Doubly Transitive Permutation Groups Which Are Not Doubly Primitive

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Communicated by Walter Feit

Received October 6, 1977

Some results on doubly transitive but not doubly primitive permutation groups are proved, giving more evidence to Atkinson's conjecture [3]. Among other results, we characterize the group $S_3(q)$ as a group satisfying the condition of the title and prove some sufficient conditions for such a group to be an automorphism group of a nontrivial block design with $\lambda = 1$.

INTRODUCTION

Recently there has been considerable interest in the structure of the one point stabilizer of a doubly transitive permutation group $G$ on a set $\Omega$. One problem is to describe those doubly transitive groups for which $G_\alpha, \alpha \in \Omega$, is imprimitive on $\Omega - \{\alpha\}$. Our general assumption is:

Hypothesis (A): $G$ is a doubly transitive permutation group on a set $\Omega$. For $\alpha \in \Omega$, $G_\alpha$ has a set $\Sigma = \{B_1, B_2, \ldots, B_t\}$, $t \geq 2$, which is a complete set of imprimitivity blocks on $\Omega - \{\alpha\}$. Let $|R_i| = h > 1$ for all $i$. Denote by $H$ the kernel of $G_\alpha$ on $\Sigma$ and by $K_i$ and $K'_i$ the subgroups of $G_\alpha$ fixing $B_i$ setwise and pointwise respectively, $1 \leq i \leq t$. Let $\beta \in B_2$. Here $|\Omega| = 1 + ht$.

M. D. Atkinson has conjectured that a group satisfying (A) is either an automorphism group of a nontrivial block design with $\lambda = 1$, or a normal extension of a Suzuki group, or must have a regular normal subgroup. We recall that the group $S_3(q)$ satisfies (A) with $H \neq 1, b = t - q$ and $G_\alpha^\Sigma$ is 2-transitive. In the first part of this article we consider groups acting "like" a Suzuki group, namely:

Hypothesis (B): Hypothesis (A) with $b = t$ (Then $|\Omega| = 1 + b^2$).

Hypothesis (C): Hypothesis (B) with $H \neq 1$.

We consider groups satisfying (C) when $G_\alpha^\Sigma$ 2-transitive and we show that if $G$ satisfies (B) then $G_{2B}^{\Sigma(P_1)}$ is never 2-transitive. C. E. Praeger ([17], [18], [19], [20] and [7]) and M. D. Atkinson characterized groups satisfying (A) with
conditions on $b$ and/or the action of $G_a$ and $G_{aB}$ on $\Sigma$. The only Suzuki group which arises in these papers is the $Sz(8)$ in [3]. We prove:

**Theorem 1.** Let $G$ satisfy (C) and assume that $G_a^Z$ is doubly primitive. Then $G$ is a normal extension of $Sz(q)$, $q = 2^{2n+1}$ for some $n$.

**Theorem 2.** Let $G$ satisfy (C) and assume that $G_a^Z$ is doubly transitive. Then, either

(i) $|\Omega|$ is odd and $G$ is a normal extension of a Suzuki group $Sz(q)$, $q = 2^{2n+1}$, or

(ii) $b = |H| = p^m$, $p$ a prime, $m > 2$, $H$ is elementary abelian and $G_a/H$ is isomorphic to a subgroup of $GL(m, p)$. Furthermore, $K_1 = 1$ and $G$ contains no regular normal subgroup.

**Theorem 3.** If $G$ satisfies (B) then $G_{aB}^{(B1)}$ is not doubly transitive.

We have the following corollaries.

**Corollary I.** Let $G$ satisfy (B). Then:

(i) If $|\Omega|$ is even then $G_a^Z$ is not triply transitive.

(ii) If $G_a^Z$ is triply transitive then $G_a$ is a rank 6 group with subdegrees: $1, b - 1, b - 1, b - 1, ((b - 2)/2)(b - 1)$. Also, $|H| = K_1 = 1$.

**Corollary II.** Let $G$ satisfy (B). Assume that $G_{aB}$ is doubly transitive and that $b - 1$ is a prime number. Then $G$ is a normal extension of $Sz(q)$, $q = 2^{2n+1}$.

**Corollary III.** Let $G$ satisfy (B). Assume that $G_{aB}$ has an orbit on $\Omega$ of size $m$. $m \geq b - 1$, $m \neq b$, on which $G_{aB}$ is doubly transitive. Then $F = \text{Fix}(G_{aB})$ contains more than two points and the $G$-translates of $F$ form a block design with $\lambda = 1$ on $\Omega$ of which $G$ is an automorphism group.

