). The equation gives connection of all coefficients with the first one \( y(0) \).

In such assumptions the unique parameter \( y(0) \) is defined via the boundary condition \( y(\epsilon_m) = 0 \) in the point \( \epsilon = -\epsilon_m = \arccos(\rho_0 / R) \).

The coefficients \( f \), \( g \) and \( y(0) \) as functions of the parameters \( r \) and \( \alpha \) are evaluated in the Appendix B. The approximate values of the coefficients are

\[ g = \frac{1}{3} \frac{r}{y^3(0)}, \]

(50)

\[ f = -\frac{1}{3} \frac{r^2}{y^7(0)}, \]

(51)
The last of them is the boundary layer thickness at the point \( \epsilon = 0 \).

4. Analysis of the solution

From Eq. (B.15) returning to the original dimensions for the boundary layer thickness in the approximation of Eq. (49) yields the form described by the relation

\[
y(0) = \sqrt{\frac{1}{2} \left( \frac{1}{6} + \frac{1}{6} \sqrt{\pi} \right)} r [\pi - 2 \arcsin(\rho_0/R)].
\]

The results of evaluation of the dimensionless boundary layer thickness \( \delta' = \delta/R \) as a function of the angle \( \epsilon \) and the parameters: dimensionless variable \( \rho_0' = \rho_0/R \) and Rayleigh number \( Ra \) is presented in Fig. 4.

The approximate solution obtained describes the structure of boundary layer and streamlines near horizontal isothermal conical surface. This solution is the first approximation (parabolic one) of the Taylor series at \( \epsilon = 0 \) (near the points of the horizontal plane that intersect the conic). The applied method allows to increase the number of terms in the series and hence to improve the accuracy of a solution. We do understand that the divergence of the series grows together with increase of \( \epsilon \). We studied a matching of the series (49) with additional asymptotic expansion at the point \( \epsilon = -\epsilon_m \). The expansion has the form \( P(\epsilon + \epsilon_m)(1 + x(\epsilon + \epsilon_m) + \beta(\epsilon + \epsilon_m)^2 \ldots) \). In the first approximation one has the function \( P(\epsilon + \epsilon_m) = A(\epsilon + \epsilon_m)^{1/4} \) that coincides with the classical solution for vertical isothermal plate. However, we do not place the results of the matching here because of the dimension of the article. Moreover, the account of this additional expansion changes the result only in small vicinity of the point \( \epsilon = -\epsilon_m \). We believe that the heat transfer coefficient also does not change much that is confirmed by our preliminary evaluations.

The construction of such a solution without the account of the factor \( P(\epsilon + \epsilon_m) \) is obviously an asymptotic one. It is important in the vicinity of the point \( \epsilon = -\epsilon_m \) where the heat transfer coefficient \( h \) which is proportional to \( 1/\delta \) that leads to the incorrect behavior and divergence while the coefficient is integrated. Here in the evaluation results given below we cut the integrals and hence normalize the heat transfer coefficient at some base angle of the cone.
Fig. 4. (a, b) Boundary layer thickness calculated from Eq. (53), (c, d) boundary layer thickness calculated from Eq. (53), (e) boundary layer thickness calculated from Eq. (53).
Going to the heat transfer coefficient evaluation results we would remind that from the knowledge of the boundary layer shape field near the heated surface one can determine the distribution of the convective heat transfer coefficient and next calculate the convective heat flow. This coefficient should be integrated over the surface that leads to the criterion relation $Nu = CR_a^n$. Nusselt number is dimensionless form of the overall heat transfer coefficient $h$ for the lateral surface of the cone, described by the relation $Nu = hD/k$, in which $D$ is the diameter of the cone base and $k$ is the thermal conductivity.

