On the determination of PL-manifolds by handles of lower dimension *

Alberto Cavicchioli

Department of Mathematics, University of Modena, Via Campi 213/B, 41100 Modena, Italy

Friedrich Hegenbarth

Department of Mathematics, University of Milano, Via Saldini 50, 20133 Milano, Italy

Received 24 September 1990

Abstract


We extend the Montesinos theorem about handle presentations of PL 4-manifolds to dimension n. For this we consider closed PL (n + 1)-manifolds of the same homotopy type as \( \#_n(S^1 \times S^n) \). Then we extend PL homeomorphisms of \( X \) over the solid handlebody.

Keywords: Handle presentation; Surgery; Homotopy type; Genus.


1. Introduction

The intention of this paper is to prove an extension to dimension \( n \) of the following well-known theorem

**Theorem 1.1** (Montesinos [14]). Each closed orientable PL 4-manifold \( M^4 \) with handle presentation

\[
M = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4
\]

is completely determined by \( H^0 \cup \lambda H^1 \cup \mu H^2 \).

* Work performed under the auspices of the G.N.S.A.G.A. of the C.N.R. and financially supported by the M.P.I. of Italy.

0166-8641/93/$06.00 © 1993 – Elsevier Science Publishers B.V. All rights reserved
For this we prove that any PL homeomorphism of \( X = \#_p(S^1 \times S^n) \) extends to the solid handlebody \( Y = \#_p(S^1 \times D^{n+1}) \) with boundary \( X \) (Section 3).

Then the proof can be completed as in [14]. As a consequence, we also give a characterization of \( S^1 \times S^n \) using the regular genus (Theorem 4.1). Using classical results of surgery theory, we also prove that any closed PL \((n + 1)\)-manifold \((n \geq 4)\) of the same homotopy type as \( X \) is PL homeomorphism to \( X \) (Section 2). This is related to results of [10] \((p = 1, n \geq 4)\) and [6] \((p = 1, n = 3)\). Moreover, any self-homotopy equivalence of \( X \) is homotopic to a PL homeomorphism (compare [17] for \( p = 1 \)).

Through the paper we work in the piecewise linear PL category (see for example [8]).

### 2. Manifolds homotopy equivalent to \( \#_p(S^1 \times S^n) \)

This section is devoted to prove the following

**Theorem 2.1.** Let \( M^{n+1} \), \( n \geq 4 \), be a closed connected PL \((n + 1)\)-manifold of the same homotopy type as \( \#_p(S^1 \times S^n) \). Then \( M \) is PL homeomorphic to \( \#_p(S^1 \times S^n) \).

This theorem is related to results of Lashof and Shaneson for \( p = 1, n \geq 4 \) (see [10, Proposition 1.1]) and Freedman and Quinn for \( p = 1, n = 3 \) (see [6]). For \( n = 2 \), the result also holds up to a connected sum with a homotopy 3-sphere (see [7, p. 57]). Furthermore, in [4] it was proved that a manifold \( M^4 \), homotopy equivalent to \( \#_p(S^1 \times S^3) \), is s-cobordant to it.

**Proof.** We apply standard techniques of surgery theory (see [13,17]).

Given a closed orientable PL \((n + 1)\)-manifold \( X^{n+1} \), \( n \geq 4 \), we have the exact surgery sequence (see [17])

\[
\begin{array}{cccc}
\Sigma(X) \otimes G/PL & \xrightarrow{\sigma_{n+2}} & L_{n+2}(\Pi_1) & \xrightarrow{\omega} & G_{n+1}(X) \\
\eta & & [X;G/PL] & \xrightarrow{\sigma_{n+2}} & L_{n+1}(\Pi_1). \\
\end{array}
\]

Recall that \( L_m(\Pi_1) \) is the \( m \)th Wall group in the oriented case and \( G_{n+1}(X) \) is the set of equivalence classes of pairs \((h, Y)\), \( Y \) an orientable closed PL \((n + 1)\)-manifold, \( h : Y \to X \) an orientation-preserving simple homotopy equivalence. Two pairs \((h', Y')\) and \((h, Y)\) are equivalent if there exists an orientation-preserving PL homeomorphism \( \tilde{a} : Y \to Y' \) such that \( h' \circ \tilde{a} \) is homotopic to \( h \).