In the last part of this paper we consider the question: What are sufficient conditions for a group satisfying (A) to act as an automorphism group of a nontrivial block design on $\Omega$ with $\lambda = 1$? One answer was given in [20] Theorem B, which states that $G_{aB}^{(B1)}$ being transitive and $b < t$, is such a condition. We consider groups with $G_{aB}^{(B1)}$ transitive for some values of $b > t$ and prove that for these values $G$ is an automorphism group of a nontrivial design with $\lambda = 1$. For example we prove that if $G_{aB}^{(B1)}$ is transitive and $t = b - 1$ or $b - 2$ or $b - 3$ then $G$ is an automorphism group of a nontrivial block design with $\lambda = 1$. The exact statement is Proposition 1 of Section 3. A result of the same type is Theorem 4 in Section 1 which is, in fact, Theorem B of [20] replacing the assumption $b < t$ by $K_1 = 1$. 


1. Preliminaries

We denote by $\text{Fix}(R)$ the set of fixed points of the subgroup $R$. The rest of our notation is standard and the reader is referred to [24] and [21] for basic information and notation about permutation groups and block designs respectively.

In this section we prove three lemmas from which the theorems will follow.

**Lemma 1.** Let $G$ satisfy (A) and $t \leq b$. Then:

(a) In any block design with $\lambda = 1$ on $\Omega$ we have that $k \leq b$.

(b) If $G_{a_0}^{\Sigma - \{\beta\}}$ is transitive then there exists an $G_{a_0}$-orbit, $\Gamma_0$, $\Gamma_0 \subseteq B_1 - \{\beta\}$, such that $|\Gamma_0| = m(t - 1)$ for some natural number $m$. Furthermore, if $g \in G_{(a, \beta)} - G_{a_0}$ and $\Gamma = \Gamma_0^g$ then $\Gamma$ is a $G_{a_0}$-orbit on $\Omega - B_1 - \{\alpha\}$ such that $|\Gamma \cap B_i| = m$ for all $i \geq 2$.

(c) If $G_{a_0}^{\Sigma - \{\beta\}}$ is transitive then $K_1^i = 1$ for all $i$.

**Proof:** (a) If $k > b$ then $k - 1 > b$ so that $r(k - 1) = bt$ implies $r \leq t \leq b < k$ contradicting Fisher's inequality.

(b) If $B_1 - \{\beta\}$ is $G_{(a, \beta)}$-invariant then Lemma 2 of [1] implies that there is a nontrivial block design on $\Omega$ with $\lambda = 1$ and $k = b + 1$, contradicting (a).

Thus, there exists a $G_{a_0}$-orbit, $\Gamma_0$, $\Gamma_0 \subseteq B_1 - \{\beta\}$, such that $\Gamma_0^g \not= B_1$ for all $g \in G_{(a, \beta)} - G_{a_0}$. Let $g \in G_{(a, \beta)} - G_{a_0}$ and let $\Gamma = \Gamma_0^g$. The set $\mathcal{A} = \{B_i \mid B_i \cap \Gamma \not= \emptyset\}$ is a $G_{a_0}$-orbit on $\Sigma - \{B_1\}$ and since $G_{a_0}^{\Sigma - \{\beta\}}$ is transitive we get that $\mathcal{A} = \Sigma - \{B_1\}$ and $|\mathcal{A}| = t - 1$. The set $\{B_i \cap \Gamma \mid i \geq 2\}$ is a complete set of imprimitivity blocks for the action of $G_{a_0}$ on $\Gamma$ so that $|B_i \cap \Gamma| = m$ for some $m$, for all $i \geq 2$ and consequently $|\Gamma| = m \cdot |\mathcal{A}| = m(t - 1) = |\Gamma_0|$.  

(c) Suppose $K_1 \not= 1$. Then $G_{a_0}$ is not faithful on $\Gamma_0$ and therefore $G_{a_0}$ is not faithful on $\Gamma$ as $G_{(a, \beta)}$ normalizes $G_{a_0}$. An element of $G_{a_0}$ fixing $\Gamma$ pointwise must fix every $B_i$, $i \geq 2$, as $\Gamma \cap B_i \not= \emptyset$ for all $i \geq 2$. It follows that $H \not= 1$. If $\text{Fix}(K_1) = B_1 \cup \{\alpha\}$ then we get a contradiction using Lemma 1.1 of [17] and part (a). Therefore $K_1$ fixes a point in $\bigcup_{i \geq 2} B_i$ so that $K_1$ fixes some $B_i$, $i \geq 2$. Since $K_1 \subseteq G_{a_0}$, $K_1$ is $1/2$-transitive on $\Sigma - \{B_1\}$ and therefore $K_1 \subseteq H$.

It follows that $H$ does not restrict faithfully to its orbits so that Proposition 4 of [15] implies that $G$ is a normal extension of $PSL(n, q)$. Since $PSL(2, q)$ is doubly primitive, $n > 2$. But then $G$ has a unique system of imprimitivity blocks with $b = q$ and $t = (q^{n-1} - 1)/(q - 1)$, contradicting $t < b$.

