Codings of separable compact subsets of the first Baire class

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Abstract

Let X be a Polish space and $\mathcal{K}$ a separable compact subset of the first Baire class on X. For every sequence $\mathbf{f} = (f_n)_n$ dense in $\mathcal{K}$, the descriptive set-theoretic properties of the set

$$\mathcal{L}_{\mathbf{f}} = \{L \in [\mathbb{N}] : (f_n)_{n \in L} \text{ is pointwise convergent}\}$$

are analyzed. It is shown that if $\mathcal{K}$ is not first countable, then $\mathcal{L}_{\mathbf{f}}$ is $\mathbf{\Pi}_1^1$-complete. This can also happen even if $\mathcal{K}$ is a pre-metric compactum of degree at most two, in the sense of S. Todorčević. However, if $\mathcal{K}$ is of degree exactly two, then $\mathcal{L}_{\mathbf{f}}$ is always Borel. A deep result of G. Debs implies that $\mathcal{L}_{\mathbf{f}}$ contains a Borel cofinal set and this gives a tree-representation of $\mathcal{K}$. We show that classical ordinal assignments of Baire-1 functions are actually $\mathbf{\Pi}_1^1$-ranks on $\mathcal{K}$. We also provide an example of a $\mathbf{\Sigma}_1^1$ Ramsey-null subset A of $[\mathbb{N}]$ for which there does not exist a Borel set $B \supseteq A$ such that the difference $B \setminus A$ is Ramsey-null.
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1. Introduction

Let X be a Polish space. A Rosenthal compact on X is a subset of real-valued Baire-1 functions on X, compact in the pointwise topology. Standard examples of such compacta include the Helly space (the space of all non-decreasing functions from the unit interval into itself), the split interval (the lexicographical ordered product of the unit interval and the two-element ordering) and the ball of the double dual of a separable Banach space not containing ℓ_1. That the later space is indeed a compact subset of the first Baire class follows from the famous Odell–Rosenthal theorem [17], which states that the ball of the double dual of a separable Banach space with the weak* topology consists only of Baire-1 functions if and only if the space does not contain ℓ_1. Actually this result motivated H.P. Rosenthal to initiate the study of compact subsets of the first Baire class in [21]. He showed that all such compacta are sequentially compact. J. Bourgain, D.H. Fremlin and M. Talagrand proved that Rosenthal compacta are Fréchet spaces [6]. We refer to [3,18] and [22] for thorough introductions to the theory, as well as its applications in analysis.

Separability is the crucial property that divides this class in two. As S. Todorčević has pointed out in [25], while non-separable Rosenthal compacta can be quite pathological, the separable ones are all "definable". This is supported by the work of many researchers, including G. Godefroy [8], A. Krawczyk [11], W. Marciszewski [13], and R. Pol [19], and is highlighted in the remarkable dichotomies and trichotomies of [25].

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Our starting point of view is how we can code separable compact subsets of the first Baire class by members of a standard Borel space. Specifically, by a *code* of a separable Rosenthal compact $\mathcal{K}$ on a Polish space X, we mean a standard Borel space C and a surjection $C \ni c \mapsto f_c \in \mathcal{K}$ such that for all $a \in \mathbb{R}$ the relation

$$(c, x) \in R_a \Leftrightarrow f_c(x) > a$$

is Borel in $C \times X$. In other words, inverse images of sub-basic open subsets of $\mathcal{K}$ are Borel in C uniformly in X.

There is a natural object one associates to every separable Rosenthal compact $\mathcal{K}$ and which can serve as a coding of $\mathcal{K}$. More precisely, for every dense sequence $\mathbf{f} = (f_n)_n$ in $\mathcal{K}$ one defines

$$\mathcal{L}_{\mathbf{f}} = \{L \in [\mathbb{N}] : (f_n)_{n \in L} \text{ is pointwise convergent}\}.$$

The Bourgain–Fremlin–Talagrand theorem [6] implies that $\mathcal{L}_{\mathbf{f}}$ totally describes the members of $\mathcal{K}$, in the sense that for every accumulation point f of $\mathcal{K}$ there exists $L \in \mathcal{L}_{\mathbf{f}}$ such that f is the pointwise limit of the sequence $(f_n)_{n \in L}$. Moreover, for every $f \in \mathcal{K}$ one also defines

$$\mathcal{L}_{\mathbf{f}, f} = \{L \in [\mathbb{N}] : (f_n)_{n \in L} \text{ is pointwise convergent to } f\}.$$

Both $\mathcal{L}_{\mathbf{f}}$ and $\mathcal{L}_{\mathbf{f}, f}$ have been studied in the literature. In [11], Krawczyk proved that $\mathcal{L}_{\mathbf{f}, f}$ is Borel if and only if f is a G_δ point of $\mathcal{K}$. The set $\mathcal{L}_{\mathbf{f}}$ (more precisely the set $\mathcal{L}_{\mathbf{f}} \setminus \mathcal{L}_{\mathbf{f}, f}$) has been also considered by Todorčević in [25], in his solution of characters of points in separable Rosenthal compacta.

There is an awkward fact concerning $\mathcal{L}_{\mathbf{f}}$, namely that $\mathcal{L}_{\mathbf{f}}$ can be non-Borel. However, a deep result of G. Debs [7] implies that $\mathcal{L}_{\mathbf{f}}$ always contains a Borel cofinal set and this subset of $\mathcal{L}_{\mathbf{f}}$ can serve as a coding. This leads to the following tree-representation of separable Rosenthal compacta.

Proposition A. *Let $\mathcal{K}$ be a separable Rosenthal compact. Then there exist a countable tree T and a sequence $(g_t)_{t \in T}$ in $\mathcal{K}$ such that the following hold.*

(1) *For every $\sigma \in [T]$ the sequence $(g_{\sigma|n})_{n \in \mathbb{N}}$ is pointwise convergent.*

(2) *For every $f \in \mathcal{K}$ there exists $\sigma \in [T]$ such that f is the pointwise limit of the sequence $(g_{\sigma|n})_{n \in \mathbb{N}}$.*

It is natural to ask when the set $\mathcal{L}_{\mathbf{f}}$ is Borel or, equivalently, when $\mathcal{L}_{\mathbf{f}}$ can serve itself as a coding (it is easy to see that $\mathcal{L}_{\mathbf{f}}$ and $\mathcal{L}_{\mathbf{f}, f}$ are always $\mathbf{\Pi}_1^1$). In this direction, the following is shown.

Theorem B. *Let $\mathcal{K}$ be a separable Rosenthal compact.*

(1) *If $\mathcal{K}$ is not first countable, then for every dense sequence $\mathbf{f} = (f_n)_n$ in $\mathcal{K}$ the set $\mathcal{L}_{\mathbf{f}}$ is $\mathbf{\Pi}_1^1$-complete.*

(2) *If $\mathcal{K}$ is pre-metric of degree exactly two, then for every dense sequence $\mathbf{f} = (f_n)_n$ in $\mathcal{K}$ the set $\mathcal{L}_{\mathbf{f}}$ is Borel.*

Part (1) above is based on a result of Krawczyk. In part (2), $\mathcal{K}$ is said to be a pre-metric compactum of degree exactly two if there exist a countable subset D of X and a countable subset $\mathcal{D}$ of $\mathcal{K}$ such that at most two functions in $\mathcal{K}$ coincide on D and moreover for every $f \in \mathcal{K} \setminus \mathcal{D}$ there exists $g \in \mathcal{K}$ with $f \neq g$ and such that g coincides with f on D. This is a subclass of the class of pre-metric compacta of degree at most two, as it is defined by Todorčević in [25]. We notice that part (2) of Theorem B cannot be lifted to all pre-metric compacta of degree at most two, as there are examples of such compacta for which the set $\mathcal{L}_{\mathbf{f}}$ is $\mathbf{\Pi}_1^1$-complete.