If we set \( X^{n+1} = \#_p(S^1 \times S^n) \), then it suffices to prove \( G_{n+1}(X) = 1 \). For this, we prove that:

1. \( \sigma_{n+1} \) and \( \sigma_{n+2} \) are surjective, i.e., the sequence

\[
\begin{array}{cccc}
* & \xrightarrow{\eta} & G_{n+1}(X) & \xrightarrow{\sigma_{n+1}} & L_{n+1}(\Pi_1) \\
\end{array}
\]

is exact.
(2) The Abelian groups \([X; G/\text{PL}]\) and \(L_{n+1}(\Pi_1)\) are isomorphic.

In order to prove (1), we have two cases:

Case 1.1: \(n = 4k + \alpha\), \(\alpha = 0, 2\).

Since \(X^{n+1}\) is orientable, any imbedded sphere \(\tilde{f} : S^1 \to X\) has trivial normal bundle, i.e., \(\tilde{f}\) extends to an imbedding \(f : S^1 \times D^n \to X\). Let \(f_1, f_2, \ldots, f_p : S^1 \times D^n \to X\) be disjoint imbeddings such that

\[
\tilde{f}_1 = f_1|_{S^1 \times \emptyset}, \tilde{f}_2 = f_2|_{S^1 \times \emptyset}, \ldots, \tilde{f}_p = f_p|_{S^1 \times \emptyset}
\]

represent a set of generators of \(\pi_1(X)\) (by general position this is always possible).

There exists a normal map \((\psi, b, \psi_\alpha : (W, \partial W) \to (D^{4k+\alpha}, S^{4k+\alpha-1})\), such that \(\psi_\alpha|_{\partial W}\) is a homotopy equivalence (implying \(\partial W = \text{PL} S^{4k+\alpha-1}\) for \(k > 1\)) and \(\sigma_\alpha(\psi_\alpha) = 1 \in L_{4k+\alpha}(1) \approx L_{4k+\alpha}(Z) \approx Z\). Here \(Z\) denotes the cyclic group \(\mathbb{Z}/\alpha\mathbb{Z}\).

For \(k > 1\) we refer to [1] (see [1, Theorems V.2.9 and V.2.11]) and for \(k = 1\) we assume \(W = \| E_8 \| \setminus \text{(4-ball)}\) (see [6]).

Let \(\tilde{W}\) be the manifold obtained from \(W\) by capping off the sphere \(\partial W\) with an \((4k + \alpha)\)-ball. By the addition property (see [1, 11.1.4, p. 32]) we obtain a normal map \(\psi_\alpha : \tilde{W} \to S^n\) such that \(\sigma(\psi_\alpha) = 1\). Let \(N_i\) \((i = 1, 2, \ldots, p)\) be the \((n+1)\)-manifold obtained from \(X\) by deleting \(f_i(S^1 \times \emptyset)\) and attaching \(S^1 \times \tilde{W}\) by an obvious identification of their boundaries. In this way we obtain a normal map of degree one

\[
\xi_i : N_i \to X = X \setminus \bigcup_{i=1}^p f_i(S^1 \times \emptyset) \cup S^1 \times S^n \setminus S^1 \times \emptyset
\]

for \(i = 1, 2, \ldots, p\). Then the surgery obstruction \(\sigma_{n+1}(\xi_i)\) is exactly the \(i\)th generator of \(L_{n+1}(\Pi_1) \approx L_{n+1}(\ast_p Z) \approx \ast_p \mathbb{Z}\) (see [2]). Here \(\ast_p Z\) denotes the free product of \(p\) factors isomorphic to \(Z\). Thus the homomorphism \(\sigma_{n+1}\) is onto as requested.

Case 1.2: \(n = 4k + \beta\), \(\beta = 1, 3\).