As a corollary we can state now Theorem B of [20] replacing the assumption $b < t$ by $K_1 \not= 1$:

**Theorem 4.** Suppose $G$ satisfies (A) and $K_1 \not= 1$. If $G_{a_0}^{\Sigma - \{\beta\}}$ is transitive then the $G$-translates of $B_1 \cup \{\alpha\}$ form a nontrivial block design with $\lambda = 1$ of which $G$ is an automorphism group.
**Proof.** Lemma 1 implies that $b < t$ and the result follows from Theorem B of [20].

The next two lemmas consider groups satisfying (B).

**Lemma 2.** Let $G$ satisfy (B). Then:

(a) If $b$ is a prime power and $G$ contains a regular normal subgroup then $|\Omega| = p$, $p$ a prime, and $|G| = (p - 1)p$, $G$ a Frobenius group.

(b) $G_{a,\Sigma}^\Sigma$ is doubly transitive if and only if $G_{a,\Sigma}^{\Sigma-\{b_1\}}$ is transitive.

(c) If $G_{a,\Sigma}^\Sigma$ is doubly transitive then $(K_{a,\Sigma})^\Sigma$ is doubly transitive.

(d) Assume that $G_{a,\Sigma}^\Sigma$ is doubly transitive, then the $G_{a,\Sigma}$-orbits $\Gamma_0$ and $\Gamma$ of Lemma 1(b) are of size $b - 1$ each. There exists another $G_{a,\Sigma}$-orbit $\Gamma_1$ of size $b - 1$ on $\Omega - B_1 - \{\alpha\}$, $\Gamma_1 \not= \Gamma$, such that for all $i \geq 2$, $|B_i \cap \Gamma_1| = |B_i \cap \Gamma_1'| = 1$.

**Proof:** (a) Here $b^2 + 1 = p^n$ for some prime $p$ and some $n$. Suppose $n > 1$. If $b$ is odd $p = 2$ and $b^2 + 1 = 1$ (mod 4). But $2^n - 1 = 2$ (mod 4), a contradiction. Thus $b = 2^m$ for some $m$. Now $2^{2m} = (p - 1)(p^{2m - 1} + p^{2m - 2} + \cdots + 1)$. It follows that $n$ is even. Thus $p^n = 0$ or 1 (mod 3). If $p^n = 1$ (mod 3), $p^n - 1 = 2^{2m} = 0$ (mod 3), a contradiction. Thus $p = 3$ and $3^{n} - 1 = 2^{2m}$. Since $n$ is even $3^n - 1 = 0$ or 3 (mod 5) while $2^{2m} = 4^m = 1$ or 4 (mod 5), a contradiction. We conclude that $n = 1$, $b = 1 = p$ and $G$ is a Frobenius group of order $(p - 1)p$ by [24] 4.4 which implies that $G$ is solvable and by [24] 11.6.

(b) If $G_{a,\Sigma}^{\Sigma-\{b_1\}}$ is transitive, so is $(K_{a,\Sigma})^{\Sigma-\{b_1\}}$ because $G_{a,\Sigma} = (K_{a,\Sigma})_{b_1}$, and so $G_{a,\Sigma}^\Sigma$ is doubly transitive. Conversely, if $G_{a,\Sigma}^\Sigma$ is doubly transitive, $K_{a,\Sigma}^{\Sigma-\{b_1\}}$ is transitive of degree $b - 1$ and since $|K_{a,\Sigma}^{\Sigma-\{b_1\}} : G_{a,\Sigma}^{\Sigma-\{b_1\}}|$ divides $b$, [24] 17.1 implies that $G_{a,\Sigma}$ is transitive on $\Sigma - \{B_1\}$.

(c) and (d) Since $b = t$, the orbits $\Gamma_0$ and $\Gamma$ of Lemma 1 are both of size $b - 1$ and $m = 1$. Thus $\Gamma_0 = B_1 - \{\beta\}$, $G_{a,\Sigma}$ is transitive on $B_1 - \{\beta\}$ and $(K_{a,\Sigma})^\Sigma$ is doubly transitive as $K_{a,\Sigma}$ is clearly transitive on $B_1$. The existence of another $G_{a,\Sigma}$-orbit, $\Gamma_1$, of size $b - 1$ which is $G_{a,\Sigma}$-invariant follows from Lemma 3 of [1]. Since $\Gamma_0$ and $\Gamma$ are not invariant under $G_{a,\Sigma}$ we have that $\Gamma_0 \not= \Gamma_1$, $\Gamma \not= \Gamma_1$. Now $\{B_i | B_i \cap \Gamma_1 \neq \emptyset\}$ is a $G_{a,\Sigma}$-orbit on $\Sigma - \{B_1\}$ and so $|\Gamma_1 \cap B_i| = 1$ and $|\Gamma \cap B_i|$ for all $i \geq 2$.

**Lemma 3.** Let $G$ satisfy (B). Assume that $G_{a,\Sigma}^\Sigma$ is doubly transitive. Let $\gamma \in B_1 - \{\beta\}$ and $2 \leq j \leq b$. By Lemma 2(d) $|B_j \cap \Gamma| = |B_j \cap \Gamma_1| = 1$, $\Gamma$ and $\Gamma_1$ as in Lemma 2.