We proceed now to discuss some applications of the above approach. It is well-known that to every real-valued Baire-1 function on a Polish space X one associates several (equivalent) ordinal rankings measuring the discontinuities of the function. An extensive study of them is done by A.S. Kechris and A. Louveau in [10]. An important example is the separation rank α. We have the following boundedness result concerning this index.

Theorem C. *Let X be a Polish space and $\mathbf{f} = (f_n)_n$ a sequence of Borel real-valued functions on X. Let*

$$\mathcal{L}_{\mathbf{f}}^1 = \{L \in [\mathbb{N}] : (f_n)_{n \in L} \text{ is pointwise convergent to a Baire-1 function}\}.$$

Then for every $C \subseteq \mathcal{L}_{\mathbf{f}}^1$ Borel, we have

$$\sup\{\alpha(f_L) : L \in C\} < \omega_1$$

where, for every $L \in C$, f_L denotes the pointwise limit of the sequence $(f_n)_{n \in L}$.

The proof of Theorem C actually is based on the fact that the separation rank is a parameterized $\mathbf{\Pi}_1^1$-rank. Theorem C, combined with the result of Debs, gives a proof of the boundedness result of [2]. Historically the first result of this form is due to J. Bourgain [5]. We should point out that in order to give a descriptive set-theoretic proof of Bourgain's result one does not need to invoke Debs' theorem.

Theorem C can also be used to provide natural counterexamples to the following approximation question in Ramsey theory. Namely, given a $\mathbf{\Sigma}_1^1$ subset A of $[\mathbb{N}]$ can we always find a Borel set $B \supseteq A$ such that the difference $B \setminus A$ is Ramsey-null? A.W. Miller had also asked whether there exists an analytic set which is not equal to Borel modulo Ramsey-null (see [15], Problem 1.6*). We show the following.

Proposition D. *There exists a $\mathbf{\Sigma}_1^1$ Ramsey-null subset A of $[\mathbb{N}]$ for which there does not exist a Borel set $B \supseteq A$ such that the difference $B \setminus A$ is Ramsey-null.*

2. Preliminaries

For any Polish space X, by $K(X)$ we denote the hyperspace of all compact subsets of X, equipped with the Vietoris topology. By $\mathcal{B}_1(X)$ (respectively $\mathcal{B}(X)$) we denote the space of all real-valued Baire-1 (respectively Borel) functions on X. By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the natural numbers, and by $[\mathbb{N}]$ the set of all infinite subsets of $\mathbb{N}$ (which is clearly a Polish subspace of $2^{\mathbb{N}}$). For every $L \in [\mathbb{N}]$, by $[L]$ we denote the set of all infinite subsets of L. For every function $f : X \to \mathbb{R}$ and every $a \in \mathbb{R}$ we set $[f > a] = \{x : f(x) > a\}$. The set $[f < a]$ has the obvious meaning.

Our descriptive set-theoretic notation and terminology follows [9]. So $\mathbf{\Sigma}_1^1$ stands for the analytic sets, and $\mathbf{\Pi}_1^1$ for the co-analytic. A set is said to be $\mathbf{\Pi}_1^1$-true if it is co-analytic non-Borel. If X, Y are Polish spaces, $A \subseteq X$ and $B \subseteq Y$, we say that A is Wadge (Borel) reducible to B if there exists a continuous (Borel) map $f : X \to Y$ such that $f^{-1}(B) = A$. A set A is said to be $\mathbf{\Pi}_1^1$-complete if it is $\mathbf{\Pi}_1^1$ and any other co-analytic set is Borel reducible to A. Clearly any $\mathbf{\Pi}_1^1$-complete set is $\mathbf{\Pi}_1^1$-true. The converse is also true under large cardinal hypotheses (see [14] or [16]). If A is $\mathbf{\Pi}_1^1$, then a map $\phi : A \to \omega_1$ is said to be a $\mathbf{\Pi}_1^1$-rank on A if there are relations $\leq_\Sigma, \leq_\Pi$ in $\mathbf{\Sigma}_1^1$ and $\mathbf{\Pi}_1^1$ respectively, such that for any $y \in A$

$$\phi(x) \leq \phi(y) \Leftrightarrow x \leq_\Sigma y \Leftrightarrow x \leq_\Pi y.$$

Notice that if A is Borel reducible to B via a Borel map f and ϕ is a $\mathbf{\Pi}_1^1$-rank on B, then the map $\psi : A \to \omega_1$ defined by $\psi(x) = \phi(f(x))$ is a $\mathbf{\Pi}_1^1$-rank on A.

Trees. If A is a non-empty set, by $A^{<\mathbb{N}}$ we denote the set of all finite sequences of A. We view $A^{<\mathbb{N}}$ as a tree equipped with the (strict) partial order $\sqsubset$ of extension. If $s \in A^{<\mathbb{N}}$, then the length $|s|$ of s is defined to be the cardinality of the set $\{t : t \sqsubset s\}$. If $s, t \in A^{<\mathbb{N}}$, then by $s^\frown t$ we denote their concatenation. If $A = \mathbb{N}$ and $L \in [\mathbb{N}]$, then by $[L]^{<\mathbb{N}}$ we denote the increasing finite sequences in L. For every $x \in A^{\mathbb{N}}$ and every $n \geq 1$ we let $x|n = (x(0), \ldots, x(n-1)) \in A^{<\mathbb{N}}$ while $x|0 = (\varnothing)$. A tree T on A is a downwards closed subset of $A^{<\mathbb{N}}$. The set of all trees on A is denoted by $\mathrm{Tr}(A)$. Hence

$$T \in \mathrm{Tr}(A) \Leftrightarrow \forall s, t \in A^{<\mathbb{N}} \ (t \sqsubset s \ \wedge \ s \in T \Rightarrow t \in T).$$

For a tree T on A, the body $[T]$ of T is defined to be the set $\{x \in A^{\mathbb{N}} : x|n \in T \text{ for all } n \in \mathbb{N}\}$. A tree T is called pruned if for every $s \in T$ there exists $t \in T$ with $s \sqsubset t$. It is called well-founded if for every $x \in A^{\mathbb{N}}$ there exists n such that $x|n \notin T$, equivalently if $[T] = \varnothing$. The set of well-founded trees on A is denoted by $\mathrm{WF}(A)$. If T is a well-founded tree we let $T' = \{t : \exists s \in T \text{ with } t \sqsubset s\}$. By transfinite recursion, one defines the iterated derivatives $T^{(\xi)}$ of T. The order $o(T)$ of T is defined to be the least ordinal ξ such that $T^{(\xi)} = \varnothing$. If S, T are well-founded trees, then a map $\phi : S \to T$ is called monotone if $s_1 \sqsubset s_2$ in S implies that $\phi(s_1) \sqsubset \phi(s_2)$ in T. Notice that in this case $o(S) \leq o(T)$. If A, B are sets, then we identify every tree T on $A \times B$ with the set of all pairs $(s, t) \in A^{<\mathbb{N}} \times B^{<\mathbb{N}}$ such that $|s| = |t| = k$ and $((s(0), t(0)), \ldots, (s(k-1), t(k-1))) \in T$. If $A = \mathbb{N}$, then we shall simply denote by Tr and WF the sets of all trees and well-founded trees on $\mathbb{N}$ respectively. The set WF is $\mathbf{\Pi}_1^1$-complete and the map $T \to o(T)$ is a $\mathbf{\Pi}_1^1$-rank on WF. The same also holds for $\mathrm{WF}(A)$ for every countable set A.

