



Properties of the density for a three-dimensional stochastic wave equation

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Abstract

We consider a stochastic wave equation in space dimension three driven by a noise white in time and with an absolutely continuous correlation measure given by the product of a smooth function and a Riesz kernel. Let $p_{t,x}(y)$ be the density of the law of the solution $u(t, x)$ of such an equation at points $(t, x) \in]0, T] \times \mathbb{R}^3$. We prove that the mapping $(t, x) \mapsto p_{t,x}(y)$ owns the same regularity as the sample paths of the process $\{u(t, x), (t, x) \in]0, T] \times \mathbb{R}^3\}$ established in [R.C. Dalang, M. Sanz-Solé, Hölder–Sobolev regularity of the solution to the stochastic wave equation in dimension three, Mem. Amer. Math. Soc., in press]. The proof relies on Malliavin calculus and more explicitly, the integration by parts formula of [S. Watanabe, Lectures on Stochastic Differential Equations and Malliavin Calculus, Tata Inst. Fund. Res./Springer-Verlag, Bombay, 1984] and estimates derived from it.

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1. Introduction

In this paper we consider the stochastic wave equation in space dimension $d = 3$,

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$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right)u(t, x) = \sigma(u(t, x))\dot{F}(t, x) + b(u(t, x)),$$

$$u(0, x) = v_0(x), \quad \frac{\partial}{\partial t}u(0, x) = \tilde{v}_0(x), \tag{1}$$

where $t \in]0, T]$ for a fixed $T > 0$, $x \in \mathbb{R}^3$, and Δ denotes the Laplacian on \mathbb{R}^3 . The functions σ and b are Lipschitz continuous, and the process \dot{F} is the formal derivative of a Gaussian random field, white in time and correlated in space defined as follows.

Let

$$f(x) = \varphi(x)k_\beta(x), \tag{2}$$

where φ is a smooth positive function and k_β denotes the Riesz kernel $k_\beta(x) = |x|^{-\beta}$, $\beta \in]0, 2[$. We shall assume that f defines a tempered measure and

$$\int_{\mathbb{R}^3} \frac{\mu(d\xi)}{1 + |\xi|^2} < \infty, \tag{3}$$

where $\mu = \mathcal{F}^{-1}(f)$ and \mathcal{F} denotes the Fourier transform operator. This condition is satisfied for instance for densities of the form (2) with $\varphi(x) = \exp(-\sigma^2|x|^2/2)$ and $\beta \in]0, 2[$ (see [5]).

Let $\mathcal{D}(\mathbb{R}^4)$ be the space of Schwartz test functions (see [16]). Then, on some probability space, there exists a Gaussian process $F = (F(\varphi), \varphi \in \mathcal{D}(\mathbb{R}^4))$ with mean zero and covariance functional defined by

$$E(F(\varphi)F(\psi)) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^3} dx f(x)(\varphi(s) * \tilde{\psi}(s))(x), \tag{4}$$

where $\tilde{\psi}(s, x) = \psi(s, -x)$.

Riesz kernels are a class of singular correlation functions which have already appeared in several papers related with the stochastic heat and wave equations, for instance in [2,3,7–9].

We recall that the fundamental solution $G(t)$ associated to the wave operator $\mathcal{L} = \frac{\partial^2}{\partial t^2} - \Delta$ in dimension three is given by $G(t) = \frac{1}{4\pi t}\sigma_t$, $t > 0$, where σ_t denotes the uniform surface measure on the sphere of radius $t \in [0, T]$, hence with total mass $4\pi t^2$. The properties of $G(t)$ together with the particular form of the covariance of the noise play a crucial role in giving a rigorous formulation to the initial value problem (1). Here, we shall follow the same formulation as in [5] which for the purpose of existence and uniqueness of solution of (1) introduces a localization of the SPDE by means of a set related with the past light cone, as follows. Let D be a bounded domain in \mathbb{R}^3 . Set

$$K_a^D(t) = \{y \in \mathbb{R}^3: d(y, D) \leq a(T - t)\}, \quad t \in [0, T], \tag{5}$$

where $a \geq 1$ and d denotes the Euclidean distance. Then, a solution to the SPDE (1) in D is an adapted, mean-square continuous stochastic process $(u(t)1_{K_a^D(t)}, t \in [0, T])$ with values in $L^2(\mathbb{R}^3)$, satisfying

$$\begin{aligned}
 u(t, \cdot)1_{K_a^D(t)}(\cdot) &= 1_{K_a^D(t)}(\cdot) \left(\frac{d}{dt}G(t) * v_0 + G(t) * \tilde{v}_0 \right) (\cdot) \\
 &+ 1_{K_a^D(t)}(\cdot) \int_0^t \int_{\mathbb{R}^3} G(t-s, \cdot - y) \sigma(u(s, y)) 1_{K_a^D(s)}(y) M(ds, dy) \\
 &+ 1_{K_a^D(t)}(\cdot) \int_0^t ds G(t-s) * (b(u(s, \cdot)) 1_{K_a^D(s)}(\cdot)), \tag{6}
 \end{aligned}$$

almost surely, for any $t \in [0, T]$, where we consider the stochastic integral defined in [4] and M denotes the martingale measure derived from F (see [3]).

The following result is a quotation of [5, Theorem 4.11] and will be invoked repeatedly in this paper.

Theorem 1.1. *Assume that:*

- (a) *the covariance density is of the form (2) with the covariance factor φ bounded and positive, $\varphi \in C^1(\mathbb{R}^d)$ and $\nabla\varphi \in C_b^\delta(\mathbb{R}^d)$, for some $\delta \in]0, 1]$;*
- (b) *the initial values v_0, \tilde{v}_0 are such that $v_0 \in C^2(\mathbb{R}^3)$, and Δv_0 and \tilde{v}_0 are Hölder continuous with orders $\gamma_1, \gamma_2 \in]0, 1]$, respectively;*
- (c) *the coefficients σ and b are Lipschitz.*

Then for any $q \in [2, \infty[$ and $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$, there is $C > 0$ such that for $(t, x), (\bar{t}, y) \in [0, T] \times D$,

$$E(|u(t, x) - u(\bar{t}, y)|^q) \leq C(|t - \bar{t}| + |x - y|)^{\alpha q}. \tag{7}$$

In particular, almost surely, the stochastic process $(u(t, x), (t, x) \in [0, T] \times D)$ solution of (6) has α -Hölder continuous sample paths, jointly in (t, x) , and

$$\sup_{(t,x) \in [0,T] \times D} E(|u(t, x)|^q) < \infty \tag{8}$$

for any $q \in [1, \infty[$.

In this paper we are interested in studying the properties of the density of the solution of (6) as a function of $(t, x) \in]0, T] \times D$, where D is a bounded subset of \mathbb{R}^3 . We shall denote this density by $p_{t,x}(y)$. We shall prove that $(t, x) \rightarrow p_{t,x}(y)$ is jointly Hölder continuous, uniformly in y on compact sets.

This question is trivial in the very particular case where the initial conditions v_0, \tilde{v}_0 and the coefficient b vanish, and the coefficient σ is a constant function. In fact, with these assumptions and $\sigma = 1$ the solution to Eq. (6) is a Gaussian process, centered, stationary in the space variable, and with

$$\sigma_t^2 := E|u(t, x)|^2 = \int_0^t ds \int_{\mathbb{R}^3} \mu(d\xi) \frac{\sin^2(s|\xi|)}{|\xi|^2}.$$

From the expression $p_{t,x}(y) = \frac{1}{\sqrt{2\pi\sigma_t}} \exp(-\frac{|y|^2}{2\sigma_t^2})$ it is not difficult to prove that

$$|p_{t,x}(y) - p_{\bar{t},x}(y)| \leq C(t_0, t_1, D)|t - \bar{t}|,$$

for $0 < t_0 \leq t < \bar{t} \leq t_1, D \subset \mathbb{R}^3$.

However, in the general situation that we are considering in this article, the problem becomes much more involved.

Suppose that v_0, \tilde{v}_0 are null functions, assume also that the covariance of the process F is given by (4) with $dx f(x)$ replaced by $\Gamma(dx)$, where Γ is a non-negative, tempered, non-negative definite measure. Set $\mu = \mathcal{F}^{-1}(\Gamma)$. We introduce an assumption on μ , denoted by (H_η) , saying that $\int_{\mathbb{R}^3} \frac{\mu(d\xi)}{(1+|\xi|^2)^\eta} < \infty$, for some value of $\eta \in]0, 1]$. Assume that the coefficients σ and b are of class C^1 with bounded derivatives and that (H_η) holds for some $\eta \in (0, 1)$. Then, the existence of the density $p_{t,x}$ at any fixed point $(t, x) \in]0, T] \times D$ has been established in [13]. Moreover, assuming that σ and b are C^∞ functions with bounded derivatives of order greater or equal to one, and that (H_η) holds for some $\eta \in (0, \frac{1}{2})$, it is proved in [14] that $y \mapsto p_{t,x}(y)$ is a C^∞ function. We refer the reader to [15] for results on applications of Malliavin calculus to the analysis of probability laws of SPDEs.

In [12], it is shown that the extension of Walsh’s integral introduced in [2] does not require for the integrands any stationary property in the spatial variable. As a consequence of this fact, the results of [2,13,14] and [15] concerning the stochastic wave equation can be formulated with non-null deterministic initial conditions. In addition, the solution of the equation in this setting coincides with the solution to (6). Furthermore, in the particular case of absolutely continuous covariance measures $\Gamma(dx) = f(x) dx$ satisfying (3) the existence and smoothness of the density $p_{t,x}$ are proved in [12] under the weaker assumption (H_1) .

Hence, on the basis of the above mentioned references and remarks, we can write the next statement, which together with Theorem 1.1 are the starting point of our work.