Let $\Gamma \cap B_j = \{\tau\}$ and $\Gamma_1 \cap B_j = \{\rho\}$. Then $G_{a,\Sigma} \cap K_j$ fixes both $\tau$ and $\rho$. Thus $G_{a,\Sigma} \cap K_j \subseteq (K_{a,\Sigma})_{\tau,\rho}$. On the other hand, since $(K_{a,\Sigma})_{\tau,\rho}$ is doubly transitive...
and \( G_{aB} \) is transitive on \( \Sigma - \{B_i\} \) (see Lemma 2) we have that: \(| K_j : G_{aB} \cap K_j | = | K_1 : G_{aB} | \cdot | G_{aB} : G_{aB} \cap K_j | = b(b - 1) = | K_j : (K_j)_{r_\rho} | \). Hence \( G_{aB} \cap K_j = (K_j)_{r_\rho} \).

Now let \( g \in G_{aA} - K_1 \) be such that \( B_1^g = B_j \). Then \( (K_1)^g = K_j \) and \( ((K_1)_\rho)^g = (K_j)_\rho \) and the double transitivity of \( K_1^B \) implies that \( ((K_1)_\rho)^g \) is conjugate in \( K_j = (K_1)^g \) to \( (K_j)_{r_\rho} \). Let \( k \in K_1 \) be such that \( ((K_1)_\rho)^{g^{-1}k}\rho = (K_j)_{r_\rho} \). Since \( (K_j)_{r_\rho} \) fixes a point \( \beta \in B_1 \), \( (K_1)_\rho = G_{aB} \) fixes a point in \( (B_1)^{g^{-1}k^{-1}} = B_i \) for some \( i \). Since \( k \in K_1 \) and \( g \notin K_1 \), \( i > 1 \). Thus \( G_{aB} \) fixes \( B_i \) as a set and since its order is equal to \( | G_{aB} \cap K_1 | \) we have that \( G_{aB} \cap K_j = (K_j)_{r_\rho} \), where \( \eta, \eta \) are in \( B_i \). Let \( h \in G_{aB} \) be such that \( (B_i)^h = B_j \). Then if \( \delta = \gamma h \) we have that \( G_{aB} \cap K_j = G_{aB} \) and \( \delta \in B_1 - \{\beta\} \).

2. Proof of Theorems

Proof of Theorems 1 and 2. Here we assume that \( H \neq 1 \). If \( H_2 \neq 1 \) then \( H_2 \) fixes the point in \( \Gamma \cap B_1 \) for all \( i \geq 2 \), \( \Gamma \) as in Lemma 2. Then by [14] \( B_1 \) we have a block design with \( \lambda = 1 \) on \( \Omega \) with \( k \geq b + 1 \), contradicting Lemma 1(a). Therefore \( H_2 = 1 \). Since \( (K_1)^{b_1} \) is doubly transitive (see Lemma 2), \( H \) is a normal regular subgroup of \( K_1 \) because \( K_1 = (K_1)^{b_1} \) by Lemma 1(c). Therefore \( | H | = b = p^n \) for some prime \( p \), \( n \geq 1 \), and \( H \) is elementary abelian. If \( G \) contains a regular normal subgroup then \( G_{aB} = 1 \) by Lemma 2(a) contradicting our assumption. Thus \( G \) contains no regular normal subgroup and in particular \( n > 1 \) (See [6]). If \( | \Omega | \) is odd, that is, \( b \) is even, O'Nan's theorem ([16]) implies that \( G \) is a normal extension of one of the following: \( S_\mathcal{Z}(q) \), \( PSU_3(n, q) \) for \( n \geq 2 \), \( PSU_3(2^k) \). But \( PSL_2(q) \) is doubly primitive and \( PSL(n, q) \), \( n \geq 3 \) and \( PSU_3(2^k) \) do not satisfy \( b = t \). Thus (i) of Theorem 2 holds.

Assume that \( G_{aB} \) is doubly primitive. This implies that \( K_1 \) is primitive on \( \Sigma - \{B_i\} \) and since \( K_2 = G_{aB} \) so that \( G_{aB} \simeq (K_2)^{\Sigma - \{B_i\}} \), \( G_{aB} \) is a primitive permutation group on \( \Sigma - \{B_i\} \). Lemma 3 implies that the permutation representations of \( G_{aB} \) on \( \Sigma - \{B_i\} \) and on \( B_1 - \{\beta\} \) are equivalent so that \( G_{aB} \) is primitive on \( B_1 - \{\beta\} \). Thus \( K_1 \) is a doubly primitive group on \( B_1 \) forcing \( p = 2 \) (see [24] 11.3). The previous paragraph implies that \( G \) is a normal extension of \( S_\mathcal{Z}(q) \), proving Theorem 1.