Let \(\alpha = 1 + \beta\) and let \(\hat{\psi}_\beta\) be as above. Then we take the identity \(\text{Id}_X : X \to X\) and form the connected sum

\[
\hat{\psi}_\beta \# \text{Id}_X : \hat{W} \# X \to S^{n+1} \# \hat{W} = X.
\]

The additivity of surgery obstructions and the equality \(\sigma(\text{Id}_X) = 0\) imply that

\[
\sigma(\hat{\psi}_\beta \# \text{Id}_X) = \sigma(\hat{\psi}_\beta) = 1 \in L_{n+1}(\Pi_1) \approx L_{n+1}(\ast_p Z) \approx \ast_p \mathbb{Z} (1)
\]

\[
\approx \begin{cases} \mathbb{Z}_2, & \text{if } \alpha = 2, \\ \mathbb{Z}, & \text{if } \alpha = 4 \end{cases}
\]

(see [2]). Thus the map \(\sigma_{n+1}\) is onto. A similar proof also holds for \(\sigma_{n+2}\) by taking products with \(I = [0, 1]\).

(2) Let \(f, g : X \to G/\text{PL}\) be two maps. Then \(f\) and \(g\) are homotopic if there are no obstructions lying in \(H^q(X; \pi_q(G/\text{PL}))\) (see [15]). Recall that

\[
\pi_q(G/\text{PL}) = \begin{cases} 0, & q \equiv 1, 3 \pmod{4}, \\ \mathbb{Z}_2, & q \equiv 2 \pmod{4}, \\ \mathbb{Z}, & q \equiv 0 \pmod{4} \end{cases}
\]

(see [16]).
If \( n = 4k + \alpha, \alpha = 0, 2, \) then the only obstructions lie in \( H^n(X; \pi_n(G/PL)) = \oplus \pi_n(G/PL). \)

If \( n = 4k + \beta, \beta = 1, 3, \) then the only obstructions lie in

\[
H^{n+1}(X; \pi_{n+1}(G/PL)) = \pi_{n+1}(G/PL).
\]

Because any obstruction can be realized by a map \( X \to G/PL, \) one obtains

\[
[X; G/PL] \simeq \begin{cases} 
\oplus \pi_n(G/PL), & n \text{ even}, \\
\pi_{n+1}(G/PL), & n \text{ odd},
\end{cases}
\]
hence \( [X; G/PL] = L_{n+1}(\Pi_1) \) as Abelian groups.

Thus the epimorphism \( \sigma_{n+1} : [X; G/PL] \to L_{n+1}(\Pi_1) \) must be injective. This completes the proof of the theorem. \( \square \)

The above proof yields the following extension of [17, Lemma 16.2, p. 232].

**Corollary 2.2.** Any self-homotopy equivalence of \( \#_p(S^1 \times S^n), n \geq 4, \) is homotopic to a PL homeomorphism.

**3. Extending homeomorphisms of \( \#_p(S^1 \times S^n) \)**

Set \( X = \#_p(S^1 \times S^n) \) for \( n \geq 4. \) Let \( \text{Aut}(X) \) (respectively \( \text{Aut}_0(X) \)) be the group of (respectively orientation-preserving) PL self-homeomorphisms of \( X, \) \( \text{SE}_0(X) \) the group of homotopy classes of orientation-preserving self-homotopy equivalences of \( X \) and \( \text{Aut}(\Pi_1) \) the group of automorphisms of \( \Pi_1(X). \) We have the following canonical commutative diagram

\[
\begin{array}{ccc}
\text{Aut}_0(X) & \xrightarrow{\pi} & \text{SE}_0(X) \\
& \downarrow{\iota} & \downarrow{\theta_0} \\
& \text{Aut}(X) & \xrightarrow{\theta} \text{Aut}(\Pi_1)
\end{array}
\]

Laudenbach and Poenaru (see [12, p. 340]) proved that \( \theta \) is surjective. Corollary 2.2 implies the surjectivity of \( \pi. \) Now it is very easy to see that \( \theta_0 \) is also epi. Indeed, for any \( \xi \in \text{Aut}(\Pi_1) \) there exists \( f \in \text{Aut}(X) \) such that \( f \cdot \xi = \xi. \) If \( \text{deg}(f) = 1, \) then \( [f] \in \text{SE}_0(X) \) and \( \theta_0[f] = \xi. \) Otherwise, we compose \( f \) with the map

\[
r' = \#(1d_{S^1} \times r) \circ r' : X \to X,
\]

where \( r : S^n \to S^n \) is the reflection on the 1st coordinate. Then \( [f \circ r'] \in \text{SE}_0(X) \) and \( \theta_0[f \circ r'] = \xi. \)