The separation rank. Let X be a Polish space. Given $A, B \subseteq X$ one associates with them a derivative on closed sets, by $F'_{A,B} = \overline{F \cap A} \cap \overline{F \cap B}$. By transfinite recursion, we define the iterated derivatives $F_{A,B}^{(\xi)}$ of F and we set

$\alpha(F, A, B)$ to be the least ordinal ξ with $F_{A,B}^{(\xi)} = \varnothing$ if such an ordinal exists, otherwise we set $\alpha(F, A, B) = \omega_1$. Now let $f : X \to \mathbb{R}$ be a function. For each pair $a, b \in \mathbb{R}$ with $a < b$ let $A = [f < a]$ and $B = [f > b]$. For every $F \subseteq X$ closed let $F_{f,a,b}^{(\xi)} = F_{A,B}^{(\xi)}$ and $\alpha(f, F, a, b) = \alpha(F, A, B)$. Let also $\alpha(f, a, b) = \alpha(f, X, a, b)$. The separation rank of f is defined by

$$\alpha(f) = \sup\{\alpha(f, a, b) : a, b \in \mathbb{Q}, a < b\}.$$

The basic fact is the following (see [10]).

Proposition 1. *A function f is Baire-1 if and only if $\alpha(f) < \omega_1$.*

3. Codings of separable Rosenthal compacta

Let X be a Polish space and $\mathbf{f} = (f_n)_n$ a sequence of Borel real-valued functions on X. Assume that the closure $\mathcal{K}$ of $\{f_n\}_n$ in $\mathbb{R}^X$ is a compact subset of $\mathcal{B}(X)$. Let us consider the set

$$\mathcal{L}_{\mathbf{f}} = \{L \in [\mathbb{N}] : (f_n)_{n \in L} \text{ is pointwise convergent}\}.$$

For every $L \in \mathcal{L}_{\mathbf{f}}$, by f_L we shall denote the pointwise limit of the sequence $(f_n)_{n \in L}$. Notice that $\mathcal{L}_{\mathbf{f}}$ is $\mathbf{\Pi}_1^1$. As the pointwise topology is not effected by the topology on X, we may (and we will) assume that each f_n is continuous (and so $\mathcal{K}$ is a separable Rosenthal compact). By a result of H.P. Rosenthal [21], we get that $\mathcal{L}_{\mathbf{f}}$ is cofinal. That is, for every $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that $L \in \mathcal{L}_{\mathbf{f}}$. Also the celebrated Bourgain–Fremlin–Talagrand theorem [6] implies that $\mathcal{L}_{\mathbf{f}}$ totally describes $\mathcal{K}$. However, most important for our purposes is the fact that $\mathcal{L}_{\mathbf{f}}$ contains a Borel cofinal set. This is a consequence of the following theorem of G. Debs [7] (which itself is the classical interpretation of the effective version of the Bourgain–Fremlin–Talagrand theorem, proved by G. Debs in [7]).

Theorem 2. *Let Y, X be Polish spaces and $(g_n)_n$ be a sequence of Borel functions on $Y \times X$ such that for every $y \in Y$ the sequence $(g_n(y, \cdot))_n$ is a sequence of continuous functions relatively compact in $\mathcal{B}(X)$. Then there exists a Borel map $\sigma : Y \to [\mathbb{N}]$ such that for any $y \in Y$, the sequence $(g_n(y, \cdot))_{n \in \sigma(y)}$ is pointwise convergent.*

Let us show how Theorem 2 implies the existence of a Borel cofinal subset of $\mathcal{L}_{\mathbf{f}}$. Given $L, M \in [\mathbb{N}]$ with $L = \{l_0 < l_1 < \cdots\}$ and $M = \{m_0 < m_1 < \cdots\}$ their increasing enumerations, let $L * M = \{l_{m_0} < l_{m_1} < \cdots\}$. Clearly $L * M \in [L]$ and moreover the function $(L, M) \mapsto L * M$ is continuous. Let $(f_n)_n$ be as in the beginning of the section and let $Y = [\mathbb{N}]$. For every $n \in \mathbb{N}$ define $g_n : [\mathbb{N}] \times X \to \mathbb{R}$ by

$$g_n(L, x) = f_{l_n}(x)$$

where l_n is the nth element of the increasing enumeration of L. The sequence $(g_n)_n$ satisfies all the hypotheses of Theorem 2. Let $\sigma : [\mathbb{N}] \to [\mathbb{N}]$ be the Borel function such that for every $L \in [\mathbb{N}]$ the sequence

$$(g_n(L, \cdot))_{n \in \sigma(L)} = (f_n)_{n \in L * \sigma(L)}$$

is pointwise convergent. The function $L \to L * \sigma(L)$ is Borel and so the set

$$A = \{L * \sigma(L) : L \in [\mathbb{N}]\}$$

is an analytic cofinal subset of $\mathcal{L}_{\mathbf{f}}$. By separation we get that there exists a Borel cofinal subset of $\mathcal{L}_{\mathbf{f}}$. The cofinality of this set in conjunction with the Bourgain–Fremlin–Talagrand theorem give us the following corollary.

Corollary 3. *Let X be a Polish space and $(f_n)_n$ a sequence of Borel functions on X which is relatively compact in $\mathcal{B}(X)$. Then there exists a Borel set $C \subseteq [\mathbb{N}]$ such that for every $c \in C$ the sequence $(f_n)_{n \in c}$ is pointwise convergent and for every accumulation point f of $(f_n)_n$ there exists $c \in C$ with $f = \lim_{n \in c} f_n$.*

In what follows we will say that the set C obtained by Corollary 3 is a *code* of $(f_n)_n$. If $\mathcal{K}$ is a separable Rosenthal compact and $(f_n)_n$ is a dense sequence in $\mathcal{K}$, then we will say that C is the code of $\mathcal{K}$. Notice that the codes depend on the dense sequence. If $c \in C$, then by f_c we shall denote the function coded by c. That is f_c is the pointwise limit of the sequence $(f_n)_{n \in c}$.

The following lemma captures the basic definability properties of the set of codes. Its easy proof is left to the reader.