To complete the proof of Theorem 2 we may assume that \( p \) is odd and we prove that (ii) holds. Let \( C = C_{G_3}(H) \). Then \( C \leq G_3 \). Suppose that \( C > H \). Then \( 1 \neq C^{\Sigma} < G_{aB} \) so that \( C^{\Sigma} \) is transitive. But \( C^{\Sigma} \cap K_2^\Sigma \simeq (C \cap K_2)/H = H/H = 1 \) since \( H \) is a regular normal subgroup of \( K_1 \). Hence \( C^{\Sigma} \) is a regular normal subgroup of \( G_{aB} \) and consequently \( C \) is a \( p \)-group, \( C \leq G_3 \) and \( C \) is transitive on \( \Omega - \{x\} \). It follows that \( G \) is a normal extension of one of the groups of Theorem C' of [12]. But \( PSL_2(q) \) is doubly primitive, the degree of \( S_\mathcal{Z}(q) \) is odd, in \( PSU_3(3, q) \), \( b \neq t \) and in the case of a group of \( C \)-type, the degree is not a square plus 1. This contradiction implies that \( C = H \) and therefore
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Given \( G_a/H \) is isomorphic to a subgroup of \( GL(n, p) \). Finally if \( n = 2 \), \( p^2 \) does not divide \( G_a/H \cong G_a^x \) contradicting the fact that \( G_a^x \) is transitive of degree \( b = p^2 \). Hence \( n > 2 \) as claimed.

Proof of Theorem 3. Assume that \( G_{a\delta} \) is doubly transitive on \( \Sigma - \{B_1\} \). Then so is \( K_1 \) and consequently \( G_a^x \) is triply transitive. If \( H \neq 1 \) then \( G_a^x \) is a normal extension of \( S_5 \) by Theorem 1. But then \( G_a^x \) is not triply transitive, a contradiction. Thus \( H = 1 \). Also \( K_1 = 1 \) by Lemma 1(e). Lemma 3 implies that the permutation representations of \( G_{a\delta} \) on \( B_1 - \{\beta\} \) and on \( \Sigma - \{B_1\} \) are equivalent, in particular \( G_{a\delta} \) is doubly transitive on \( B_1 - \{\beta\} \) so that \((K_1)^{B_1} = K_1 \) is triply transitive. Now, a theorem of Cameron ([5]) implies that either \( K_1 \) has a regular normal subgroup on \( B_1 \) or \( b = 6 \) and \( K_1 \cong PGL(2, 5) \). If \( N \) is a normal regular subgroup of \( K_1 \), \( |N| = b \). On the other hand since \((K_1)^{B_1} = K_1 \) is doubly transitive, \( N \) is transitive on \( \Sigma - \{B_1\} \) forcing \( b - 1 \) \| \( |N| = b \) which is impossible. Thus \( b = 6 \) and \( |\Omega| = 37 \). But then \( G \) is doubly primitive (See [13] p. 528), a contradiction.

Proof of Corollary 1. Assume that \( G_a^x \) is triply transitive. As in the proof of Th. 3 (and by Lemma 2(b)), \( H = K_1 = 1 \). Theorem 3 implies that \( G_{a\delta} \cap K_2 \) is not transitive on \( \Sigma - \{B_1, B_2\} \). Now

\[
|K_1 \cap K_2 : G_{a\delta} \cap K_2| = \frac{|K_1 : G_{a\delta} \cap K_2|}{|K_1 : K_1 \cap K_2|} = \frac{b(b-1)}{b-1} = b
\]

(See Lemma 2 and 3.). Since \( K_1 \cap K_2 \) is transitive on \( \Sigma - \{B_1, B_2\} \) the orbits of \( G_{a\delta} \cap K_2 \) on \( \Sigma - \{B_1, B_2\} \) are of size divisible by \((b-2)\{b, b-2\} \) (See [24] 17.1) and consequently \( G_{a\delta} \) has two orbits of size \((b-2)/2\) on it. It follows that \( b \) is even proving (i).