In order to prove the theorem of Montesinos for \( n \geq 3 \) we have to extend any self-homeomorphism of \( X = \#_p(S^1 \times S^n) \) to the solid handlebody \( Y = \#_p(S^1 \times D^{n+1}). \) For convenience we assume that \( Y \) is canonically imbedded in \( R^{n+2}. \) Without loss of generality we can restrict our attention to orientation-preserving
homeomorphisms. In fact, there is a self-homeomorphism of $Y$ which is orientation-reversing in the boundary $X$ as shown above.

From the previous commutative diagram we obtain the following exact sequence

$$1 \to \text{Ker } \tilde{\theta} \to \text{Aut}_0(X) \xrightarrow{\tilde{\theta}^{-1} \circ \pi} \text{Aut}(\Pi_1) \to 1,$$

i.e., $\text{Aut}_0(X)/\text{Ker } \tilde{\theta} \simeq \text{Aut}(\Pi_1)$.

First we prove

**Proposition 3.1.** Let $f : X \to X$ be a PL homeomorphism which is homotopic to the identity $\text{Id}_X$, i.e., $f \in \text{Ker } \tilde{\theta}$. Then $f$ extends to a self-homeomorphism of $Y$.

**Proof.** Let $f : X \to X$ be a PL homeomorphism homotopic to the identity. Let $Y$ be the boundary connected sum $\#_p (S^i \times D^{n+1})$. Obviously the boundary of $Y$ is $X$.

We have to prove that $f$ extends over the solid handlebody $Y$. Form the manifolds $M = Y \cup_{\text{Id}} Y$ and $N = Y \cup_f Y$. Obviously $M$ is PL homeomorphic to $\#_p (S^1 \times S^{n+1})$.

Furthermore $N$ is homotopy equivalent to $M$ since $f$ is homotopic to the identity.

Let $i_1 : Y \to M$ and $j_1 : Y \to N$ (respectively $i_2 : Y \to M$ and $j_2 : Y \to N$) be the canonical inclusions of $Y$ into the first (respectively second) copy of it. For simplicity we identify $Y = i_1(Y)$ with $Y = j_1(Y)$ so that $M \cap N = Y$.

Note that

$$f = j_2 |_{\bar{X}} \circ j_1 |_X.$$

We can identify the fundamental groups of $M$ and $N$ by the natural inclusions $i_1 : Y \subset M$ and $j_1 : Y \subset N$. By Theorem 2.1 there is a homeomorphism $g : M \to N$.

Since any automorphism of $\Pi_1(M)$ is induced by a PL homeomorphism $M \to M$, we can always assume that the induced automorphism $g_* : \Pi_1(M) \to \Pi_1(N) = \Pi_1(M)$ is the identity.

Let $S^1_i$ be the canonical $i$th $S^1$-factor of $Y$ for $i = 1, 2, \ldots, p$. Then the 1-sphere $\Sigma^1_i = g(S^1_i)$ is homotopic to $S^1_i$, hence they are also isotopic as dim $Y \geq 4$. Then we isotope $g$ to a map $h$ which sends the 1-dimensional graph $G = \bigvee_{i=1}^p S^1_i$ (one-point union) in $Y$ to itself via the identity. We can assume that the map $h$ is also the identity on a regular neighborhood $W$ of $G$ in $Y = M \cap N$ (see Fig. 1). Since $h^{-1}(j_1(Y)) = h^{-1}(Y)$ is also a regular neighborhood of $G$, there exists a PL homeomorphism $\psi : M \to M$ and a regular neighborhood $W' \subset \text{int } W$ such that

1. $\psi |_{W'} = \text{identity}$,
2. $\psi(h^{-1}(Y)) = Y$,
3. $\psi$ is isotopic to the identity.

Let $k = h \circ \psi^{-1}$. Then we have $k(Y) = Y$, $k |_{W'} = \text{identity}$ and $k(i_2(Y)) = j_2(Y)$.