Lemma 4. *Let X and $(f_n)_n$ be as in Corollary 3 and let C be a code of $(f_n)_n$. Then for every $a \in \mathbb{R}$ the following relations*

(i) $(c, x) \in R_a \Leftrightarrow f_c(x) > a$,
(ii) $(c, x) \in R_a' \Leftrightarrow f_c(x) \geq a$,
(iii) $(c_1, c_2, x) \in D_a \Leftrightarrow |f_{c_1}(x) - f_{c_2}(x)| > a$

are Borel.

The existence of codings of separable Rosenthal compacta gives us the following tree-representation of them.

Proposition 5. *Let $\mathcal{K}$ be a separable Rosenthal compact. Then there exist a countable tree T and a sequence $(g_t)_{t \in T}$ in $\mathcal{K}$ such that the following hold.*

(1) *For every $\sigma \in [T]$ the sequence $(g_{\sigma|n})_{n \in \mathbb{N}}$ is pointwise convergent.*
(2) *For every $f \in \mathcal{K}$ there exists $\sigma \in [T]$ such that f is the pointwise limit of the sequence $(g_{\sigma|n})_{n \in \mathbb{N}}$.*

Proof. Let $(f_n)_n$ be a dense sequence in $\mathcal{K}$. We may assume that for every $n \in \mathbb{N}$ the set $\{m : f_m = f_n\}$ is infinite. This extra condition guarantees that for every $f \in \mathcal{K}$ there exists $L \in \mathcal{L}_\mathbf{f}$ such that $f = f_L$. Let C be the codes of $(f_n)_n$. Now we shall use a common unfolding trick. As C is Borel in $2^\mathbb{N}$ there exists $F \subseteq 2^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ closed such that $C = \mathrm{proj}_{2^\mathbb{N}} F$. Let T be the unique (downwards closed) pruned tree on $2 \times \mathbb{N}$ such that $F = [T]$. This will be the desired tree. It remains to define the sequence $(g_t)_{t \in T}$. Set $g_{(\varnothing, \varnothing)} = f_0$. Let $t = (s, w) \in T$ and $k \geq 1$ with $s \in 2^{<\mathbb{N}}$, $w \in \mathbb{N}^{<\mathbb{N}}$ and $|s| = |w| = k$. Define $n_t \in \mathbb{N}$ to be $n_t = \max\{n < k : s(n) = 1\}$, if the set $\{n < k : s(n) = 1\}$ is non-empty, and $n_t = 0$ otherwise. Finally set $g_t = f_{n_t}$. It is easy to check that for every $\sigma \in [T]$ the sequence $(g_{\sigma|n})_n$ is pointwise convergent, and so (1) is satisfied. That (2) is also satisfied follows from the fact that for every $f \in \mathcal{K}$ there exists $L \in \mathcal{L}_\mathbf{f}$ with $f = f_L$ and the fact that C is cofinal. $\square$

Remark 1. (1) We should point out that Corollary 3, combined with J.H. Silver's theorem (see [14] or [24]) on the number of equivalence classes of co-analytic equivalence relations, gives an answer to the cardinality problem of separable Rosenthal compacta, a well-known fact that can also be derived by the results of [25] (see also [2], Remark 3). Indeed, let $\mathcal{K}$ be one and let C be the set of codes of $\mathcal{K}$. Define the following equivalence relation on C, by

$$c_1 \sim c_2 \Leftrightarrow f_{c_1} = f_{c_2} \Leftrightarrow \forall x \ f_{c_1}(x) = f_{c_2}(x).$$

Then $\sim$ is a $\mathbf{\Pi}_1^1$ equivalence relation. Hence, by Silver's dichotomy, either the equivalence classes are countable or perfectly many. The first case implies that $|\mathcal{K}| = \aleph_0$, and the second one that $|\mathcal{K}| = 2^{\aleph_0}$.
(2) Although the set C of codes of a separable Rosenthal compact $\mathcal{K}$ is considered to be a Borel set which describes $\mathcal{K}$ efficiently, when it is considered as a subset of $[\mathbb{N}]$ it can be chosen to have rich structural properties. In particular, it can be chosen to be hereditary (i.e. if $c \in C$ and $c' \in [c]$, then $c' \in C$) and invariant under finite changes. To see this, start with a code C_1 of $\mathcal{K}$, i.e. a Borel cofinal subset of $\mathcal{L}_\mathbf{f}$. Let

$$\Phi = \big\{(F, G) : (F \subseteq \mathcal{L}_\mathbf{f}) \wedge (G \cap C_1 = \varnothing) \wedge$$
$$[\forall L, M \ (L \in F \wedge M \subseteq L \Rightarrow M \notin G)] \wedge$$
$$[\forall L, M, s \ (L \in F \wedge (L \bigtriangleup M = s) \Rightarrow M \notin G)]\big\}.$$

Let also $A = \{N : \exists L \in C_1 \ \exists s \in [\mathbb{N}]^{<\mathbb{N}} \ \exists M \in [L] \text{ with } N \bigtriangleup M = s\}$. Then A is $\mathbf{\Sigma}_1^1$ and clearly $\Phi(A, \sim A)$. As Φ is $\mathbf{\Pi}_1^1$ on $\mathbf{\Sigma}_1^1$, hereditary and continuous upward in the second variable, by the dual form of the second reflection theorem (see [9], Theorem 35.16), there exists $C \supseteq A$ Borel with $\Phi(C, \sim C)$. Clearly C is as desired.
(3) We notice that the idea of coding subsets of function spaces by converging sequences appears also in [4], where a representation result of $\mathbf{\Sigma}_2^1$ subsets of $C([0, 1])$ is proved.

4. A boundedness result

4.1. Determining $\alpha(f)$ by compact sets

Let X be a Polish space and $f : X \to \mathbb{R}$ a Baire-1 function. The aim of this subsection is to show that the value $\alpha(f)$ is completely determined by the derivatives taken over compact subsets of X (notice that this is trivial if X is compact metrizable). Specifically we have the following.

Proposition 6. *Let X be a Polish space, $f : X \to \mathbb{R}$ Baire-1 and $a < b$ reals. Then $\alpha(f, a, b) = \sup\{\alpha(f, K, a, b) :$ $K \subseteq X$ compact\}.*

The proof of Proposition 6 is an immediate consequence of the following lemmas. In what follows, all balls in X are taken with respect to some compatible complete metric ρ of X.