Since the representations of \( G_{a\delta} \) on \( B_1 - \{\beta\} \) and on \( \Sigma - \{B_1\} \) are equivalent (Lemma 3), \( G_{a\delta} = (K_1)^{\beta} \) is a rank 3 group on \( B_1 - \{\beta\} \) with subdegrees 1, \((b-2)/2\), \((b-2)/2\). Let \( \Gamma \) and \( \Gamma_1 \) be as in Lemma 2. Let \( \{\beta_i\} = \Gamma \cap K_1 \) and \( \{\gamma_i\} = \Gamma \cap K_i \) for \( i \geq 2 \). Then \((K_1)^{\beta_i, \gamma_i} \) has two orbits on \( B_1 - \{\beta_i, \gamma_i\} \) of size \((b-2)/2\) each. As in the proof of Lemma 3 we can show that \( G_{a\delta} \cap K_i = (K_i)^{\beta_i, \gamma_i} \). Let \( A_2 \) and \( A_2 \) be the two \((K_2)^{\beta, \gamma_i}\)-orbits on \( B_1 - \{\beta_2, \gamma_i\} \). No element of \( G_{a\delta} \) maps a point of \( A_2 \) into \( A_2 \), for if \( g \in G_{a\delta} \) does, \( g \) fixes \( B_2 \) and then \( g \in G_{a\delta} \cap K_2 = (K_2)^{\beta_2, \gamma_2} \) for which \( A_2 \) and \( A_2 \) are different orbits. This implies that for any \( g_1, g_2 \in G_{a\delta} \), no element of \( G_{a\delta} \) maps a point of \( A_2 \) into \( A_2 \) and in particular \( A_2^g \cap A_2^g = \emptyset \). It follows that if \( g \in G_{a\delta} \) is such that \( B_2^g = B_1 \) then \( A_2^g \) and \( A_2^g \) are the two \( G_{a\delta} \cap K_i = (K_i)^{\beta_i, \gamma_i}\)-orbits on \( B_1 - \{\beta_i, \gamma_i\} \). Since \( G_{a\delta}^x \) is transitive, this implies that \( G_{a\delta} \) is transitive on both \( \Delta = \bigcup g \in G_{a\delta} A_2^g \) and \( \Lambda = \bigcup g \in G_{a\delta} A_2^g \) and that \(|\Delta| = |\Lambda| = (b-2)/2(b-1)\). The argument above shows that \( \Lambda \) and \( \Delta \) are \( G_{a\delta}\)-invariant as \( \Delta \cup \Lambda = \Omega = \{\alpha, \beta\} - \Gamma_0 - \Gamma - \Gamma_1 \) and therefore they are \( G_{a\delta}\)-orbits as desired.

Proof of Corollary 2. Let \( b = p + 1 \), \( p \) a prime. Then \( p \neq 2 \) by [6]. Recall
that $K_1 = 1$ by Lemma 1(c). If $G_{a\beta}$ is nonsolvable, $G_{a\beta}$ is doubly transitive ([24] 11.7) on $B_1 - \{\beta\}$ and by Lemma 3 we get that $G_{a\beta}^{\sigma_1(B_1)}$ is doubly transitive, contradicting Theorem 3. Thus $G_{a\beta}$ is solvable so that either $|G_{a\beta}| = p$ or $G_{a\beta}$ is a Frobenius group. Assume first that $(K_1)^{B_1} = K_1$ contains no regular normal subgroup. Then $(K_1)^{B_1}$ is a Zassenhaus group of degree $1 + p$. By [8], [11] and [22] we have that $K_1 \simeq PSL(2, p)$. Since $|K_1 : K_1 \cap K_2| = p$, [9] p. 214 clearly implies that $p = 5, 7, 11$. If $p = 5$, $|\Omega| = 37$ and $G$ is doubly primitive. If $p = 7, 11$, $|G_{a\beta}|$ is odd and $|\Omega| = 65$ or 145. Since in $PSU(3, 2^k)$, $b \neq t$ [4] implies that $p = 7$ and $G \simeq S_8(8)$.

Therefore, $(K_1)^{B_1}$ contains a regular normal subgroup $N$, $|N| = 1 + p$. Since $K_1$ is transitive on $\Sigma - \{B_1\}$ of degree $p$, $N$ is half transitive and so $N$ is trivial on $\Sigma - \{B_1\}$. Thus $H \neq 1$ and the result follows from Theorem 2(ii).

Proof of Corollary 3. Let $\Lambda$ be a $G_{a\beta}$ orbit of size $m$, $m \geq b - 1$, $m \neq b$, on which $G_{a\beta}$ is doubly transitive. If $\Lambda \subseteq B_1$, then $\Lambda = B_1 - \{\beta\}$ and Lemma 1 and Lemma 2 of [1] implies that $A^p, g \in G_{a\beta}$, is another $G_{a\beta}$-orbit on which $G_{a\beta}$ is doubly transitive and $A^p \cap B_1 = \emptyset$. Thus we may as well assume that $A \cap B_1 = \emptyset$. Let $i > 1$ be such that $B_i \cap A \neq \emptyset$. Since $B_i \cap A$ is an imprimitivity block for the action of $G_{a\beta}$ on $A$ and since $G_{a\beta}$ is primitive we have that $|B_i \cap A| = m$ or 1. If for all $i > 1$, $|B_i \cap A| \neq m$ then $|B_i \cap A| = 1$ for all $i > 1$ and $m = b - 1$. Then $G_{a\beta}^{\sigma_1(B_1)}$ is doubly transitive, contradicting Theorem 3. Hence there is an $i > 1$ such that $|B_i \cap A| = m$ and so $m = b - 1$ and $A \subseteq B_i$. It follows that $G_{a\beta}$ fixes $B_i$ and thus it fixes the only point of $B_1 - A$.

Now the result follows from [1] Lemma 1.