Theorem 1.2. *Assume assumptions (a) and (b) of Theorem 1.1. Suppose also that σ and b are C^∞ functions with bounded derivatives of order greater or equal to one, and $\inf\{|\sigma(z)|, z \in \mathbb{R}\} \geq \sigma_0 > 0$. Then, for any fixed $(t, x) \in]0, T] \times D$, the law of the real valued random variable $u(t, x)$ solution to (6) has a density $p_{t,x} \in C^\infty$.*

The main purpose of this paper is to prove that with the assumptions of this theorem, for any $y \in \mathbb{R}$, the mapping

$$(t, x) \in]0, T] \times \mathbb{R}^3 \mapsto p_{t,x}(y)$$

is jointly α -Hölder continuous with $\alpha \in]0, \inf(\gamma_1, \gamma_2, \frac{2-\beta}{2}, \frac{1+\delta}{2})$ [(see Theorem 2.1 in Section 1).

For stochastic differential equations and some finite-dimensional stochastic evolution systems with an underlying semigroup structure one can find results of this type for instance in [6]. For SPDEs the problem has not been yet very much explored. To the best of our knowledge, this issue has only been studied for the stochastic heat equation in spatial dimension $d = 1$ in [10] and for the wave equation with $d = 2$ in [9] (see [1] and [7] for the existence and regularity of the density for these two types of SPDEs). It is worthy noticing that in these two references, the Hölder degree regularity of $p_{t,x}(y)$ in (t, x) is better than for the sample paths of the solution process $u(t, x)$, while in the equation under consideration we obtain the same order. As it will

become clear from the proof, the reason is the rather degenerate character of the fundamental solution of the wave equation in dimension three.

The method of our proof is based on the integration by parts formula of Malliavin calculus, as in the above mentioned references. We next give the main ideas and steps of the proof. Fix $y \in \mathbb{R}$ and let $(g_{n,y}, n \geq 1)$ be a sequence of smooth functions converging pointwise to the Dirac delta function $\delta_{\{y\}}$. Fix $t, \bar{t} \in]0, T]$, $x, \bar{x} \in D$ and assume that we can prove

$$\sup_{n \in \mathbb{N}, y \in K} |E[g_{n,y}(u(t, x)) - g_{n,y}(u(\bar{t}, \bar{x}))]| \leq C[|t - \bar{t}|^{\beta_1} + |x - \bar{x}|^{\beta_2}], \tag{9}$$

for some $\beta_1, \beta_2 > 0$, where $K \subset \mathbb{R}$. Then, since $p_{t,x}(y) = E[\delta_{\{y\}}(u(t, x))]$ (see [17, Theorem 1.12] for a rigorous meaning of this identity), by passing to the limit as $n \rightarrow \infty$, we will have joint Hölder continuity of the mapping $(t, x) \in]0, T] \times D \mapsto p_{t,x}(y) \in \mathbb{R}$ with degree β_1 in t and β_2 in x , uniformly in $y \in K$.

An estimate like (9) is obtained by the following procedure. For simplicity we write g instead of $g_{n,y}$. We first consider a Taylor expansion of $g(u(\bar{t}, \bar{x}))$ around $u(t, x)$ up to a certain order r_0 chosen in such a way to obtain optimal values of β_1 and β_2 . Then for any $r \leq r_0$, we estimate terms of the type

$$|E[g^{(r)}(u(t, x))(u(t, x) - u(\bar{t}, \bar{x}))^r]|,$$

and the term corresponding to the rest in the Taylor expansion, whose structure is similar. For this we use the version of the integration by parts formula for one-dimensional random variables given in [17, Lemma 2, p. 54] (see also [11, Eqs. (2.29)–(2.31)]) which we now quote as a lemma.

Lemma 1.3. *On an abstract Wiener space (Ω, \mathbf{H}, P) , we consider two real-valued random variables ξ and Z such that $\xi \in \mathbb{D}^\infty$, $\|D\xi\|_{\mathbf{H}}^{-1} \in \bigcap_{p \geq 2} L^p(\Omega)$, $Z \in \mathbb{D}^\infty$. Let g be a function in C^r , for some $r \geq 1$. Denote by \tilde{g} the antiderivative of g . Then, the following formula holds:*

$$E(g^{(r)}(\xi)Z) = E(\tilde{g}(\xi)H_{r+1}(Z, \xi)), \tag{10}$$

where $H_r, r \geq 1$, is defined recursively by

$$H_1(Z, \xi) = \delta\left(\frac{ZD\xi}{\|D\xi\|_{\mathcal{H}}^2}\right),$$

$$H_{r+1}(Z, \xi) = \delta\left(H_r(Z, \xi)\frac{ZD\xi}{\|D\xi\|_{\mathcal{H}}^2}\right), \quad r \geq 1.$$

In this lemma, δ stands for the adjoint operator of the Malliavin derivative, also termed divergence operator or Skorohod integral and we have used the notations of [11] and [15], as we shall do throughout the paper when referring to notions and results of Malliavin calculus.

The abstract Wiener space that we shall consider here is the one associated with the Gaussian process F restricted to the time interval $[0, T]$, as is described in [15, Section 6.1]. For the sake of completeness and its further use, we recall that $\mathbf{H} := \mathcal{H}_T$, $\mathcal{H}_T = L^2([0, T]; \mathcal{H})$ and that \mathcal{H} is the completion of the inner product space consisting of test functions endowed with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} dx f(x)(\varphi * \tilde{\psi})(x).$$

Assume that the function \tilde{g} in Lemma 1.3 is bounded. From (10) it clearly follows that

$$|E[g^{(r)}(\xi)Z]| \leq \|\tilde{g}\|_{\infty} \|H_{r+1}(Z, \xi)\|_{L^1(\Omega)}. \tag{11}$$

Furthermore, as a consequence of the continuity property of the Skorohod integral and the assumptions on ξ , for any $r \geq 1, k \geq 1$ and $p \in (1, \infty)$,

$$\|H_r(Z, \xi)\|_{k,p} \leq C \|Z\|_{k+r, 4^r p}$$

(see [10, Corollary 4.1]). Consequently, under the previous assumptions from (11) we obtain

$$|E[g^{(r)}(\xi)Z]| \leq C \|\tilde{g}\|_{\infty} \|Z\|_{r+1, 4^{r+1}}. \tag{12}$$

Let us recall that for a natural number k and a real number $p \in [1, \infty[$,

$$\|Z\|_{k,p} = \|Z\|_{L^p(\Omega)} + \sum_{r=1}^k \|D^r Z\|_{L^p(\Omega; \mathcal{H}_T^{\otimes r})}.$$

We shall apply (12) mainly to $\xi := u(t, x)$ and $Z := (u(t, x) - u(\bar{t}, \bar{x}))^r$, for natural values of r . Under the hypotheses of Theorem 2.1 the assumptions of Lemma 1.3 are satisfied (see [14] and [15, Chapters 7 and 8]). Thus we face the problem of giving upper bounds for $\|u(t, x) - u(\bar{t}, \bar{x})\|_{r+1, 4^{r+1}}$.

Malliavin derivatives of the solution of (1) satisfy evolution equations (see [15, Theorem 7.1] and [14]). Indeed, for $x \in D$, and a natural number $k \geq 1, (D_{\tau,*}^k u(t, x), (t, x) \in [0, T] \times D)$ is a $\mathcal{H}_T^{\otimes k}$ -valued process satisfying $D_{\tau,*} u(t, x) = 0$ for $\tau > t$, and for $\tau \leq t$ it is the solution of the evolution equation

$$\begin{aligned} D_{\tau,*}^k u(t, x) &= Z_{\tau,*}^k(t, x) + \int_0^t \int_{\mathbb{R}^3} G(t-s, x-y) \\ &\quad \times (\Gamma^k(\sigma, u(s, y)) + \sigma'(u(s, y))D_{\tau,*}^k u(s, y))M(ds, dy) + \int_0^t ds \int_{\mathbb{R}^3} G(s, dy) \\ &\quad \times (\Gamma^k(b, u(s, y)) + b'(u(t-s, x-y))D_{\tau,*}^k u(t-s, x-y)). \end{aligned} \tag{13}$$

In this equation, $(Z^k(t, x), (t, x) \in [0, T] \times D)$ is a $\mathcal{H}_T^{\otimes k}$ -valued stochastic process and for a given function $g \in \mathcal{C}^k$ and a random variable $X \in \mathbb{D}^{k,2}$, $\Gamma^k(g, X) = D^k(g(X)) - g'(X)D^k X$.

The solution of (13) satisfies

$$\sup_{(t,x) \in [0,T] \times D} \|u(s, y)\|_{k,p} < +\infty \tag{14}$$

for any $p \in [1, \infty[$ (see [15, Theorem 7.1]).

In the next section, we shall make use of the explicit form of (13) for $k = 1$. In this case

$$Z^1_{\tau,*}(t, x) = G(t - \tau, x - *)\sigma(u(\tau, *)) \tag{15}$$

and $\Gamma^1(g, X) = 0$.

With some effort, using the tools on stochastic integration of Hilbert-valued processes developed in [15] it can be proved that the conclusions of Theorem 1.1 also apply to the $\mathcal{H}_T^{\otimes k}$ -valued stochastic process solution to (13). More precisely, for any $k \geq 1$, $q \in [2, \infty[$ and $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$, there is $C > 0$ such that for $(t, x), (\bar{t}, y) \in [0, T] \times D$,

$$\|u(t, x) - u(\bar{t}, y)\|_{k,q} \leq C(|t - \bar{t}| + |x - y|)^\alpha. \tag{16}$$

Hence, with the Hölder continuity property on $u(t, x)$ and its Malliavin derivatives we may be able to prove (9) for specific values of β_1, β_2 .

We shall fix what is the top order r_0 in the Taylor expansion of $g(u(\bar{t}, \bar{x}))$. Clearly, the lower exponents β_i should come from the first order term. However, in the examples studied so far, terms of first and second order give the same exponent. For Eq. (6) the situation is different. Already at the first order level of the expansion, we shall see that the contribution of the pathwise integral involving the coefficient b is of the same order than the Hölder continuity exponent given in Theorem 1.1. Clearly, the second order term would provide twice the Hölder continuity degree. Therefore, a Taylor expansion of first order gives the best possible result. However, to conclude whether the regularity of the density $p_{t,x}$ in (t, x) is the same as that of the sample paths of $u(t, x)$, we have to check that the contribution to the first order term in the Taylor expansion of the stochastic integral is not worse than that of the pathwise integral. This explains the strategy of the proof of the main result in the next section.