Lemma 7. *Let X, f and $a < b$ be as in Proposition 6. Let also $F \subseteq X$ closed, $x \in X$ and $\xi < \omega_1$ be such that $x \in F_{f,a,b}^{(\xi)}$. Then for every $\varepsilon > 0$, if we let $C = F \cap \overline{B(x, \varepsilon)}$, we have $x \in C_{f,a,b}^{(\xi)}$.*

Proof. Fix F and ε as above. For notational simplicity let $U = B(x, \varepsilon)$ and $C = F \cap \overline{B(x, \varepsilon)}$. By induction we shall show that

$$F^{(\xi)} \cap U \subseteq C^{(\xi)}$$

where $F^{(\xi)} = F_{f,a,b}^{(\xi)}$ and similarly for C. This clearly implies the lemma. For $\xi = 0$ is straightforward. Suppose that the lemma is true for every $\xi < \zeta$. Assume that $\zeta = \xi + 1$ is a successor ordinal. Let $y \in F^{(\xi+1)} \cap U$. As U is open, we have

$$y \in \overline{F^{(\xi)} \cap U \cap [f < a]} \cap \overline{F^{(\xi)} \cap U \cap [f > b]}.$$

By the inductive assumption we get that

$$y \in \overline{C^{(\xi)} \cap [f < a]} \cap \overline{C^{(\xi)} \cap [f > b]} = C^{(\xi+1)}$$

which proves the case of successor ordinals. If ζ is limit, then

$$F^{(\zeta)} \cap U = \bigcap_{\xi < \zeta} F^{(\xi)} \cap U \subseteq \bigcap_{\xi < \zeta} C^{(\xi)} = C^{(\zeta)}$$

and the lemma is proved. $\square$

Lemma 8. *Let X, f and $a < b$ be as in Proposition 6. Let also $F \subseteq X$ closed, $x \in X$ and $\xi < \omega_1$ be such that $x \in F_{f,a,b}^{(\xi)}$. Then there exists $K \subseteq F$ countable compact such that $x \in K_{f,a,b}^{(\xi)}$.*

Proof. Again for notational simplicity for every $C \subseteq X$ closed and every $\xi < \omega_1$ we let $C^{(\xi)} = C_{f,a,b}^{(\xi)}$. The proof is by induction on countable ordinals, as before. For $\xi = 0$ the lemma is obviously true. Suppose that the lemma has been proved for every $\xi < \zeta$. Let $F \subseteq X$ closed and $x \in F^{(\zeta)}$. Notice that one of the following alternatives must occur.

(A1) $f(x) < a$ and there exists a sequence $(y_n)_n$ such that $y_n \neq y_m$ for $n \neq m$, $f(y_n) > b$, $y_n \in F^{(\xi_n)}$ and $y_n \to x$;
(A2) $f(x) > b$ and there exists a sequence $(z_n)_n$ such that $z_n \neq z_m$ for $n \neq m$, $f(z_n) < a$, $z_n \in F^{(\xi_n)}$ and $z_n \to x$;
(A3) there exist two distinct sequences $(y_n)_n$ and $(z_n)_n$ such that $y_n \neq y_m$ and $z_n \neq z_m$ for $n \neq m$, $f(y_n) < a$, $f(z_n) > b$, $y_n, z_n \in F^{(\xi_n)}$ and $y_n \to x$, $z_n \to x$,

where above the sequence $(\xi_n)_n$ of countable ordinals is as follows.

(C1) If $\zeta = \xi + 1$, then $\xi_n = \xi$ for every n.
(C2) If ζ is limit, then $(\xi_n)_n$ is an increasing sequence of successor ordinals with $\xi_n \nearrow \zeta$.

We shall treat the alternative (A1) (the other ones are similar). Let $(r_n)_n$ be a sequence of positive reals such that $\overline{B(y_n, r_n)} \cap \overline{B(y_m, r_m)} = \varnothing$ if $n \neq m$ and $x \notin \overline{B(y_n, r_n)}$ for every n. Let $C_n = F \cap \overline{B(y_n, r_n)}$. By Lemma 7, we get that $y_n \in C_n^{(\xi_n)}$. By the inductive assumption, there exists $K_n \subseteq C_n \subseteq F_n$ countable compact such that $y_n \in K_n^{(\xi_n)}$. Finally let $K = \{x\} \cup (\bigcup_n K_n)$. Then K is countable compact and it is easy to see that $x \in K^{(\zeta)}$. $\square$

Remark 2. Notice that the proof of Lemma 8 actually shows that

$$\alpha(f, a, b) = \sup\{\alpha(f, K, a, b) : K \subseteq X \text{ countable compact}\}.$$

Moreover observe that if $\alpha(f, a, b)$ is a successor ordinal, then the above supremum is attainted.

4.2. The main result

This subsection is devoted to the proof of the following result.

Theorem 9. *Let X be a Polish space and $\mathbf{f} = (f_n)_n$ a sequence of Borel real-valued functions on X. Let*

$$\mathcal{L}^1_{\mathbf{f}} = \{L \in [\mathbb{N}] : (f_n)_{n \in L} \text{ is pointwise convergent to a Baire-1 function}\}.$$

Then for every $C \subseteq \mathcal{L}^1_{\mathbf{f}}$ Borel, we have

$$\sup\{\alpha(f_L) : L \in C\} < \omega_1$$

where, for every $L \in C$, f_L denotes the pointwise limit of the sequence $(f_n)_{n \in L}$.

For the proof of Theorem 9 we will need the following theorem, which gives us a way of defining parameterized $\mathbf{\Pi}^1_1$-ranks (see [9], page 275).

Theorem 10. *Let Y be a standard Borel space, X a Polish space and $\mathbb{D} : Y \times K(X) \to K(X)$ be a Borel map such that for every $y \in Y$, $\mathbb{D}_y$ is a derivative on $K(X)$. Then the set*

$$\Omega_{\mathbb{D}} = \{(y, K) : \mathbb{D}_y^{(\infty)}(K) = \varnothing\}$$

is $\mathbf{\Pi}^1_1$ and the map $(y, K) \to |K|_{\mathbb{D}_y}$ is a $\mathbf{\Pi}^1_1$-rank on $\Omega_{\mathbb{D}}$.

We continue with the proof of Theorem 9.

Proof of Theorem 9. Let $C \subseteq \mathcal{L}^1_{\mathbf{f}}$ Borel arbitrary. Fix $a, b \in \mathbb{R}$ with $a < b$. Define $\mathbb{D} : C \times K(X) \to K(X)$ by

$$\mathbb{D}(L, K) = \overline{K \cap [f_L < a]} \cap \overline{K \cap [f_L > b]}$$

where f_L is the pointwise limit of the sequence $(f_n)_{n \in L}$. It is clear that for every $L \in C$ the map $K \to \mathbb{D}(L, K)$ is a derivative on $K(X)$ and that $\alpha(f_L, K, a, b) = |K|_{\mathbb{D}_L}$. We will show that $\mathbb{D}$ is Borel. Define $A, B \in C \times K(X) \times X$ by

$$(L, K, x) \in A \Leftrightarrow (x \in K) \wedge (f_L(x) < a)$$

and

$$(L, K, x) \in B \Leftrightarrow (x \in K) \wedge (f_L(x) > b).$$

It is easy to check that both A and B are Borel. Also let $\tilde{A}, \tilde{B} \subseteq C \times K(X) \times X$ be defined by

$$(L, K, x) \in \tilde{A} \Leftrightarrow x \in \overline{A_{(L,K)}}$$

and

$$(L, K, x) \in \tilde{B} \Leftrightarrow x \in \overline{B_{(L,K)}},$$

where $A_{(L,K)} = \{x : (L, K, x) \in A\}$ is the section of A (and similarly for B). Notice that for every $(L, K) \in C \times K(X)$ we have $\mathbb{D}(L, K) = \tilde{A}_{(L,K)} \cap \tilde{B}_{(L,K)}$. As $\tilde{A}_{(L,K)}$ and $\tilde{B}_{(L,K)}$ are compact (being subsets of K), by Theorem 28.8 in [9], it is enough to show that the sets $\tilde{A}$ and $\tilde{B}$ are Borel. We will need the following easy consequence of the Arsenin–Kunugui theorem (the proof is left to the reader).