3. Automorphism Groups of Block Designs

In [20] Theorem B it is shown that $G_{a\beta}^{\sigma_1(B_1)}$ being transitive is a sufficient condition for a group satisfying (A) to be an automorphism group of a nontrivial block design on $\Omega$ with $\lambda = 1$, provided $b < t$. For $b = t$ this condition is not sufficient as the example of $Sz(q)$ shows. Corollary 3 gives a sufficient condition for this case. In this section we consider the above condition for some values of $t < b$ and show that it is a sufficient condition for them. We prove:

Proposition 1. Let $G$ satisfy (A) and assume that $G_{a\beta}^{\sigma_1(B_1)}$ is transitive. Suppose that one of the following holds:

(a) $t > b - 4$ and $t \neq b$
(b) $t = b - 4, t \neq 5, 11, 16, 21$.
(c) $t = b - 5, t \neq 5$, and either $t > 121$ or $t < 121$ and $6 \neq t$.

Then $G$ is an automorphism group of a nontrivial block design with $\lambda = 1$ on $\Omega$. Moreover, if $t = b - 1$ then $|\text{Fix}(G_{a\beta})| = 3$ so that design has $k = 3$. 

First we prove a lemma.

**Lemma 4.** Let $G$ satisfy (A). Let $\Lambda$ be a $G_{aB}$ orbit of size $k$ on $B_1 - \{\beta\}$. Assume that:

(a) $\Lambda$ is the only orbit of $G_{aB}$ on $B_1 - \{\beta\}$ of size dividing $k!$.
(b) $G_{aB}$ is not faithful on $\Lambda$ and $G_{aB}^{e - \{B_1\}}$ is transitive.
(c) $t - 1 > k!$

Then $G$ is an automorphism group of a non-trivial block design on $\Omega$ with $\lambda = 1$.

**Proof:** Let $N$ be the kernel of $G_{aB}$ on $\Lambda$. Let $g \in G$ such that $Ng \subseteq G_{aB}$. Since $|G_{aB} : N^\theta|$ divides $k!$, [24] 17.2 implies the size of each orbit of $N^\theta$ on $\Sigma - \{B_1\}$ is at least $(t - 1)/k! > 1$. It follows that $N^\theta$ fixes no block other than $B_1$ and therefore $\text{Fix}(N^\theta) \cap (\Omega - B_1 - \{a\}) = \emptyset$ and $\text{Fix}(N^\theta) \subseteq B_1 \cup \{a\}$. Let $\Delta$ be a $G_{aB}$-orbit on $B_1 - \{\beta\}$ on which $N^\theta$ fixes a point $\theta$. Then $N^\theta \subseteq G_{aB}$ and since $|G_{aB} : G_{aB}^\theta| = |\Delta|$ we get that $|\Delta|$ divides $|G_{aB} : N^\theta|$.

By assumption $\Delta = \Lambda$ and therefore $\text{Fix}(N^\theta) \cap (B_1 - \{\beta\}) \subseteq \Lambda$. Since $|\text{Fix}(N^\theta)| = |\text{Fix}(N)|$, $N^\theta = N$ and $N$ is a weakly closed subgroup of $G_{aB}$ in $G$. Now the result follows from [1] Lemma 1.

**Proof of Proposition 1.** If $b < t$ the result is Theorem B of [20]. For $t < b$, $G_{aB}$ has an orbit, $I_0$, on $B_1$ of size $m(t - 1)$ and $K_1 = 1$ (see Lemma 1). If $t = b - 1$ then $m = 1$ unless $t = 2$. Hence, if $t = 2$, $(K_1)^{b_1}$ is a rank 3 group with subdegrees 1, 1, $t - 1$. Since $t - 1 > 1$ and $G_{aB}^{e - \{B_1\}}$ is transitive we get that $|\text{Fix}(G_{aB})| = 3$ and by Lemma 1 of [1] we're done. The case $t = 2$ is impossible, for if $t = 2$, $b = 3$ and $G_{aB} = 1$ (by [7] Proposition 2.1).

By Lemma 1 of [1] we can assume that $|\text{Fix}(G_{aB})| = 2$. Also by assumption $G_{aB} \neq 1$.

Assume that $t = b - 2$. If $t > 3$, $m = 1$ and $(K_1)^{b_1} = K_1$ is a rank 3 group with subdegrees 1, 2, $t - 1$. As $t - 1 > 1$ $|G_{aB}|$, $G_{aB}$ is not faithful on the orbit of size 2 and the result follows from Lemma 4. If $t = 2$, $b = 4$ then [2] implies the result and if $t = 3$, $b = 5$ then [1] implies it.

Let $t = b - 3$. If $t - 1 > 6$, $m = 1$ and $(K_1)^{b_1} = K_1$ is a rank 3 group with subdegrees 1, 3, $t - 1$. Again $|G_{aB}| > 6$ so that $G_{aB}$ is not faithful on the orbit of size 3. Using Lemma 4 we're done. If $t = 2, 3, 4, 5, 7, |\Omega| = 11, 19, 29, 41, 71$ respectively and $G$ is doubly primitive (see [13] p. 528). If $t = 6$, $b = 9$ and the subdegrees are 1, 5, 3. Since $4 < 9$, [16] Lemma 4 implies that $(K_1)^{b_1}$ is primitive. Let $\Lambda$ be the $G_{aB}$ orbit of size 3, then $5 < |G_{aB}^\Lambda|$, contradicting [24] Th. 18.4.