2. Main result

Throughout this section D denotes a fixed bounded domain of \mathbb{R}^3 and C will be any positive finite constant. We assume that (3) holds. Our purpose is to prove the following theorem.

Theorem 2.1. *Assume that:*

- (a) *the covariance density is of the form (2) and the covariance density factor φ is bounded and positive, $\varphi \in C^1(\mathbb{R}^d)$ and $\nabla\varphi \in C_b^\delta(\mathbb{R}^d)$ for some $\delta \in]0, 1]$;*
- (b) *the initial values v_0, \tilde{v}_0 are such that $v_0 \in C^2(\mathbb{R}^3)$, and Δv_0 and \tilde{v}_0 are Hölder continuous with orders $\gamma_1, \gamma_2 \in]0, 1]$, respectively;*
- (c) *the coefficients σ and b are C^∞ functions with bounded derivatives of order greater or equal to one, and there exists $\sigma_0 > 0$ such that $\inf\{|\sigma(z)|, z \in \mathbb{R}\} \geq \sigma_0$.*

Then the mapping

$$(t, x) \in]0, T] \times D \mapsto p_{t,x}(y)$$

is α -Hölder continuous jointly in (t, x) with $\alpha \in]0, \inf(\gamma_1, \gamma_2, \frac{2-\beta}{2}, \frac{1+\delta}{2})[$, uniformly in $y \in \mathbb{R}$.

Proof. Fix $y \in \mathbb{R}$ and let $(g_n, y, n \geq 1)$ be a sequence of regular functions converging pointwise to $\delta_{\{y\}}$ as $n \rightarrow \infty$; for example, a sequence of Gaussian kernels with mean y and variances converging to zero. We may assume that the corresponding antiderivatives $\tilde{g}_{n,y}$ are uniformly bounded by 1. To simplify the notation, we shall write g instead of $g_{n,y}$.

Step 1 (Time increments). For $(t, x) \in [0, T] \times D$ we consider the Taylor expansion

$$E[g(u(t+h, x)) - g(u(t, x))] = E[g'(u(t, x))(u(t+h, x) - u(t, x)) + E[g''(\tilde{u}(t, x, h))(u(t+h, x) - u(t, x))^2], \tag{17}$$

where $h > 0$ and $\tilde{u}(t, x, h)$ denotes a random variable lying on the segment determined by $u(t+h, x)$ and $u(t, x)$.

First order term. Set

$$T_1(t, x, h) = |E[g'(u(t, x))(u(t+h, x) - u(t, x))]|.$$

We aim to prove that

$$\sup_{(t,x) \in [0,T] \times D} T_1(t, x, h) \leq Ch^\alpha, \tag{18}$$

with $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$.

By using Eq. (6), we write $T_1(t, x, h) \leq \sum_{i=1}^3 T_{1,i}(t, x, h)$, with

$$\begin{aligned} T_{1,1}(t, x, h) &= \left| E \left[g'(u(t, x)) \left[\left(\frac{d}{dt} G(t+h) * v_0 + G(t+h) * \tilde{v}_0 \right) (x) - \left(\frac{d}{dt} G(t) * v_0 + G(t) * \tilde{v}_0 \right) (x) \right] \right] \right|, \\ T_{1,2}(t, x, h) &= \left| E \left[g'(u(t, x)) \left[\int_0^{t+h} ds \int_{\mathbb{R}^3} G(t+h-s, dz) b(u(s, x-z)) - \int_0^t ds \int_{\mathbb{R}^3} G(t-s, dz) b(u(s, x-z)) \right] \right] \right|, \\ T_{1,3}(t, x, h) &= \left| E \left[g'(u(t, x)) \left[\int_0^{t+h} \int_{\mathbb{R}^3} G(t+h-s, x-z) \sigma(u(s, x-z)) M(ds, dz) - \int_0^t \int_{\mathbb{R}^3} G(t-s, x-z) \sigma(u(s, x-z)) M(ds, dz) \right] \right] \right|. \end{aligned}$$

In fact, by our choice of (t, x) all the indicator functions in (6) take the value 1.

We shall apply repeatedly the inequality (12) with $r = 1$, $\xi := u(t, x)$ and different choices of Z . To begin with, we take

$$Z := \left(\frac{d}{dt} G(t+h) * v_0 + G(t+h) * \tilde{v}_0 \right)(x) - \left(\frac{d}{dt} G(t) * v_0 + G(t) * \tilde{v}_0 \right)(x).$$

Since Z is deterministic, $\|Z\|_{k,p} = |Z|$, for any k and p . Then, applying (12) and [5, Lemma 4.9] yields

$$\sup_{(t,x) \in [0,T] \times D} T_{1,1}(t, x, h) \leq Ch^{\gamma_1 \wedge \gamma_2}. \tag{19}$$

We next study the term $T_{1,2}(t, x, h)$. Let

$$T_{1,2,1}(t, x, h) = \left| E \left[g'(u(t, x)) \int_t^{t+h} ds \int_{\mathbb{R}^3} G(t+h-s, dz) b(u(s, x-z)) \right] \right|.$$

We apply (12) to $Z := \int_t^{t+h} ds \int_{\mathbb{R}^3} G(t+h-s, dz) b(u(s, x-z))$ and consider the measure on $[t, t+h] \times \mathbb{R}^3$ given by $dsG(t+h-s, dz)$ with total mass $\frac{h^2}{2}$ and an arbitrary $p \in [1, \infty[$. By applying Minkowski’s inequality, we obtain

$$\begin{aligned} & \left\| \int_t^{t+h} ds \int_{\mathbb{R}^3} G(t+h-s, dz) b(u(s, x-z)) \right\|_{2,p} \\ & \leq \int_t^{t+h} ds \int_{\mathbb{R}^3} G(t+h-s, dz) \|b(u(s, x-z))\|_{2,p}. \end{aligned}$$

By the chain rule of Malliavin calculus,

$$\|b(u(s, y))\|_{2,p} \leq C(1 + \|u(s, y)\|_{2,p} + \|u(s, y)\|_{2,2p}^2).$$

Consequently,

$$\sup_{(s,y) \in [0,T] \times D^{2T}} \|b(u(s, y))\|_{2,p} < \infty \tag{20}$$

where for a bounded set $D \subset \mathbb{R}^3$ and $a \geq 0$, we denote by $D^a = \{z \in \mathbb{R}^3; d(z, D) \leq a\}$ and we have applied (14).

Thus, we have proved

$$\sup_{(t,x) \in [0,T] \times D} \left\| \int_t^{t+h} ds \int_{\mathbb{R}^3} G(t+h-s, dz) b(u(s, x-z)) \right\|_{2,p} \leq Ch^2,$$

for any $p \in [1, \infty[$, and consequently,

$$\sup_{(t,x) \in]0, T] \times D} T_{1,2,1}(t, x, h) \leq Ch^2. \tag{21}$$

Set

$$T_{1,2,2}(t, x, h) = \left\| E \left[g'(u(t, x)) \int_0^t ds \int_{\mathbb{R}^3} b(u(s, x - z)) [G(t + h - s, dz) - G(t - s, dz)] \right] \right\|,$$

that according to (12) we can bound as follows,

$$T_{1,2,2}(t, x, h) \leq C \left\| \int_0^t ds \int_{\mathbb{R}^3} b(u(s, x - z)) [G(t + h - s, dz) - G(t - s, dz)] \right\|_{2,4^2}.$$

We have

$$\begin{aligned} & \int_0^t ds \int_{\mathbb{R}^3} b(u(s, x - z)) G(t + h - s, dz) \\ &= \int_0^t ds (t + h - s) \int_{B_1(0)} G(1, dz) b(u(s, x - (t + h - s)z)), \\ & \int_0^t ds \int_{\mathbb{R}^3} b(u(s, x - z)) G(t - s, dz) \\ &= \int_0^t ds (t - s) \int_{B_1(0)} G(1, dz) b(u(s, x - (t - s)z)), \end{aligned} \tag{22}$$

as can be easily checked by applying the change of variables $z \mapsto \frac{z}{t+h-s}$ and $z \mapsto \frac{z}{t-s}$, respectively. Then, by the triangular inequality we obtain for any $p \in [1, \infty[$

$$\begin{aligned} & \left\| \int_0^t ds \int_{\mathbb{R}^3} b(u(s, x - z)) [G(t + h - s, dz) - G(t - s, dz)] \right\|_{2,p} \\ & \leq Ch \left\| \int_0^t ds \int_{B_1(0)} G(1, dz) b(u(s, x - (t + h - s)z)) \right\|_{2,p} \\ & \quad + C \left\| \int_0^t ds \int_{B_1(0)} G(1, dz) [b(u(s, x - (t + h - s)z)) - b(u(s, x - (t - s)z))] \right\|_{2,p}. \end{aligned} \tag{23}$$

For the study of the first term in the right-hand side of (23), we apply Minkowski’s inequality and then (20). This yields

$$\left\| \int_0^t ds \int_{B_1(0)} G(1, dz) b(u(s, x - (t + h - s)z)) \right\|_{2,p} < C.$$

The Lipschitz property of b and (7), (16), (14) yield

$$\|b(u(s, x - (t + h - s)z)) - b(u(s, x - (t - s)z))\|_{2,p} \leq Ch^\alpha,$$

with $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$. Consequently, after having applied Minkowski’s inequality we see that the second term of the right-hand side of (23) is bounded by Ch^α , uniformly in $(t, x) \in [0, T] \times D$.

Thus, we have proved

$$\sup_{(t,x) \in [0,T] \times D} T_{1,2,2}(t, x, h) \leq Ch^\alpha,$$

and along with (21) we obtain

$$\sup_{(t,x) \in [0,T] \times D} T_{1,2}(t, x, h) \leq Ch^\alpha, \tag{24}$$

with $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$.