Lemma 11. *Let Z be a standard Borel space, X a Polish space and $F \subseteq Z \times X$ Borel with K_σ sections. Then the set $\tilde{F}$ defined by*

$$(z, x) \in \tilde{F} \Leftrightarrow x \in \overline{F_z}$$

is a Borel subset of $Z \times X$.

By our assumptions, for every $L \in C$ the function f_L is Baire-1 and so for every $(L, K) \in C \times K(X)$ the sections $A_{(L,K)}$ and $B_{(L,K)}$ of A and B respectively are K_σ. Hence, by Lemma 11, we get that $\tilde{A}$ and $\tilde{B}$ are Borel.

By the above we conclude that $\mathbb{D}$ is a Borel map. By Theorem 10, the map $(L, K) \to |K|_{\mathbb{D}_L}$ is a $\mathbf{\Pi}_1^1$-rank on $\Omega_{\mathbb{D}}$. By Proposition 1 and the fact that $C \subseteq \mathcal{L}_{\mathbf{f}}^1$, we get that for every $(L, K) \in C \times K(X)$ the transfinite sequence $\left(\mathbb{D}_L^{(\xi)}(K)\right)_{\xi < \omega_1}$ must be stabilized at $\varnothing$ and so $\Omega_{\mathbb{D}} = C \times K(X)$. As $\Omega_{\mathbb{D}}$ is Borel, by boundedness we have

$$\sup\{|K|_{\mathbb{D}_L} : (L, K) \in C \times K(X)\} < \omega_1.$$

It follows that

$$\sup\{\alpha(f_L, K, a, b) : (L, K) \in C \times K(X)\} < \omega_1.$$

By Proposition 6, we get

$$\sup\{\alpha(f_L, a, b) : L \in C\} < \omega_1.$$

This completes the proof of the theorem. $\quad\square$

4.3. Consequences

Let us recall some definitions from [2]. Let X be a Polish space, $(f_n)_n$ a sequence of real-valued functions on X and let $\mathcal{K}$ be the closure of $\{f_n\}_n$ in $\mathbb{R}^X$. We will say that $\mathcal{K}$ is a (separable) quasi-Rosenthal if every accumulation point of $\mathcal{K}$ is a Baire-1 function and moreover we will say that $\mathcal{K}$ is Borel separable if the sequence $(f_n)_n$ consists of Borel functions. Combining Theorem 9 with Corollary 3 we get the following result of [2].

Theorem 12. *Let X be a Polish space and $\mathcal{K}$ a Borel separable quasi-Rosenthal compact on X. Then*

$$\sup\{\alpha(f) : f \in Acc(\mathcal{K})\} < \omega_1$$

where $Acc(\mathcal{K})$ denotes the accumulation points of $\mathcal{K}$. In particular, if $\mathcal{K}$ is a separable Rosenthal compact on X, then

$$\sup\{\alpha(f) : f \in \mathcal{K}\} < \omega_1.$$

Besides boundedness, the implications of Theorem 9 and the relation between the separation rank and the Borelness of $\mathcal{L}_{\mathbf{f}}$ are more transparently seen when X is a compact metrizable space. In particular we have the following.

Proposition 13. *Let X be a compact metrizable space and $\mathcal{K}$ a separable Rosenthal compact on X. Let $\mathbf{f} = (f_n)_n$ be a dense sequence in $\mathcal{K}$ and $a < b$ reals. Then the map $L \to \alpha(f_L, a, b)$ is a $\mathbf{\Pi}_1^1$-rank on $\mathcal{L}_{\mathbf{f}}$ if and only if the set $\mathcal{L}_{\mathbf{f}}$ is Borel.*

Proof. First assume that $\mathcal{L}_{\mathbf{f}}$ is not Borel. By Theorem 12, we have that

$$\sup\{\alpha(f_L, a, b) : L \in \mathcal{L}_{\mathbf{f}}\} < \omega_1$$

and so the map $L \to \alpha(f_L, a, b)$ cannot be a $\mathbf{\Pi}_1^1$-rank on $\mathcal{L}_{\mathbf{f}}$, as $\mathcal{L}_{\mathbf{f}}$ is $\mathbf{\Pi}_1^1$-true. Conversely, assume that $\mathcal{L}_{\mathbf{f}}$ is Borel. By the proof of Theorem 9, we have that the map $(L, K) \to |K|_{\mathbb{D}_L}$ is a $\mathbf{\Pi}_1^1$-rank on $\mathcal{L}_{\mathbf{f}} \times K(X)$. It follows that the relation

$$L_1 \preceq L_2 \Leftrightarrow \alpha(f_{L_1}, a, b) \leq \alpha(f_{L_2}, a, b) \Leftrightarrow |X|_{\mathbb{D}_{L_1}} \leq |X|_{\mathbb{D}_{L_2}}$$

is Borel in $\mathcal{L}_{\mathbf{f}} \times \mathcal{L}_{\mathbf{f}}$. This implies that the map $L \to \alpha(f_L, a, b)$ is a $\mathbf{\Pi}_1^1$-rank on $\mathcal{L}_{\mathbf{f}}$, as desired. $\quad\square$

Remark 3. Although the map $L \to \alpha(f_L, a, b)$ is not always a $\mathbf{\Pi}_1^1$-rank on $\mathcal{L}_{\mathbf{f}}$, it is easy to see that it is a $\mathbf{\Pi}_1^1$-rank on the codes C of $\mathcal{K}$, as the relation

$$c_1 \preceq c_2 \Leftrightarrow \alpha(f_{c_1}, a, b) \leq \alpha(f_{c_2}, a, b) \Leftrightarrow |X|_{\mathbb{D}_{c_1}} \leq |X|_{\mathbb{D}_{c_2}}$$

is Borel in $C \times C$ for every pair $a < b$ of reals. Hence, when X is compact metrizable space, we could say that the separation rank is a $\mathbf{\Pi}_1^1$-rank "in the codes".

We proceed to discuss another application of Theorem 9 which deals with the following approximation question in Ramsey theory. Recall that a set $N \subseteq [\mathbb{N}]$ is called Ramsey-null if for every $s \in [\mathbb{N}]^{<\mathbb{N}}$ and every $L \in [\mathbb{N}]$ with $s < L$, there exists $L' \in [L]$ such that $[s, L'] \cap N = \varnothing$. As every analytic set is Ramsey [23], it is natural to ask the following. Is it true that for every analytic set $A \subseteq [\mathbb{N}]$ there exists $B \supseteq A$ Borel such that $B \setminus A$ is Ramsey-null? As we will show, the answer is no, and a counterexample can be found which is in addition Ramsey-null.

To this end we will need some notations from [3]. Let X be a separable Banach space. By $X^{**}_{\mathcal{B}_1}$ we denote the set of all Baire-1 elements of the ball of the second dual X^{**} of X. We will say that X is α-universal if

$$\sup\{\alpha(x^{**}) : x^{**} \in X^{**}_{\mathcal{B}_1}\} = \omega_1.$$

We should point out that there exist non-universal (in the classical sense) separable Banach spaces which are α-universal (see [1]). We have the following.