Suppose that $t = b - 4$. If $t = 2$, $|\Omega| = 13$ and $G$ is a normal extension of $PSL(3, 3)$ for which $t \neq b - 4$. If $t = 3$, $b = 7$ and [7] (2.1) yields a contradiction. If $t = 4$, $b = 8$ and the proposition holds by [3]. Now, $t \neq 5$ by assumption and if $t = 6$, $|\Omega| = 61$ and $G$ is doubly primitive (see [13] p. 528).
Hence we can assume that $t > 6$. Since $m = 1$, the subdegrees of $(K_1)^B_1$ are either $1, 2, 2, t - 1$ or $1, 4, 3, t - 1$. In the first case, let $A_1, A_2$ be the two orbits of size $2$ and $N_1, N_2$ the kernels of $G_{a\beta}$ on $A_1, A_2$, respectively. Since $t - 1 > 5$, $N_1 \cap N_2 \neq 1$ since $N_1 \cap N_2 = 1$ would imply $|G_{a\beta}| = 4$. Then as in the proof of Lemma 4 it can be shown that $N_1 \cap N_2$ is a weakly closed subgroup of $G_{a\beta}$ in $G$. Then, we're done by Lemma 1 of [1]. In the case $1, 4, t - 1$ we get the proposition using Lemma 4 if $t - 1 > 24$. Assume that $(K_1)^B_1$ is primitive. Then for some number $\mu \neq 0$ and $\lambda$ we have that $\mu(t - 1) = 4(3 - \lambda)$ (see [10] Lemma 5, Cor. 3). Then $t - 1 = 6, 8, 12$ so that $b$ is a prime and $|\Omega|$ is not a prime power. This contradicts [7] (2.1). If $(K_1)^B_1$ is imprimitive then $5 | b$ (see [10] Lemma 4) so that for $6 < t \leq 25$ we have $t = 11, 16, 21$. These are excluded in the assumption.

Finally, let $t = b - 5$. If $t = 2, 6, b$ is a prime and we get a contradiction using [7] (2.1). If $t = 3, b = 8$ and [3] implies the result. If $t = 4$, $|\Omega| = 37$ and $G$ is doubly primitive. Since $t \neq 5$ by assumption we can assume that $t > 6$ so that $m = 1$ and the subdegrees of $(K_1)^B_1$ are $1, 5, t - 1$ or $1, 2, 3, t - 1$. The proposition holds for the second case by Lemma 4. In the case $1, 5, t - 1$ we use Lemma 4 to prove the result for $t - 1 > 120$. If $(K_1)^B_1$ is imprimitive $6 | b$ ([10] Lemma 4) contradicting our assumption. Hence $(K_1)^B_1$ is primitive and so $\mu(t - 1) = 5(4 - \lambda)$, $\lambda$ and $\mu$ as in [10], also $\mu \neq 0$ ([10]) Lemma 5, Cor. 3). Thus, $t - 1 = 10, 15, 20$.

If $t = 21, b = 26$ and the subdegrees are $1, 5, 20$, contradicting [24] Th. 31.2. If $t = 16$, $\mu = \frac{1}{3}(4 - \lambda)$ so that $\lambda = \mu = 1$ contradicting Lemma 7 of [10]. If $t = 11, b = 16$ and $K_1 = (K_2)^B_1$ has a regular normal subgroup $N$ (See [23] p. 179). Since $|N| = 16, N \leq K_1, N$ is half transitive on the $15$ points of $\sum \setminus \{B_1\}$ forcing $N \subseteq H$. Hence $H \neq 1$. If $H_{16} = 1, H = N$ and we get a contradiction using [16]. Thus $H_{16} \neq 1$ and since $m = 1, H_{16}$ fixes the point of $\Gamma \cap B_i$ for all $i \geq 2$. Now [14] $B_1$ is used to obtain the result.

\textit{Note added in proof.} The proof of Lemma 4 holds if $N^g \not\subseteq H$ because then $|G_{a\beta}^{\Sigma-(B_2)}|; (N^g)^{\Sigma-(B_2)} |$ divides $\lambda l$. The case $N^g \subseteq H$ is impossible because then $|N^g|$ divides $|H \cap G_{a\beta}|$ so that $|G_{a\beta}^{\Sigma-(B_1)}| = |G_{a\beta}|/|G_{a\beta} \cap H|$ divides $|G_{a\beta} : N|$ which divides $\lambda l$. Since $t - 1$ divides $|G_{a\beta}^{\Sigma-(B_2)}|$ we get that $t - 1 | \lambda l$ a contradiction.

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