Let us remark that in [10] and [9] the contribution of the term analogous to $T_{1,2}(t, x, h)$ is a power of h of higher order than the Hölder degree of the solution. In fact, for the heat equation and the wave equation in spatial dimension two, by integration of the increments $G(t + h - s, dz) - G(t - s, dz)$ we get powers of h . For the wave equation in dimension three, such an approach is not possible. Instead, “increments” of $G(t - s, dz)$ are transferred to increments of $b(u(s, x - z))$ (this is the role played by the change of variables that we have performed to obtain (22)) and after this, we can conclude by applying the Lipschitz property of b and the Hölder continuity of the sample paths.

The inequality (24) tell us that we are not going to improve the Hölder degree of the mapping $t \in]0, T] \mapsto p_{t,x}(y)$ in more than the given α . But it might happen that the contribution of $T_{1,3}(t, x, h)$ makes the overall estimation worse. We next carry out a careful analysis of this term and prove that its contribution in terms of powers of h is the same as $T_{1,2}(t, x, h)$.

We write $T_{1,3}(t, x, h) = T_{1,3,1}(t, x, h) + T_{1,3,2}(t, x, h)$ with

$$T_{1,3,1}(t, x, h) = \left| E \left[g'(u(t, x)) \int_t^{t+h} \int_{\mathbb{R}^3} \sigma(u(s, y)) G(t + h - s, x - y) M(ds, dy) \right] \right|,$$

$$T_{1,3,2}(t, x, h) = \left| E \left[g'(u(t, x)) \int_0^t \int_{\mathbb{R}^3} \sigma(u(s, y)) [G(t+h-s, x-y) - G(t-s, x-y)] M(ds, dy) \right] \right|.$$

The term $T_{1,3,1}(t, x, h)$ vanishes, since the random variable $g'(u(t, x))$ is adapted to the natural filtration generated by the martingale measure M .

In contrast with $T_{1,2}$, for the analysis of $T_{1,3,2}(t, x, h)$ we do not start by applying (12), which actually would lead to worse results (see Remark 2.2); instead, we apply [15, Proposition 3.9]. Since the mathematical expectation of a Skorohod integral is zero, we obtain

$$\begin{aligned} T_{1,3,2}(t, x, h) &= |E\langle g''(u(t, x)) D_{\cdot,*} u(t, x), \sigma(u(\cdot, *)) \\ &\quad \times [G(t+h-\cdot, x-*) - G(t-\cdot, x-*)] 1_{]0,t[}(\cdot) \rangle_{\mathcal{H}_T} | \\ &= |E[g''(u(t, x)) \langle D_{\cdot,*} u(t, x), \sigma(u(\cdot, *)) \\ &\quad \times [G(t+h-\cdot, x-*) - G(t-\cdot, x-*)] 1_{]0,t[}(\cdot) \rangle_{\mathcal{H}_T}]|. \end{aligned}$$

For any $(\tilde{t}, \tilde{x}) \in [0, T] \times \mathbb{R}^3$, we define

$$B_{\cdot,*}^h(\tilde{t}, \tilde{x}) = \sigma(u(\cdot, *)) [G(\tilde{t}+h-\cdot, \tilde{x}-*) - G(\tilde{t}-\cdot, \tilde{x}-*)].$$

With this notation, and by applying (12) to $T_{1,3,2}(t, x, h)$ we see that

$$T_{1,3,2}(t, x, h) \leq C \|\langle D_{\cdot,*} u(t, x), B_{\cdot,*}^h(t, x) \rangle_{\mathcal{H}_T}\|_{3,4^3}. \tag{25}$$

We shall consider norms $\|\cdot\|_{k,p}$ for arbitrary $k \geq 1$ and $p \in [1, \infty[$, instead of $\|\cdot\|_{3,4^3}$. By virtue of (13) and (15), we write

$$\begin{aligned} &\|\langle D_{\cdot,*} u(t, x), B_{\cdot,*}^h(\tilde{t}, \tilde{x}) \rangle_{\mathcal{H}_T}\|_{k,p} \\ &\leq C (\|B_1^h(t, x; \tilde{t}, \tilde{x})\|_{k,p} + \|B_2^h(t, x; \tilde{t}, \tilde{x})\|_{k,p} + \|B_3^h(t, x; \tilde{t}, \tilde{x})\|_{k,p}), \end{aligned}$$

where

$$\begin{aligned} B_1^h(t, x; \tilde{t}, \tilde{x}) &= \langle G(t-\cdot, x-*) \sigma(u(\cdot, *)), B_{\cdot,*}^h(\tilde{t}, \tilde{x}) \rangle_{\mathcal{H}_T}, \\ B_2^h(t, x; \tilde{t}, \tilde{x}) &= \left\langle \int_0^t \int_{\mathbb{R}^3} G(t-s, x-y) \sigma'(u(s, y)) D_{\cdot,*} u(s, y) M(ds, dy), B_{\cdot,*}^h(\tilde{t}, \tilde{x}) \right\rangle_{\mathcal{H}_T}, \\ B_3^h(t, x; \tilde{t}, \tilde{x}) &= \left\langle \int_0^t ds \int_{\mathbb{R}^3} G(s, dy) b'(u(t-s, x-y)) D_{\cdot,*} u(t-s, x-y), B_{\cdot,*}^h(\tilde{t}, \tilde{x}) \right\rangle_{\mathcal{H}_T}. \end{aligned}$$

Consider the change of variables $(s, y) \rightarrow (t-s, \frac{y}{t-s})$; Fubini's theorem along with Minkowski's inequality yield

$$\begin{aligned} \|B_3^h(t, x; \tilde{t}, \tilde{x})\|_{k,p} &= \left\| \int_0^t ds(t-s) \int_{\mathbb{R}^3} G(1, dy) b'(u(s, x - (t-s)y)) \right. \\ &\quad \left. \times \langle D_{*,*}u(s, x - (t-s)y), B_{*,*}^h(\tilde{t}, \tilde{x}) \rangle_{\mathcal{H}_T} \right\|_{k,p} \\ &\leq C \|b'\|_\infty \int_0^t ds \int_{\mathbb{R}^3} G(1, dy) \|\langle D_{*,*}u(s, x - (t-s)y), B_{*,*}^h(\tilde{t}, \tilde{x}) \rangle_{\mathcal{H}_T}\|_{k,p} \\ &\leq C \int_0^t ds \sup_{y \in K_a^D(s)} \|\langle D_{*,*}u(s, y), B_{*,*}^h(\tilde{t}, \tilde{x}) \rangle_{\mathcal{H}_T}\|_{k,p}. \end{aligned}$$

The last inequality is obtained as follows. By definition of the sets $K_a^D(t)$, it is obvious that for any $(t, x) \in [0, T] \times D$, x belongs to $K_a^D(t)$. Since the support of the measure $G(s, dy)$ is the boundary of the ball centered at zero and with radius s , the y -variable in the above integrals belongs to $K_a^D(s)$.

Hence, by Gronwall’s lemma

$$\begin{aligned} &\sup_{x \in K_a^D(t)} \|\langle D_{*,*}u(t, x), B_{*,*}^h(\tilde{t}, \tilde{x}) \rangle_{\mathcal{H}_T}\|_{k,p} \\ &\leq C \sup_{x \in K_a^D(t)} (\|B_1^h(t, x; \tilde{t}, \tilde{x})\|_{k,p} + \|B_2^h(t, x; \tilde{t}, \tilde{x})\|_{k,p}). \end{aligned} \tag{26}$$

Our aim is to prove that

$$\sup_{(t,x) \in [0,T] \times D} \|\langle D_{*,*}u(t, x), B_{*,*}^h(t, x) \rangle_{\mathcal{H}_T}\|_{k,p} \leq Ch^\alpha, \tag{27}$$

with $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$, for $k = 0, 1, 2, 3$, where by convention $\|\cdot\|_{0,p} = \|\cdot\|_p$.

To illustrate the method and simplify the presentation, we shall consider in (26) the norm $\|\cdot\|_{1,p}$ instead of $\|\cdot\|_{3,p}$. That is, we shall deal only with derivatives up to the first order. Thus let us first prove (27) for $k = 0$, that means for the $L^p(\Omega)$ -norm. For this, we start by studying the $L^p(\Omega)$ -norm of $B_2^h(t, x; \tilde{t}, \tilde{x})$.

To shorten the notation, set $\mathcal{D}(s, y; \tilde{t}, \tilde{x}) = \langle D_{*,*}u(s, y), B_{*,*}^h(\tilde{t}, \tilde{x}) \rangle_{\mathcal{H}_T}$. From Burkholder’s inequality it follows that

$$\begin{aligned} &\|B_2^h(t, x; \tilde{t}, \tilde{x})\|_p^p \\ &= E \left| \int_0^t \int_{\mathbb{R}^3} G(t-s, x-y) \sigma'(u(s, y)) \mathcal{D}(s, y; \tilde{t}, \tilde{x}) M(ds, dy) \right|^p \end{aligned}$$

$$\begin{aligned}
 &\leq CE \left(\int_0^t ds \int_{\mathbb{R}^3} dy \int_{\mathbb{R}^3} d\bar{y} G(t-s, x-y) \sigma'(u(s, y)) f(y-\bar{y}) \right. \\
 &\quad \left. \times G(t-s, x-\bar{y}) \sigma'(u(s, \bar{y})) \mathcal{D}(s, y; \tilde{t}, \tilde{x}) \mathcal{D}(s, \bar{y}; \tilde{t}, \tilde{x}) \right)^{\frac{p}{2}} \\
 &\leq C \int_0^t ds \sup_{y \in K_a^D(s)} \{ E | \langle D_{\cdot,*} u(s, y), B_{\cdot,*}^h(\tilde{t}, \tilde{x}) \rangle_{\mathcal{H}_T} |^p \}. \tag{28}
 \end{aligned}$$

Therefore, (26) with $k = 0$ and Gronwall’s lemma yields

$$\sup_{x \in K_a^D(t)} \| \langle D_{\cdot,*} u(t, x), B_{\cdot,*}^h(\tilde{t}, \tilde{x}) \rangle_{\mathcal{H}_T} \|_p \leq C \sup_{x \in K_a^D(t)} \| B_1^h(t, x; \tilde{t}, \tilde{x}) \|_p$$

with a constant C independent of \tilde{t} and \tilde{x} . Hence, we can fix $\tilde{t} = t$ and $\tilde{x} = x$ in the preceding inequality and obtain

$$\sup_{t \in [0, T]} \sup_{x \in K_a^D(t)} \| \langle D_{\cdot,*} u(t, x), B_{\cdot,*}^h(t, x) \rangle_{\mathcal{H}_T} \|_p \leq C \sup_{t \in [0, T]} \sup_{x \in K_a^D(t)} \| B_1^h(t, x) \|_p, \tag{29}$$

where $B_1^h(t, x)$ stands for $B_1^h(t, x; t, x)$.