On the base of the presented in this paper theory we numerically calculated the constant in Nusselt-Rayleigh relation $C = Nu/Ra^{0.25}$ for given values of the angles of the cone base: $\alpha = 0^\circ$, $10^\circ$, $20^\circ$, $30^\circ$, $40^\circ$, $45^\circ$, $50^\circ$, $60^\circ$ and $70^\circ$. The results are following: $C = 0.763$, $0.761$, $0.755$, $0.746$, $0.729$, $0.716$, $0.696$, $0.617$ and $0.435$ respectively. We verified the proposed physical and mathematical model of convective heat transfer from an isothermal horizontal conic by a comparison of the results of this model for the three cones ($\alpha = 30^\circ$, $45^\circ$ and $60^\circ$) with the experiment [9]. The theoretical solutions differs from experimental results for air: $C_{\text{exp}} = 0.749$ for $\alpha = 30^\circ$, $C_{\text{exp}} = 0.742$ for $\alpha = 45^\circ$ and $C_{\text{exp}} = 0.677$ for $\alpha = 60^\circ$ in the relative magnitude $+0.4\%$, $+3.5\%$ and $–8.2\%$ respectively.

5. Conclusions

We conclude that the mathematical modelling of the convective heat transfer from the horizontal, isothermal conical surface could be provided by the analytical solution of the Navier–Stokes and Fourier–Kirchhoff equations with typical for laminar natural convection assumptions. The use of the asymptotic expansion near the vicinity of the (medium) horizontal plane allows to derive an explicit expression for the local heat transfer coefficient. The expansion describes the phenomenon in a compact form but contain the non-integrable singularity in the starting point of the boundary layer. This discrepancy should be corrected by additional asymptotics to be matched with the first one [13]. We consider this as a perspective in our investigations. Here we neglect some small interval adjacent the point of the boundary layer beginning when integrate over the cone surface, deriving the expression for the Nusselt-Rayleigh relation. The comparison with the known case (round plate) and experiments give a reasonable approval of our modelling. We consider this as a verification of the proposed approach including the mathematical aspects of the model.

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Appendix A

Differentiating Eq. (2) gives a tangent vector projection to the curve $S$ (Fig. 2)

$$\frac{dx}{d\epsilon} = \frac{d\rho}{d\epsilon} \sin \epsilon + \rho \cos \epsilon \quad \frac{dy}{d\epsilon} = \frac{d\rho}{d\epsilon} \cos \epsilon - \rho \sin \epsilon \quad \frac{dz}{d\epsilon} = \frac{d\rho}{d\epsilon} \tan \alpha.$$  \hspace{1cm} (A.1)
Because the ratio of coordinates of the vector $\tau(8)$ and $(A.1)$ should be constant $\tau_x(d\epsilon/dx)\big/\tau_y(d\epsilon/dy)$, $\tau_z(d\epsilon/dz)$ so one can found the differential equation $(A.2)$ of the curve and next after its integration the relation $(A.3)$

$$\frac{d\rho}{d\epsilon} = \rho \cos^2 \alpha \tan \epsilon,$$  \hfill (A.2)

and after integration

$$\rho = \rho_0 (\cos \epsilon)^{-\cos^2 \alpha},$$  \hfill (A.3)

where $\rho_0$ is the integration constant that have the geometrical sense of the distance between the parallel vertical planes. One of them contained the symmetry axis and the second one cuts the base of conic in points $\epsilon = \pm \epsilon_m$. The family of the curves $S$ can be also parametrized by the distance $\rho_0$ (see Fig. 2).

Introduction of the relation $\rho = l \cos \alpha$ into (A.3) gives

$$l = \frac{\rho_0 (\cos \epsilon)^{-\cos^2 \alpha}}{\cos \alpha}.$$  \hfill (A.4)

For a differential of Eq. (A.4) one has:

$$dl = \rho_0 (\cos \epsilon)^{-\cos^2 \alpha - 1} \sin \epsilon \cos \alpha \, d\epsilon.$$  \hfill (A.5)

From the Pythagorean theorem $dl^2 = dl^2 + (\rho \, d\epsilon)^2$ (see Fig. 5) and (A.5) one can find

$$\frac{d\epsilon}{d\tau} = \frac{(\cos \epsilon)^{\cos^2 \alpha + 1}}{\rho_0 \sqrt{1 - \sin^2 \epsilon \sin^2 \alpha}}.$$  \hfill (A.6)

![Fig. 5. The graphical relation between $dl$, $\rho \, d\epsilon$ and $d\tau$ on the curve $S$.](image)
Appendix B