Proposition 14. *There exists a Σ^1_1 Ramsey-null subset A of $[\mathbb{N}]$ for which there does not exist a Borel set $B \supseteq A$ such that the difference $B \setminus A$ is Ramsey-null.*

Proof. Let X be a separable α-universal Banach space and fix a norm dense sequence $\mathbf{f} = (x_n)_n$ in the ball of X (it will be convenient to assume that $x_n \neq x_m$ if $n \neq m$). Let

$$\mathcal{L}_{\mathbf{f}} = \{L \in [\mathbb{N}] : (x_n)_{n \in L} \text{ is weak* convergent}\}.$$

Clearly $\mathcal{L}_{\mathbf{f}}$ is $\mathbf{\Pi}^1_1$. Moreover, notice that $\mathcal{L}_{\mathbf{f}} = \mathcal{L}^1_{\mathbf{f}}$ according to the notation of Theorem 9.

Let $x^{**} \in X^{**}_{\mathcal{B}_1}$ arbitrary. By the Odell–Rosenthal theorem (see [3] or [17]), there exists $L \in \mathcal{L}_{\mathbf{f}}$ such that $x^{**} = w^* - \lim_{n \in L} x_n$. It follows that

$$\sup\{\alpha(x^{**}) : x^{**} \in X^{**}_{\mathcal{B}_1}\} = \sup\{\alpha(x_L) : L \in \mathcal{L}_{\mathbf{f}}\}$$

where x_L denotes the weak* limit of the sequence $(x_n)_{n \in L}$. Denote by $(e_n)_n$ the standard basis of ℓ_1 and let

$$\Lambda = \{L \in [\mathbb{N}] : \exists k \text{ such that } (x_n)_{n \in L} \text{ is } (k+1)\text{-equivalent to } (e_n)_n\}$$

where, as usual, if $L \in [\mathbb{N}]$ with $L = \{l_0 < l_1 < \cdots\}$ its increasing enumeration, then $(x_n)_{n \in L}$ is $(k+1)$-equivalent to $(e_n)_n$ if for every $m \in \mathbb{N}$ and every $a_0, \ldots, a_m \in \mathbb{R}$ we have

$$\frac{1}{k+1} \sum_{n=0}^{m} |a_n| \leq \Big\| \sum_{n=0}^{m} a_n x_{l_n} \Big\|_X \leq (k+1) \sum_{n=0}^{m} |a_n|.$$

Then Λ is $\mathbf{\Sigma}^0_2$. We notice that, by Bourgain's result [5] and our assumptions on the space X, the set Λ is non-empty. Let also

$$\Lambda_1 = \{N \in [\mathbb{N}] : \exists L \in \Lambda \, \exists s \in [\mathbb{N}]^{<\mathbb{N}} \text{ such that } N \triangle L = s\}.$$

Clearly Λ_1 is $\mathbf{\Sigma}^0_2$ too. Observe that both $\mathcal{L}_{\mathbf{f}}$ and Λ_1 are hereditary and invariant under finite changes. Moreover the set $\mathcal{L}_{\mathbf{f}} \cup \Lambda_1$ is cofinal. This is essentially a consequence of Rosenthal's dichotomy (see, for instance, [12]). It follows that the set $A = [\mathbb{N}] \setminus (\mathcal{L}_{\mathbf{f}} \cup \Lambda_1)$ is $\mathbf{\Sigma}^1_1$ and Ramsey-null.

We claim that A is the desired set. Assume not, i.e. there exists a Borel set $B \supseteq A$ such that the difference $B \setminus A$ is Ramsey-null. We set $C = [\mathbb{N}] \setminus (B \cup \Lambda_1)$. Then $C \subseteq \mathcal{L}_{\mathbf{f}}$ is Borel and moreover $\mathcal{L}_{\mathbf{f}} \setminus C$ is Ramsey-null. It follows that for every $x^{**} \in X^{**}_{\mathcal{B}_1}$ there exists $L \in C$ such that $x^{**} = x_L$. As C is Borel, by Theorem 9 we have that

$$\sup\{\alpha(x^{**}) : x^{**} \in X^{**}_{\mathcal{B}_1}\} = \sup\{\alpha(x_L) : L \in C\} < \omega_1$$

which contradicts the fact that X is α-universal. The proof is completed. $\quad\square$

Remark 4. (1) An example as in Proposition 14 can also be given using the convergence rank γ studied by A.S. Kechris and A. Louveau [10]. As the reasoning is the same, we shall briefly describe the argument. Let $(f_n)_n$ be a sequence of continuous function on $2^{\mathbb{N}}$ with $\|f_n\|_\infty \leq 1$ for all $n \in \mathbb{N}$ and such that the set $\{f_n : n \in \mathbb{N}\}$ is norm dense in the ball of $C(2^{\mathbb{N}})$. As in Proposition 14, consider the sets $\mathcal{L}_{\mathbf{f}}$, Λ_1 and $A = [\mathbb{N}] \setminus (\mathcal{L}_{\mathbf{f}} \cup \Lambda_1)$. Then the set A is $\mathbf{\Sigma}^1_1$ and Ramsey-null. That A cannot be covered by a Borel set B such that the difference $B \setminus A$ is Ramsey-null follows essentially by the following facts.

(F1) The map $(g_n)_n \mapsto \gamma\big((g_n)_n\big)$ is a $\mathbf{\Pi}_1^1$-rank on the set $\mathrm{CN} = \{(g_n)_n \in C(2^{\mathbb{N}})^{\mathbb{N}} : (g_n)_n \text{ is pointwise convergent}\}$ (see [9], page 279). Hence the map

$$\mathcal{L}_{\mathbf{f}} \ni L = \{l_0 < l_1 < \cdots\} \mapsto \gamma\big((f_{l_n})_n\big)$$

is a $\mathbf{\Pi}_1^1$-rank on $\mathcal{L}_{\mathbf{f}}$.

(F2) For every $\Delta \in \mathbf{\Delta}_2^0$, there exists $L \in \mathcal{L}_{\mathbf{f}}$ such that the sequence $(f_n)_{n \in L}$ is pointwise convergent to χ_Δ. By Proposition 1 in [10], we get that $\alpha(\chi_\Delta) \leq \gamma\big((f_n)_{n \in L}\big)$. It follows that

$$\sup\{\gamma\big((f_n)_{n \in L}\big) : L \in \mathcal{L}_{\mathbf{f}}\} \geq \sup\{\alpha(\chi_\Delta) : \Delta \in \mathbf{\Delta}_2^0\} = \omega_1.$$

(2) For the important special case of a separable Rosenthal compact $\mathcal{K}$ defined on a compact metrizable space X and having a dense set of continuous functions, Theorem 12 has originally been proved by J. Bourgain [5]. We should point out that in this case one does not need Corollary 3 in order to carry out the proof. Let us briefly explain how this can be done. So assume that X is compact metrizable and $\mathbf{f} = (f_n)_n$ is a sequence of continuous functions dense in $\mathcal{K}$. Fix $a, b \in \mathbb{Q}$ with $a < b$ and let $A_n = [f_n \leq a]$ and $B_n = [f_n \geq b]$. For a given $M \in [\mathbb{N}]$ let as usual

$$\liminf_{n \in M} A_n = \bigcup_n \bigcap_{k \geq n, k \in M} A_k$$

and similarly for $\liminf_{n \in M} B_n$. Observe the following.