By the very definition of the inner product in \mathcal{H}_T we have

$$\begin{aligned}
 \| B_1^h(t, x) \|_p^p &= E \left| \int_0^t dr \int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^3} d\eta G(t-r, x-\xi) \sigma(u(r, \xi)) f(\xi-\eta) \right. \\
 &\quad \left. \times [G(t+h-r, x-\eta) - G(t-r, x-\eta)] \sigma(u(r, \eta)) \right|^p.
 \end{aligned}$$

We consider each one of the terms in the difference of the right-hand side of this inequality and apply respectively the change of variables

$$(\xi, \eta) \rightarrow \left(x - \xi, (x - \eta) \frac{t-r}{t+h-r} \right), \quad (\xi, \eta) \rightarrow (x - \xi, x - \eta).$$

With this, the increments in time of the measure G are transferred to increments of σ and f . More precisely, we obtain

$$\| B_1^h(t, x) \|_p^p \leq C (T_{1,3,2,1}(t, x, h) + T_{1,3,2,2}(t, x, h))$$

with

$$\begin{aligned}
 T_{1,3,2,1}(t, x, h) &= E \left| \int_0^t dr \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(t-r, d\xi) G(t-r, d\eta) \sigma(u(r, x-\xi)) \right. \\
 &\quad \times \frac{t+h-r}{t-r} f\left(\frac{t+h-r}{t-r}\eta - \xi\right) \\
 &\quad \left. \times \left[\sigma\left(u\left(r, x - \frac{t+h-r}{t-r}\eta\right)\right) - \sigma(u(r, x-\eta)) \right] \right|^p, \\
 T_{1,3,2,2}(t, x, h) &= E \left| \int_0^t dr \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(t-r, d\xi) G(t-r, d\eta) \sigma(u(r, x-\xi)) \sigma(u(r, x-\eta)) \right. \\
 &\quad \left. \times \left(\frac{t+h-r}{t-r} f\left(\frac{t+h-r}{t-r}\eta - \xi\right) - f(\eta - \xi) \right) \right|^p.
 \end{aligned}$$

For the analysis of $T_{1,3,2,1}(t, x, h)$ we consider the measure with support on $[0, T] \times B_{t-r}(0) \times B_{t-r}(0)$ defined by

$$\nu(dr; d\xi, d\eta) := dr G(t-r, d\xi) G(t-r, d\eta) \left| \frac{t+h-r}{t-r} f\left(\frac{t+h-r}{t-r}\eta - \xi\right) \right|. \tag{30}$$

Following the steps of the proof of [5, Lemma 6.3] we obtain

$$\sup_{0 \leq t \leq t+h \leq T} \int_0^t \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} \nu(dr; d\xi, d\eta) < \infty.$$

Then, we can write

$$\begin{aligned}
 T_{1,3,2,1}(t, x, h) &= E \left| \int_0^t \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} \nu(dr; d\eta, d\xi) \sigma(u(r, x-\xi)) \right. \\
 &\quad \left. \times \left[\sigma\left(u\left(r, x - \frac{t+h-r}{t-r}\eta\right)\right) - \sigma(u(r, x-\eta)) \right] \right|^p
 \end{aligned}$$

and apply Hölder’s inequality with respect to the measure $\nu(dr; d\eta, d\xi)$. This yields

$$\begin{aligned}
 T_{1,3,2,1}(t, x, h) &\leq C \int_0^t \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} \nu(dr; d\eta, d\xi) \\
 &\quad \times E \left| \sigma(u(r, x-\xi)) \left[\sigma\left(u\left(r, x - \frac{t+h-r}{t-r}\eta\right)\right) - \sigma(u(r, x-\eta)) \right] \right|^p.
 \end{aligned}$$

We now apply Schwarz’ inequality to the factor containing the expectation. Since the coefficient σ is a Lipschitz function, by (7) and (8) we obtain

$$\begin{aligned}
 T_{1,3,2,1}(t, x, h) &\leq C \sup_{|\eta|=t-r} \left(E \left| \sigma \left(u \left(r, x - \frac{t+h-r}{t-r} \eta \right) \right) - \sigma(u(r, x - \eta)) \right|^{2p} \right)^{\frac{1}{2}} \\
 &\leq C \sup_{|\eta|=t-r} \left| \frac{h}{t-r} \eta \right|^{\alpha p} \leq Ch^{\alpha p},
 \end{aligned}
 \tag{31}$$

with $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$ and a constant C not depending on $(t, x) \in [0, T] \times D$ for an arbitrary D .

To study $T_{1,3,2,2}(t, x, h)$ we consider the measure on $[0, t] \times B_{t-r}(0) \times B_{t-r}(0)$ given by

$$\begin{aligned}
 \mu^h(dr; d\xi, d\eta) &= dr G(t-r, d\xi)G(t-r, d\eta) \\
 &\quad \times \left| \frac{t+h-r}{t-r} f \left(\frac{t+h-r}{t-r} \eta - \xi \right) - f(\eta - \xi) \right|.
 \end{aligned}$$

We also consider two additional measures with the same support as $\mu^h(dr; d\xi, d\eta)$ obtained by applying the triangular inequality to the expression

$$\left| \frac{t+h-r}{t-r} f \left(\frac{t+h-r}{t-r} \eta - \xi \right) - f(\eta - \xi) \right|.$$

They are defined by

$$\begin{aligned}
 \mu_1^h(dr; d\xi, d\eta) &= dr G(t-r, \xi)G(t-r, d\eta) \frac{h}{t-r} f(\eta - \xi), \\
 \mu_2^h(dr; d\xi, d\eta) &= dr G(t-r, \xi)G(t-r, d\eta) \frac{t+h-r}{t-r} \\
 &\quad \times \left| f \left(\frac{t+h-r}{t-r} \eta - \xi \right) - f(\eta - \xi) \right|.
 \end{aligned}
 \tag{32}$$

With these new ingredients, $T_{1,3,2,2}(t, x, h) \leq C(T_{1,3,2,2,1} + T_{1,3,2,2,2})$, where

$$\begin{aligned}
 T_{1,3,2,2,1} &= E \left| \int_0^t \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} \mu_1^h(dr; d\xi, d\eta) \sigma(u(r, x - \xi)) \sigma(u(r, x - \eta)) \right|^p, \\
 T_{1,3,2,2,2} &= E \left| \int_0^t \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} \mu_2^h(dr; d\xi, d\eta) \sigma(u(r, x - \xi)) \sigma(u(r, x - \eta)) \right|^p.
 \end{aligned}$$

We next check that $\int_0^t \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} \mu_1^h(dr; d\xi, d\eta) < Ch$. Indeed, owing to (2) and by the change of variable $r \mapsto t - r$, we have

$$\begin{aligned} & \int_0^t \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} \mu_1^h(dr; d\xi, d\eta) \\ & \leq Ch \int_0^t \frac{dr}{r} \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} G(r, d\xi)G(r, d\eta)k_\beta(\xi - \eta) \\ & = Ch \int_0^t \frac{dr}{r} \int_{\mathbb{R}^3} d\xi \frac{|\mathcal{F}G(r)(\xi)|^2}{|\xi|^{3-\beta}} \leq Ch, \end{aligned}$$

uniformly in $t \in [0, T]$, where in the last inequality we have applied [5, Lemma 2.3] with $b = 1$.

Consequently, Hölder’s inequality, the linear growth of the coefficient σ and the property (8) yield

$$T_{1,3,2,2,1} \leq Ch^p, \tag{33}$$

uniformly in $(t, x) \in [0, T] \times D$.

The next step consists of proving that $\mu_2^h(dr; d\xi, d\eta)$ defines a finite measure as well, and in giving an estimate of its total mass in terms of powers of h . For this, we consider the inequality

$$\begin{aligned} & \left| f\left(\frac{t+h-r}{t-r}\eta - \xi\right) - f(\eta - \xi) \right| \\ & \leq \left| \varphi\left(\frac{t+h-r}{t-r}\eta - \xi\right) - \varphi(\eta - \xi) \right| k_\beta(\eta - \xi) \\ & \quad + \left| \varphi\left(\frac{t+h-r}{t-r}\eta - \xi\right) \right| \left| k_\beta\left(\frac{t+h-r}{t-r}\eta - \xi\right) - k_\beta(\eta - \xi) \right|, \end{aligned} \tag{34}$$

which is a consequence of (2) and the triangular inequality. The properties of φ together with [5, Lemma 2.3] yield

$$\begin{aligned} & \int_0^t dr \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} G(t-r, d\xi)G(t-r, d\eta) \frac{t+h-r}{t-r} \\ & \quad \times \left| \varphi\left(\frac{t+h-r}{t-r}\eta - \xi\right) - \varphi(\eta - \xi) \right| k_\beta(\eta - \xi) \\ & \leq Ch \int_0^t \frac{dr}{t-r} \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} G(t-r, d\xi)G(t-r, d\eta)k_\beta(\eta - \xi) \\ & \leq Ch, \end{aligned} \tag{35}$$

uniformly in $t \in [0, T]$.