Let us evaluate the first derivative of \(y(\varepsilon)\) at the point \((\varepsilon = 0)\). We start from Eq. (44) at this point

\[
\left[ y^3(\varepsilon) \frac{\partial y(\varepsilon)}{\partial \varepsilon} \right]_{\varepsilon=0} = \frac{1}{9} y^3(0) \left[ \frac{\partial y(\varepsilon)}{\partial \varepsilon} \right]_{\varepsilon=0} (27r + 8y(0) \sin \alpha) = r^2, \tag{B.1}
\]

and solve Eq. (44) with respect to \(g = \left[ \frac{\partial y(\varepsilon)}{\partial \varepsilon} \right]_{\varepsilon=0}\),

\[
g = \left[ \frac{\partial y(\varepsilon)}{\partial \varepsilon} \right]_{\varepsilon=0} = \frac{9r^2}{(27r + 8(\sin \alpha)y(0))y^3(0)}. \tag{B.2}
\]

Next we should evaluate the second derivative of \(y(\varepsilon)\) at the point \(\varepsilon = 0\). For this aim we differentiate Eq. (B.2) and then solve the result with respect to \(f = \left[ \frac{\partial y(\varepsilon)}{\partial \varepsilon} \right]_{\varepsilon=0}\). The results of the differentiating for all coefficients of Eq. (44) are

\[
\left[ \frac{d}{d\varepsilon} E \right]_{\varepsilon=0} = \frac{5}{9} \sin \alpha, \tag{B.3}
\]

\[
\left[ \frac{d}{d\varepsilon} G \right]_{\varepsilon=0} = \frac{8}{9} g \sin \alpha, \tag{B.4}
\]

\[
\left[ \frac{d}{d\varepsilon} H \right]_{\varepsilon=0} = \sin \alpha, \tag{B.5}
\]

\[
\left[ \frac{d}{d\varepsilon} F \right]_{\varepsilon=0} = \frac{1}{40} r(1 - \cos^2 \alpha). \tag{B.6}
\]

The results of the substitution into terms of Eq. (44) are

\[
\left[ \frac{d}{d\varepsilon} \left( y^4(\varepsilon) E \right) \right]_{\varepsilon=0} = \frac{5}{9} y^4(0) (\sin \alpha) f,
\]

\[
\left[ \frac{d}{d\varepsilon} 3y^3(\varepsilon) \left( E \frac{\partial y(\varepsilon)}{\partial \varepsilon} \right)^2 \right]_{\varepsilon=0} = 3y^3(0) g^2 \frac{5}{9} \sin \alpha,
\]

\[
\left[ \frac{d}{d\varepsilon} y^3(\varepsilon) G \frac{\partial y(\varepsilon)}{\partial \varepsilon} \right]_{\varepsilon=0} = \frac{1}{9} y^2(0) (32g^2 \sin \alpha y(0) + 8y^2(0)f \sin \alpha + 81g^2 r + 27y(0)f r),
\]

\[
\left[ \frac{d}{d\varepsilon} (y^5(\varepsilon) H) \right]_{\varepsilon=0} = y^5(0) \sin \alpha,
\]

\[
\left[ \frac{d}{d\varepsilon} (y^4(\varepsilon) F) \right]_{\varepsilon=0} = y^4(0) r(1 - \cos^2 \alpha),
\]
and from the right side one has

\[
\left[ \frac{d}{d\epsilon} \left( -r^2 ((\sin \alpha \sin \epsilon)^2 - 1) \cos^{-2\cos^2 \alpha} \epsilon \right) \right]_{\epsilon=0} = 0.
\]

Collecting the new terms of the result of the differentiation of Eq. (44) one arrives at

\[
\left( \frac{2187}{64} y^7(0) r^3 - \frac{13}{9} y^{10}(0) \sin \alpha \cos^2 \alpha + \frac{2349}{64} y^8(0)(\sin \alpha) r^2 + \frac{51}{4} y^9(0) r \right.
\]
\[
\left. - \frac{51}{4} y^9(0) r \cos \alpha + \frac{13}{9} y^{10}(0) \sin \alpha \right) f + \frac{729}{64} y^8(0) r^3 - \frac{729}{64} y^8(0) r^3 \cos^2 \alpha
\]
\[
- \frac{27}{4} y^9(0) r^2 \cos^2 \alpha \sin \alpha + y^{10}(0) r \cos^4 \alpha + \frac{1161}{64} y^9(0)(\sin \alpha) r^2 + \frac{423}{64} r^3(\sin \alpha) y(0)
\]
\[
+ \frac{729}{64} r^5 + y^{11}(0) \sin \alpha - y^{11}(0) \sin \alpha \cos^2 \alpha + \frac{31}{4} y^{10}(0) r - \frac{35}{4} y^{10}(0) r \cos^2 \alpha = 0. \tag{B.7}
\]