The homeomorphism $k$ defines two homeomorphisms

$$k_1 = j_1^{-1} \circ k \circ i_1 : Y \to Y$$

and

$$k_2 = j_2^{-1} \circ k \circ i_2 : Y \to Y.$$
We have

\[ k_2|_{\alpha Y} \circ k_1|_{\alpha Y}^{-1} = j_2^{-1} \circ k \circ i_2 \circ i_1^{-1} \circ k^{-1} \circ j_1 = j_2^{-1} \circ j_1 = f. \]

This implies that \( k_2 \circ k_1^{-1} \) is an extension of \( f \) to \( Y \). \( \square \)
Now we are going to extend an arbitrary self-homeomorphism of $X$ to $Y$. By [11, p. 339] and [12, Section 5.3], the group $\text{Aut}_0(X)/\text{Ker } \tilde{\theta} = \text{Aut}(H_f)$ is generated by sliding 1-handles, twisting 1-handles, permuting 1-handles and rotations. We recall that all these generators extend to $Y$ as proved in [11,12].

If $f \in \text{Aut}_0(X)$, then $\tilde{\theta}(f)$ can be represented by a homeomorphism $g : X \to X$ which extends to a homeomorphism $G : X \to Y$. Since $g^{-1} \circ f \not\in \text{Ker } \tilde{\theta}$, the composition $g^{-1} \circ f$ is homotopic to the identity $\text{Id}_X$. By Proposition 3.1 the map $g^{-1} \circ f$ extends to a homeomorphism $H : Y \to Y$. Then $F = G \circ H : Y \to Y$ is an extension of the given homeomorphism $f$.

Summarizing we have proved the following

**Proposition 3.2.** Any self-homeomorphism $f : X \to X$ extends to the solid handlebody $Y$.

As an consequence we obtain

**Corollary 3.3.** Let $N^m$ be a compact connected PL $m$-manifold, $m \geq 4$, with boundary $\partial N = \#_p(S^1 \times S^{m-2})$. Then the manifold $N \cup \#_p(S^1 \times D^{m-1})$ is independent of the way of pasting the boundaries together.

For $m = 4$, this result was proved in [14, Theorem 2].

**Proof.** Let $f, \ g : X \to X$, $X = \#_p(S^1 \times S^{m-2})$, be arbitrary PL homeomorphisms. We have to show that the manifolds $N \cup_f Y$ and $N \cup_g Y$, $Y = \#_p(S^1 \times D^{m-1})$, are PL homeomorphic. Indeed, the composition $g^{-1} \circ \text{Id}_N \circ f : X \to X$ extends to a homeomorphism $T : Y \to Y$. Then the map $\text{Id}_N \cup T : N \cup_f Y \to N \cup_g Y$ is the requested homeomorphism. □

This corollary implies the following result about handle presentations of manifolds.

**Corollary 3.4.** Each closed orientable $n$-manifold $M^n$ with handle presentation

$$M = H^0 \cup \lambda_1 H^1 \cup \cdots \cup \lambda_{n-1} H^{n-1} \cup H^n \quad (n \geq 4)$$

is completely determined up to homeomorphism by

$$H^0 \cup \lambda_1 H^1 \cup \cdots \cup \lambda_{n-2} H^{n-2}.$$

4. Epilogue

In this short section we merely mention a characterization theorem of $S^1 \times S^n$ among closed PL $(n + 1)$-manifolds by using the regular genus. We postpone the
complete treatment to a future paper, entitled “On the topological structure of genus one PL-manifolds”. For the definition of the regular genus of a closed PL-manifold we refer for example to [3]. In [3] it was proved that the unique closed PL 4-manifold of genus one is $S^1 \times S^3$. Using Corollary 3.4 and the combinatorial calculations of [3], we can prove the following result.

**Theorem 4.1.** Let $M^{n+1}$ be a closed PL $(n+1)$-manifold $(n \geq 4)$. Then the regular genus of $M$ is one if and only if $M$ is PL homeomorphic to $S^1 \times S^n$.

**Acknowledgement**

The authors wish to thank the referee for his useful suggestions to improve the paper.

**References**