Let us now consider the contribution to $\mu_2^h(dr; d\xi, d\eta)$ of the second term of the right-hand side of (34). Let $\tilde{\alpha} \in]0, 1[$, $\beta \in]0, 2[$ with $\tilde{\alpha} + \beta \in]0, 2[$. By applying [5, Lemma 2.6(a)] with $b := \tilde{\alpha}$, $a := 3 - (\tilde{\alpha} + \beta)$, $c := h$, $u := \eta - \xi$, $x := \frac{\eta}{t-r}$, we obtain

$$\begin{aligned} & \int_0^t dr \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} G(t-r, d\xi)G(t-r, d\eta) \frac{t+h-r}{t-r} \left| \varphi\left(\frac{t+h-r}{t-r}\eta - \xi\right) \right| \\ & \quad \times \left| k_\beta\left(\frac{t+h-r}{t-r}\eta - \xi\right) - k_\beta(\eta - \xi) \right| \\ & \leq \|\varphi\|_\infty \int_0^t dr \frac{t+h-r}{t-r} \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} G(t-r, d\xi)G(t-r, d\eta) Dk_\beta\left(\eta - \xi, \frac{h}{t-r}\eta\right) \\ & \leq \|\varphi\|_\infty h^{\tilde{\alpha}} \int_0^t dr \frac{t+h-r}{t-r} \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} G(t-r, d\xi)G(t-r, d\eta) \\ & \quad \times \int_{\mathbb{R}^3} dw k_{\tilde{\alpha}+\beta}(\eta - \xi - hw) \left| Dk_{3-\tilde{\alpha}}\left(w, \frac{\eta}{t-r}\right) \right|, \end{aligned} \tag{36}$$

where we have set $Dg(x, y) := g(x + y) - g(x)$ for a function $g : \mathbb{R}^3 \rightarrow \mathbb{R}$.

Our next purpose is to prove that the last integral in the above expression is bounded, uniformly in $t, t + h \in [0, T]$. For this, as in [5, Lemma 6.4] we split the integral on the w -variable in the last expression into the sum of two integrals: on a finite ball containing the origin and on the complementary of this set. In this way we obtain as an upper bound of

$$\begin{aligned} & \int_0^t dr \frac{t+h-r}{t-r} \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} G(t-r, d\xi)G(t-r, d\eta) \\ & \quad \times \int_{\mathbb{R}^3} dw k_{\tilde{\alpha}+\beta}(\eta - \xi - hw) \left| Dk_{3-\tilde{\alpha}}\left(w, \frac{\eta}{t-r}\right) \right|, \end{aligned}$$

the sum of the three terms:

$$\begin{aligned} I_1^{(1)} &= \int_0^t dr \frac{t+h-r}{t-r} \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} G(t-r, d\xi)G(t-r, d\eta) \\ & \quad \times \int_{B_2(0)} dw k_{\tilde{\alpha}+\beta}(\eta - \xi - hw) k_{3-\tilde{\alpha}}\left(w + \frac{\eta}{t-r}\right), \\ I_1^{(2)} &= \int_0^t dr \frac{t+h-r}{t-r} \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} G(t-r, d\xi)G(t-r, d\eta) \\ & \quad \times \int_{B_2(0)} dw k_{\tilde{\alpha}+\beta}(\eta - \xi - hw) k_{3-\tilde{\alpha}}(w), \end{aligned}$$

$$I_1^{(3)} = \int_0^t dr \frac{t+h-r}{t-r} \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} G(t-r, d\xi) G(t-r, d\eta) \\ \times \int_{B_2(0)^c} dw k_{\tilde{\alpha}+\beta}(\eta - \xi - hw) \left| Dk_{3-\tilde{\alpha}}\left(w, \frac{\eta}{t-r}\right) \right|.$$

Consider the change of variable $w \mapsto w + \frac{\eta}{t-r}$ and then $\eta \mapsto \frac{t+h-r}{t-r}\eta$ that we apply to $I_1^{(1)}$. By Fubini’s theorem we obtain

$$I_1^{(1)} \leq \int_{B_3(0)} dw k_{3-\tilde{\alpha}}(w) \int_0^t dr \frac{t+h-r}{t-r} \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} G(t-r, d\xi) G(t-r, d\eta) \\ \times k_{\tilde{\alpha}+\beta}\left(\frac{t+h-r}{t-r}\eta - \xi - hw\right) \\ = \int_{B_3(0)} dw k_{3-\tilde{\alpha}}(w) \int_0^t dr \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(t-r, d\xi) G(t-r+h, d\eta) k_{\tilde{\alpha}+\beta}(\eta - \xi - hw).$$

The properties of the Fourier transform and the expression of this operator applied to Riesz kernels yield, after regularization of G ,

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(t-r, d\xi) G(t-r+h, d\eta) k_{\tilde{\alpha}+\beta}(\eta - \xi - hw) \\ = \int_{\mathbb{R}^3} \mathcal{F}G(t-r)(\xi) \overline{\mathcal{F}G(t-r+h)(\xi)} k_{3-(\tilde{\alpha}+\beta)}(\cdot - hw)(\xi).$$

Hence by applying Schwarz’s inequality, the last integral is bounded by

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^3} |\mathcal{F}G(t)(\xi)|^2 k_{3-(\tilde{\alpha}+\beta)} d\xi,$$

which is known to be finite whenever $\tilde{\alpha} + \beta \in [0, 2]$ (see for instance [5, Eq. (2.5)]).

Since $k_{3-\tilde{\alpha}}(w)$ is integrable in a neighbourhood of the origin for any $\tilde{\alpha} > 0$, we finally obtain $I_1^{(1)}$, is bounded uniformly in $t, h \in [0, T]$.

Similar but simpler arguments show that the same property hold for $I_1^{(2)}$.

For any $\lambda \in [0, 1]$ set $\psi(\lambda) = k_{3-\tilde{\alpha}}(w + \lambda \frac{\eta}{t-r})$. It is easy to check that $|\psi'(\lambda)| \leq Ck_{4-\tilde{\alpha}}(w + \lambda \frac{\eta}{t-r})$. Moreover, for $|w| \geq 2$, and $|\eta| = t - r$,

$$\left| w + \lambda \frac{\eta}{t-r} \right| \geq \left| |w| - \left| \lambda \frac{\eta}{t-r} \right| \right| \geq |w| - 1 \geq \frac{|w|}{2},$$

by the triangular inequality.

Thus,

$$\begin{aligned}
 I_1^{(3)} &\leq C \int_0^t dr \frac{t+h-r}{t-r} \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} G(t-r, d\xi) G(t-r, d\eta) \\
 &\quad \times \int_{(B_2(0))^c} k_{\tilde{\alpha}+\beta}(\eta - \xi - hw) dw \int_0^1 k_{4-\tilde{\alpha}}\left(w + \lambda \frac{\eta}{t-r}\right) d\lambda \\
 &\leq C \left(\int_{(B_2(0))^c} dw k_{4-\tilde{\alpha}}(w) \right) \left(\int_0^t \frac{dr}{r} \int_{\mathbb{R}^3} |\mathcal{F}G(r)(\xi)|^2 k_{3-(\tilde{\alpha}+\beta)} d\xi \right). \tag{37}
 \end{aligned}$$

For $\tilde{\alpha} \in]0, 1[$ the integral $\int_{(B_2(0))^c} dw k_{4-\tilde{\alpha}}(w)$ is finite. Moreover, if $\tilde{\alpha} + \beta \in]0, 2[$ the last integral in (37) is also finite, owing to [5, Lemma 2.3] applied to the value $b = 1$. This lead us to conclude that $I_1^{(3)}$ is bounded uniformly in $t, h \in [0, T]$.

Summarizing, as a consequence of (35), (36) and the preceding discussion, we have proved that

$$\sup_{t \in [0, t]} \int_0^t \int_{B_{t-r}(0)} \int_{B_{t-r}(0)} \mu_2^h(dr; d\xi, d\eta) \leq Ch^{\tilde{\alpha}}, \tag{38}$$

with $\tilde{\alpha} \in]0, \frac{2-\beta}{2}[$.

We can now apply Hölder’s inequality with respect to the measure $\mu_2^h(dr; d\xi, d\eta)$. By virtue of (38), the linear growth of σ and (8) we obtain

$$\sup_{(t,x) \in [0, T] \times D} T_{1,3,2,2,2} \leq Ch^{\alpha p}, \tag{39}$$

with $\alpha \in]0, \frac{2-\beta}{2}[$.

Finally, the estimates (31), (33) and (39) imply that

$$\sup_{(t,x) \in [0, T] \times D} \|B_1^h(t, x)\|_p \leq Ch^\alpha, \tag{40}$$

and *a fortiori*

$$\sup_{(t,x) \in [0, T] \times D} \| \langle D_{\cdot, * } u(t, x), B_{\cdot, * }^h(t, x) \rangle_{\mathcal{H}_T} \|_p \leq Ch^\alpha, \tag{41}$$

with $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$. This finishes the analysis of the $\| \cdot \|_p$ contribution to the left-hand side of (27).

We next consider the $L^p(\Omega, \mathcal{H}_T)$ -norm of $D \langle D_{\cdot, * } u(t, x), B_{\cdot, * }^h(t, x) \rangle_{\mathcal{H}_T}$. As in the previous step, we shall replace $B_{\cdot, * }^h(t, x)$ by $B_{\cdot, * }^h(\tilde{t}, \tilde{x})$ with arbitrary $\tilde{t} \in [0, T]$, $\tilde{x} \in \mathbb{R}^3$. By virtue of (26) and (41) it suffices to study the $L^p(\Omega, \mathcal{H}_T)$ -norm of $DB_i^h(t, x; \tilde{t}, \tilde{x})$ for $i = 1, 2$. We start with the analysis of $B_2^h(t, x; \tilde{t}, \tilde{x})$.