The solution of Eq. (B.7) is

\[
f = \left[ \frac{\partial \delta(y)}{\partial \epsilon} \right]_{\epsilon=0} = -9 \frac{423 r^4(\sin \alpha) y(0) + 729 r^2(3^3 + r y^8(0) \sin^2 \alpha + (\sin \alpha) y^9(0))}{y^7(0)(27r + 13(\sin \alpha) y(0))(27r + 8(\sin \alpha) y(0))^2}
\]
\[
- 9 \frac{16 y^9(0)(\sin^2 \alpha)(y(0) + r \sin \alpha)(4(\sin \alpha) y(0) + 27r)}{y^7(0)(27r + 13(\sin \alpha) y(0))(27r + 8(\sin \alpha) y(0))^2}. \tag{B.8}
\]

Let us recall that \(\epsilon_m = \arccos(\rho_0/R)\) and \(r = \rho_0 K^{1/3}\).

Now we introduce the boundary condition at the edge of the cone, where the boundary layer arises

\[ y(-\epsilon_m) = 0. \tag{B.9} \]

Here we restrict ourselves by parabolic approximation for the asymptotic expansion of the solution \(y\) of the differential equation of the boundary layer (44) in the form

\[ y(\epsilon) = y(0) + g \epsilon + \frac{1}{2} f \epsilon^2. \tag{B.10} \]

Eq. (B.9) for the parameter \(y(0)\) is an algebraic equation of high order, which has no explicit solution. So we expand the equation in Taylor series with respect to the variable \(z = \frac{y(0) \sin \alpha}{r}\). The analysis of the series gives the condition of a possibility to cut the expression and to rest only the first terms it. Therefore we consider such values of parameters that \(\frac{1}{2} y(0) \sin \alpha \ll r\). In this region one has in the first approximation

\[ g = \frac{1}{3} \frac{r}{y^3(0)}, \tag{B.11} \]
\[ f = -\frac{1}{3} \frac{r^2}{y^7(0)}. \tag{B.12} \]
After substitution of (B.11) and (B.12) into Eq. (B.9) it simplifies
\[
\left[ y(0) + \frac{1}{3} \frac{r}{y^3(0) \epsilon} - \frac{1}{6} \frac{r^2}{y^7(0) \epsilon^2} \right]_{\epsilon = \arccos(\rho_0/R)} = y(0) - \frac{1}{3} \frac{r}{y^3(0) \epsilon} \arccos(\rho_0/R) \\
- \frac{1}{6} \frac{r^2}{y^7(0) \epsilon^2} \arccos^2(\rho_0/R) = 0 ,
\]
and next
\[
y^8(0) - \frac{1}{3} r \arccos(\rho_0/R) y^4(0) - \frac{1}{6} r^2 \arccos^2(\rho_0/R) y^8(0) - \frac{1}{3} r \arccos(\rho_0/R) y^4(0) \\
- \frac{1}{6} r^2 \arccos^2(\rho_0/R) = 0 .
\] (B.13)

Introducing the new variable
\[ Z = y^4(0) , \]
one goes to second order equation
\[ Z^2 - \frac{1}{3} r \arccos(\rho_0/R) Z - \frac{1}{6} r^2 \arccos^2(\rho_0/R) = 0 . \]

The solution has two roots, the first one is negative, hence non-physical, and the second is positive
\[ Z = \frac{1}{2} \left( \frac{1}{6} + \frac{1}{6} \sqrt{7} \right) r [\pi - 2 \arcsin(\rho_0/R)] . \] (B.14)

From Eq. (B.14) it follows that the boundary layer thickness at the point (\( \epsilon = 0 \)) is
\[ y(0) = Z^{1/4} = \sqrt[4]{\frac{1}{2} \left( \frac{1}{6} + \frac{1}{6} \sqrt{7} \right) r [\pi - 2 \arcsin(\rho_0/R)] } . \] (B.15)

References