By applying the differential rules of Malliavin calculus we have

$$\begin{aligned}
 D_{.*}B_2^h(t, x; \tilde{t}, \tilde{x}) &= G(t - \cdot, x - *)\sigma'(u(.*))\mathcal{D}(s, y; \tilde{t}, \tilde{x}) \\
 &\quad + \int_0^t \int_{\mathbb{R}^3} G(t - s, x - y)[\sigma''(u(s, y))D_{.*}u(s, y)\mathcal{D}(s, y; \tilde{t}, \tilde{x}) \\
 &\quad + \sigma'(u(s, y))D_{.*}\mathcal{D}(s, y; \tilde{t}, \tilde{x})]M(ds, dy).
 \end{aligned}$$

Applying Hölder’s inequality and using that σ' is bounded, we obtain, as in (28),

$$\begin{aligned}
 E\|G(t - \cdot, x - *)\sigma'(u(.*))\mathcal{D}(s, y; \tilde{t}, \tilde{x})\|_{\mathcal{H}_T}^p \\
 \leq C \int_0^t ds \sup_{y \in K_a^D(s)} \{E|\mathcal{D}(s, y; \tilde{t}, \tilde{x})|^p\}.
 \end{aligned} \tag{42}$$

For fixed \tilde{t}, \tilde{x} we consider the \mathcal{H}_T -valued process defined by

$$K(s, y; \tilde{t}, \tilde{x}) = \sigma''(u(s, y))D_{.*}u(s, y)\mathcal{D}(s, y; \tilde{t}, \tilde{x}) + \sigma'(u(s, y))D_{.*}\mathcal{D}(s, y; \tilde{t}, \tilde{x}), \tag{43}$$

$(s, y) \in [0, T] \times \mathbb{R}^3$, for which we have

$$\begin{aligned}
 E(\|K(s, y; \tilde{t}, \tilde{x})\|_{\mathcal{H}_T}^p) &\leq \|\sigma''\|_\infty^p (E(\mathcal{D}(s, y; \tilde{t}, \tilde{x})^{2p})^{\frac{1}{2}} (E\|Du(s, y)\|_{\mathcal{H}_T}^{2p})^{\frac{1}{2}} \\
 &\quad + \|\sigma'\|_\infty^p E\|D\mathcal{D}(s, y; \tilde{t}, \tilde{x})\|_{\mathcal{H}_T}^p.
 \end{aligned} \tag{44}$$

We can apply the L^p -estimates for stochastic integrals with respect to the Gaussian process M of Hilbert-valued integrands (see [15, Eq. (6.8) of Theorem 6.1] and [12, p. 289]) yielding

$$\begin{aligned}
 &\left\| \int_0^t \int_{\mathbb{R}^3} G(t - s, x - y)K(s, y; \tilde{t}, \tilde{x})M(ds, dy) \right\|_{L^p(\Omega; \mathcal{H}_T)}^p \\
 &\leq C \int_0^t ds \sup_{y \in K_a^D(s)} ((E(\mathcal{D}(s, y; \tilde{t}, \tilde{x})^{2p})^{\frac{1}{2}} + \|D\mathcal{D}(s, y; \tilde{t}, \tilde{x})\|_{L^p(\Omega; \mathcal{H}_T)}^p).
 \end{aligned} \tag{45}$$

By taking $\tilde{t} = t$ and $\tilde{x} = x$ and considering the inequalities (42), (45), we obtain

$$\begin{aligned}
 &\|DB_2^h(t, x)\|_{L^p(\Omega; \mathcal{H}_T)}^p \\
 &\leq C \int_0^t ds \sup_{y \in K_a^D(s)} [E|\langle D_{.*}u(s, y), B_{.*}^h(t, x) \rangle_{\mathcal{H}_T}|^p + (E|\langle D_{.*}u(s, y), B_{.*}^h(t, x) \rangle_{\mathcal{H}_T}|^{2p})^{\frac{1}{2}} \\
 &\quad + \|D\langle D_{.*}u(s, y), B_{.*}^h(t, x) \rangle_{\mathcal{H}_T}\|_{L^p(\Omega; \mathcal{H}_T)}^p].
 \end{aligned} \tag{46}$$

Then, (41) and Gronwall’s lemma yield

$$\begin{aligned} & \sup_{t \in [0, T]} \sup_{x \in K_a^D(t)} \|D(D_{\cdot, * } u(t, x), B_{\cdot, * }^h(t, x))\|_{\mathcal{H}_T} \|L^p(\Omega; \mathcal{H}_T)\|^p \\ & \leq C \left(\sup_{t \in [0, T]} \sup_{x \in K_a^D(t)} \|DB_1^h(t, x)\|_{L^p(\Omega; \mathcal{H}_T)}^p + h^{\alpha p} \right), \end{aligned} \tag{47}$$

with $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2} [$.

The last step of the proof consist of checking that for an arbitrary bounded set $D \subset \mathbb{R}^3$,

$$\sup_{(t,x) \in [0, T] \times D} \|DB_1^h(t, x)\|_{L^p(\Omega; \mathcal{H}_T)} \leq Ch^\alpha, \tag{48}$$

with $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2} [$.

The proof of this fact can be done following the same lines as for (40). We apply the results on the densities $\nu(dr; d\xi, d\eta)$, $\mu_1^h(dr; d\xi, d\eta)$, $\mu_2^h(dr; d\xi, d\eta)$, defined in (30), (32) respectively, proved so far. Instead of the process $\{\sigma(u(s, y)), (s, y) \in [0, T] \times \mathbb{R}^3\}$ and the $L^p(\Omega)$ -norm, we shall deal here with the \mathcal{H}_T -valued process $\{D(\sigma(u(s, y))), (s, y) \in [0, T] \times \mathbb{R}^3\}$ and the $L^p(\Omega; \mathcal{H}_T)$ -norm. In addition to (7), we should also apply (16) and (14). We leave the details to the reader.

Together with (19) and (24) this proves (18) and concludes the proof of the first step of the proof.

Remark 2.2. Applying first (12) and then estimates for the $\|\cdot\|_{2,p}$ -norm of the stochastic integral leads to

$$\begin{aligned} T_{1,3,2} & \leq C \left\| \int_0^t \int_{\mathbb{R}^3} \sigma(u(s, y)) [G(t+h-s, x-y) - G(t-s, x-y)] M(ds, dy) \right\|_{2,4^3} \\ & \leq Ch^{\frac{\alpha}{2}}. \end{aligned}$$

Thus, we loose accuracy. This may be a justification for a pretty tricky approach in the preceding proof.

The rest in the time expansion. The second and last term in (17) to be examined is

$$R(t, x, h) = |E[g''(\tilde{u}(t, x, h))(u(t+h, x) - u(t, x))^2]|.$$

We shall apply (12) to the random variables $\xi := \tilde{u}(t, x, h)$ and $Z := (u(t+h, x) - u(t, x))^2$. For this, we have to make sure that the assumptions of Lemma 1.3 are satisfied. For $Z := (u(t+h, x) - u(t, x))^2$, and the two choices of $\xi - u(t, x)$ and $u(t+h, x)$ —this has been proved in [14]. Then it suffices to remark that the norm $\|\cdot\|_{\mathcal{H}_T}$ as well as $\|\cdot\|_{\mathcal{H}_T}^{-1}$ define convex functions and use the definition of $\tilde{u}(t, x, h)$ to conclude.

Consequently,

$$R(t, x, h) \leq C \|u(t+h, x) - u(t, x)\|_{3,4^3}^2.$$

Owing to (7) and (16) we conclude that

$$\sup_{(t,x) \in [0,T] \times D} R(t, x, h) \leq Ch^{2\alpha}, \tag{49}$$

with $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$.

The estimates (18) and (49) show that

$$\sup_{(t,x) \in [0,T] \times D} |E[g(u(t+h, x)) - g(u(t, x))]| \leq Ch^\alpha,$$

with $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$. Therefore the mapping $t \in]0, T[\mapsto p_{t,x}(y)$ is Hölder continuous of degree $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$, uniformly in $y \in \mathbb{R}^3$ varying on bounded sets.

Step 2 (Space increments). Fix $t \in]0, T[$ and consider the Taylor expansion

$$\begin{aligned} E[g(u(t, \bar{x})) - g(u(t, x))] &= E[g'(u(t, x))(u(t, \bar{x}) - u(t, x))] \\ &\quad + E[g''(\hat{u}(t, x, \bar{x}))(u(t, \bar{x}) - u(t, x))^2], \end{aligned} \tag{50}$$

where $x, \bar{x} \in D$ and $\hat{u}(t, x, \bar{x})$ denotes a random variable lying on the segment determined by $u(t, \bar{x})$ and $u(t, x)$.

First order term. Our aim is to prove that

$$\sup_{t \in [0,T]} |E[g'(u(t, x))(u(t, \bar{x}) - u(t, x))]| \leq C|x - \bar{x}|^\alpha, \tag{51}$$

with $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$.

As we did for the time increments, we consider Eq. (6) and write

$$E[g'(u(t, x))(u(t, \bar{x}) - u(t, x))] = \sum_{i=1}^3 S_i(t, x, \bar{x}),$$

with

$$\begin{aligned} S_1(t, x, \bar{x}) &= \left| E \left[g'(u(t, x)) \left[\left(\frac{d}{dt} G(t) * v_0 + G(t) * \tilde{v}_0 \right) (\bar{x}) - \left(\frac{d}{dt} G(t) * v_0 + G(t) * \tilde{v}_0 \right) (x) \right] \right] \right|, \end{aligned}$$

$$\begin{aligned} S_2(t, x, \bar{x}) &= \left| E \left[g'(u(t, x)) \int_0^t ds \int_{\mathbb{R}^3} G(t-s, dz) [b(u(s, \bar{x}-z)) - b(u(s, x-z))] \right] \right|, \end{aligned}$$

$$S_3(t, x, \bar{x}) = \left| E \left[g'(u(t, x)) \int_0^t \int_{\mathbb{R}^3} [G(t-s, \bar{x}-z) - G(t-s, x-z)] \sigma(u(s, y)) M(ds, dz) \right] \right|.$$

Let us consider $S_1(t, x, \bar{x})$. As for the term $T_{1,1}(t, x, h)$, we first apply the inequality (12) and notice that

$$Z(t; x, \bar{x}) := \left(\frac{d}{dt} G(t) * v_0 + G(t) * \tilde{v}_0 \right) (\bar{x}) - \left(\frac{d}{dt} G(t) * v_0 + G(t) * \tilde{v}_0 \right) (x)$$

is deterministic. Thus, it suffices to estimate the absolute value of the random variable $Z(t; x, \bar{x})$ defined before. For this, we apply [5, Lemmas 4.2 and 4.4] which tell us that the fractional Sobolev norm of any integration degree $p \geq 2$ and differential order $\rho < \gamma_1 \wedge \gamma_2$ is bounded. Hence, since p is arbitrary, by the Sobolev embedding theorem we have that

$$\sup_{t \in [0, T]} S_1(t, x, \bar{x}) \leq \sup_{t \in [0, T]} C |Z(t; x, \bar{x})| \leq C |x - \bar{x}|^\rho, \tag{52}$$

with $\rho < \gamma_1 \wedge \gamma_2$.

We continue the proof with the study of the term $S_2(t, x, \bar{x})$. By virtue of (12), it suffices to find an upper bound of

$$\left\| \int_0^t ds \int_{\mathbb{R}^3} G(t-s, dz) [b(u(s, \bar{x}-z)) - b(u(s, x-z))] \right\|_{2,4^2}$$

in terms of a power of $|x - \bar{x}|$.

The measure on $[0, t] \times \mathbb{R}^3$ defined by $ds G(t-s, dz)$ is finite. Hence, we can apply Minkowski’s inequality and obtain for any $p \in [1, \infty[$

$$\begin{aligned} & \left\| \int_0^t ds \int_{\mathbb{R}^3} G(t-s, dz) [b(u(s, \bar{x}-z)) - b(u(s, x-z))] \right\|_{2,p} \\ & \leq \int_0^t ds \int_{\mathbb{R}^3} G(t-s, dz) \|b(u(s, \bar{x}-z)) - b(u(s, x-z))\|_{2,p} \\ & \leq C |x - \bar{x}|^\alpha, \end{aligned}$$

with $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$. The last inequality is obtained by using that b and its derivatives are Lipschitz continuous and bounded functions, and by applying (7) and (16).

Hence,

$$\sup_{t \in [0, T]} S_2(t, x, \bar{x}) \leq C |x - \bar{x}|^\alpha, \tag{53}$$

for any $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$.

To analyze $S_3(t, x, \bar{x})$ we proceed in a similar manner as for $T_{1,3}(t, x, h)$ by applying first [15, Proposition 3.9] and then (12). We obtain

$$\begin{aligned} S_3(t, x, \bar{x}) &= |E[g''(u(t, x))(D_{\cdot,*}u(t, x), \sigma(u(\cdot, *))) \\ &\quad \times [G(t - \cdot, \bar{x} - *) - G(t - \cdot, x - *)]1_{]0,t[}(\cdot)]_{\mathcal{H}_T}]| \\ &\leq C \|[D_{\cdot,*}u(t, x), \sigma(u(\cdot, *))] [G(t - \cdot, \bar{x} - *) - G(t - \cdot, x - *)]1_{]0,t[}(\cdot)]_{\mathcal{H}_T}\|_{3,4^3}. \end{aligned}$$

Notice that the last expression has a similar structure than the right-hand side of (25) where $B_{\cdot,*}^h(t, x) := G(t + h - \cdot, x - *) - G(t - \cdot, x - *)$ is replaced by $G(t - \cdot, \bar{x} - *) - G(t - \cdot, x - *)$. Hence, we can proceed as in the analysis of the time increments to see that it suffices to deduce an estimate for

$$\|[G(t - \cdot, x - *)\sigma(u(\cdot, *)), \sigma(u(\cdot, *))] [G(t - \cdot, \bar{x} - *) - G(t - \cdot, x - *)]\|_{\mathcal{H}_T}\|_{3,p},$$

for any $p \in [1, \infty[$.

To pursue the proof, we split the argument of the above expression into two terms

$$\begin{aligned} S_{3,1}(t, x, \bar{x}) &= \int_0^t dr \int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^3} d\eta G(t - r, x - \xi) \\ &\quad \times \sigma(u(r, \xi))f(\xi - \eta)\sigma(u(r, \eta))G(t - r, \bar{x} - \eta), \\ S_{3,2}(t, x, \bar{x}) &= \int_0^t dr \int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^3} d\eta G(t - r, x - \xi) \\ &\quad \times \sigma(u(r, \xi))f(\xi - \eta)\sigma(u(r, \eta))G(t - r, x - \eta), \end{aligned}$$

and we apply the change of variables $(\xi \rightarrow x - \xi, \eta \rightarrow \bar{x} - \eta)$, $(\xi \rightarrow x - \xi, \eta \rightarrow x - \eta)$, respectively. We obtain

$$\begin{aligned} &S_{3,1}(t, x, \bar{x}) - S_{3,2}(t, x, \bar{x}) \\ &= \int_0^t dr \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(t - r, d\xi)G(t - r, d\eta) f(\xi - \eta)\sigma(u(r, x - \xi)) \\ &\quad \times (\sigma(u(r, \bar{x} - \eta)) - \sigma(u(r, x - \eta))) \\ &\quad + \int_0^t dr \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(t - r, d\xi)G(t - r, d\eta) [f(x - \bar{x} - (\xi - \eta)) - f(\xi - \eta)] \\ &\quad \times \sigma(u(r, x - \xi))\sigma(u(r, \bar{x} - \eta)). \end{aligned} \tag{54}$$

By Minkowski’s inequality the $\|\cdot\|_{k,p}$ -norm of the first term in the right-hand side of (54) is bounded by

$$\begin{aligned} & \int_0^t dr \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(t-r, d\xi) G(t-r, d\eta) f(\xi - \eta) \\ & \quad \times \|\sigma(u(r, x - \xi)) [\sigma(u(r, \bar{x} - \eta)) - \sigma(u(r, x - \eta))] \|_{k,p} \\ & \leq C|x - \bar{x}|^\alpha, \end{aligned} \tag{55}$$

where the very last upper bound follows from (7), (16) and (14).

For the second term of the right-hand side of (54) we apply [5, Lemma 6.1] which implies

$$\sup_{t \in [0, T]} \int_0^t dr \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} G(r, d\xi) G(r, d\eta) |f(x - \bar{x} - (\xi - \eta)) - f(\xi - \eta)| \leq C|x - \bar{x}|^{\tilde{\alpha}},$$

with $\tilde{\alpha} \in]0, (2 - \beta) \wedge 1[$. From this and the properties (8), (14), we obtain the upper bound $C|x - \bar{x}|^{\tilde{\alpha}}$.

Hence we conclude

$$\sup_{t \in [0, T]} S_3(t, x, \bar{x}) \leq C|x - \bar{x}|^\alpha, \tag{56}$$

for any $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$. With (52)–(56), we have proved (51).

The rest term in the space expansion. The contribution of the second order term in (50) comes from the estimate

$$|E[g''(\hat{u}(t, x, \bar{x}))(u(t, \bar{x}) - u(t, x))^2]| \leq C\|g''\|_\infty h^{2\alpha},$$

which is a consequence of (7).

Hence, we have proved that for any fixed $y \in \mathbb{R}^3$ the mapping $x \in D \mapsto p_{t,x}(y)$ is Hölder continuous of degree $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$, uniformly in $t \in]0, T[$.

The proof of the theorem is now complete. \square

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References

[1] V. Bally, E. Pardoux, Malliavin calculus for white noise driven parabolic SPDEs, *Potential Anal.* 9 (1998) 27–64.
 [2] R.C. Dalang, Extending the martingale measure stochastic integral with applications to spatially homogeneous SPDE's, *Electron. J. Probab.* 4 (1999).
 [3] R.C. Dalang, N.E. Frangos, The stochastic wave equation in two spatial dimensions, *Ann. Probab.* 26 (1) (1998) 187–212.

- [4] R.C. Dalang, C. Mueller, Some non-linear SPDE's that are second order in time, *Electron. J. Probab.* 8 (1) (2003) 1–21.
- [5] R.C. Dalang, M. Sanz-Solé, Hölder–Sobolev regularity of the solution to the stochastic wave equation in dimension three, *Mem. Amer. Math. Soc.*, in press.
- [6] S. Kusuoka, D. Stroock, Applications of Malliavin calculus, Part II, *J. Fac. Sci. Univ. Tokyo, Sect. IA, Math.* 32 (1985) 1–76.
- [7] A. Millet, M. Sanz-Solé, A stochastic wave equation in two space dimensions: Smoothness of the law, *Ann. Probab.* 27 (1999) 803–844.
- [8] A. Millet, M. Sanz-Solé, Approximation and support theorem for a wave equation in two space dimensions, *Bernoulli* 6 (5) (2000) 887–915.
- [9] A. Millet, P.-L. Morien, On a stochastic wave equation in two space dimensions: Regularity of the solution and its density, *Stochastic Process. Appl.* 86 (1) (2000) 141–162.
- [10] P.-L. Morien, The Hölder and the Besov regularity of the density for the solution of a parabolic stochastic partial differential equation, *Bernoulli* 5 (2) (1999) 275–298.
- [11] D. Nualart, *The Malliavin Calculus and Related Topics*, second ed., *Probab. Appl. (N. Y.)*, Springer-Verlag, Berlin, 2006.
- [12] D. Nualart, L. Quer-Sardanyons, Existence and smoothness of the density for spatially homogeneous SPDEs, *Potential Anal.* 27 (2007) 281–299.
- [13] L. Quer-Sardanyons, M. Sanz-Solé, Absolute continuity of the law of the solution to the 3-dimensional stochastic wave equation, *J. Funct. Anal.* 206 (1) (2004) 1–32.
- [14] L. Quer-Sardanyons, M. Sanz-Solé, A stochastic wave equation in dimension 3: Smoothness of the law, *Bernoulli* 10 (1) (2004) 165–186.
- [15] M. Sanz-Solé, *Malliavin Calculus with Applications to Stochastic Partial Differential Equations*, *Fund. Sci. Math.*, EPFL Press, 2005, distributed by CCR Press.
- [16] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966.
- [17] S. Watanabe, *Lectures on Stochastic Differential Equations and Malliavin Calculus*, *Tata Inst. Fund. Res./Springer-Verlag*, Bombay, 1984.