(O1) For every $M \in [\mathbb{N}]$ the sets $\liminf_{n \in M} A_n$ and $\liminf_{n \in M} B_n$ are both $\mathbf{\Sigma}_2^0$.
(O2) If $L, M \in [\mathbb{N}]$ are such that $L \subseteq M$, then $\liminf_{n \in M} A_n \subseteq \liminf_{n \in L} A_n$ and similarly for B_n.
(O3) If $L \in \mathcal{L}_{\mathbf{f}}$, then $[f_L < a] \subseteq \liminf_{n \in L} A_n \subseteq [f_L \leq a]$ and respectively $[f_L > b] \subseteq \liminf_{n \in L} B_n \subseteq [f_L \geq b]$.

Define $\mathbb{D} : [\mathbb{N}] \times K(X) \to K(X)$ by

$$\mathbb{D}(M, K) = \overline{K \cap \liminf_{n \in M} A_n} \cap \overline{K \cap \liminf_{n \in M} B_n}.$$

By (O1) and using the same arguments as in the proof of Theorem 9, we can easily verify that $\mathbb{D}$ is Borel. As $\mathcal{L}_{\mathbf{f}}$ is cofinal, by (O2) and (O3) we can also easily verify that $\Omega_{\mathbb{D}} = [\mathbb{N}] \times K(X)$. So by boundedness we get $\sup\{|K|_{\mathbb{D}_M} : (M, K) \in [\mathbb{N}] \times K(X)\} < \omega_1$. Now using (O3) again, we finally get that $\sup\{\alpha(f, a, b) : f \in \mathcal{K}\} < \omega_1$, as desired.

5. On the descriptive set-theoretic properties of $\mathcal{L}_{\mathbf{f}}$

In this section we will show that certain topological properties of a separable Rosenthal compact $\mathcal{K}$ imply the Borelness of the set $\mathcal{L}_{\mathbf{f}}$. To this end, we recall that $\mathcal{K}$ is said to be a pre-metric compactum of degree at most two if there exists a countable subset D of X such that at most two functions in $\mathcal{K}$ agree on D (see [25]). Let us consider the following subclass.

Defintion 15. We will say that $\mathcal{K}$ is a pre-metric compactum of degree exactly two, if there exist a countable subset D of X and a countable subset $\mathcal{D}$ of $\mathcal{K}$ such that at most two functions in $\mathcal{K}$ coincide on D and moreover for every $f \in \mathcal{K} \setminus \mathcal{D}$ there exists $g \in \mathcal{K}$ with $g \neq f$ and such that g coincides with f on D.

An important example of such a compact is the split interval (but it is not the only important one — see Remark 5 below). Under the above terminology we have the following.

Theorem 16. *Let X be a Polish space and $\mathcal{K}$ a separable Rosenthal compact on X. If $\mathcal{K}$ is pre-metric of degree exactly two, then for every dense sequence $\mathbf{f} = (f_n)_n$ in $\mathcal{K}$ the set $\mathcal{L}_{\mathbf{f}}$ is Borel.*

Proof. Let $\mathbf{f} = (f_n)_n$ be a dense sequence in $\mathcal{K}$ and C be the set of codes of $(f_n)_n$. Let also $D \subseteq X$ countable and $\mathcal{D} \subseteq \mathcal{K}$ countable verifying that $\mathcal{K}$ is pre-metric of degree exactly two.

CLAIM. *There exists $\mathcal{D}' \subseteq \mathcal{K}$ countable with $\mathcal{D} \subseteq \mathcal{D}'$ and such that for every $c \in C$ with $f_c \in \mathcal{K} \setminus \mathcal{D}'$ there exists $c' \in C$ such that $f_{c'} \neq f_c$ and $f_{c'}$ coincides with f_c on D.*

Proof of the claim. Let $c \in C$ be such that $f_c \in \mathcal{K} \setminus \mathcal{D}$. Let g be the (unique) function in $\mathcal{K}$ with $g \neq f_c$ and such that g coincides with f_c on D. If there does not exist $c' \in C$ with $g = f_{c'}$, then g is an isolated point of $\mathcal{K}$. We set

$$\mathcal{D}' = \mathcal{D} \cup \{f \in \mathcal{K} : \exists g \in \mathcal{K} \text{ isolated such that } f(x) = g(x) \; \forall x \in D\}.$$

As the isolated points of $\mathcal{K}$ are countable and $\mathcal{K}$ is pre-metric of degree at most two, we get that $\mathcal{D}'$ is countable. Clearly $\mathcal{D}'$ is as desired. $\diamond$

Let $\mathcal{D}'$ be the set obtained above and put

$$\mathcal{L}_{\mathcal{D}'} = \bigcup_{f \in \mathcal{D}'} \mathcal{L}_{\mathbf{f}, f} = \bigcup_{f \in \mathcal{D}'} \{L \in [\mathbb{N}] : (f_n)_{n \in L} \text{ is pointwise convergent to } f\}.$$

As every point in $\mathcal{K}$ is G_δ, we see that $\mathcal{L}_{\mathcal{D}'}$ is Borel (actually it is $\mathbf{\Sigma}_4^0$). Consider the following equivalence relation $\sim$ on C, defined by

$$c_1 \sim c_2 \Leftrightarrow \forall x \in D \; f_{c_1}(x) = f_{c_2}(x).$$

By [Lemma 4](), the equivalence relation $\sim$ is Borel. Consider now the relation P on $C \times C \times K(X) \times X$ defined by

$$(c_1, c_2, K, x) \in P \Leftrightarrow (c_1 \sim c_2) \; \wedge \; (x \in K) \; \wedge \; (|f_{c_1}(x) - f_{c_2}(x)| > 0).$$

Again P is Borel. Moreover notice that for every $c_1, c_2 \in C$ the function $x \mapsto |f_{c_1}(x) - f_{c_2}(x)|$ is Baire-1, and so, for every $(c_1, c_2, K) \in C \times C \times K(X)$ the section $P_{(c_1, c_2, K)} = \{x \in X : (c_1, c_2, K, x) \in P\}$ of P is K_σ. By Theorem 35.46 in [9], the set $S \subseteq C \times C \times K(X)$ defined by

$$(c_1, c_2, K) \in S \Leftrightarrow \exists x \; (c_1, c_2, K, x) \in P$$

is Borel and there exists a Borel map $\phi : S \rightarrow X$ such that for every $(c_1, c_2, K) \in S$ we have $\big(c_1, c_2, K, \phi(c_1, c_2, K)\big) \in P$. By the above claim, we have that for every $c \in C \setminus \mathcal{L}_{\mathcal{D}'}$ there exist $c' \in C$ and $K \in K(X)$ such that $(c, c', K) \in S$. Moreover, observe that the set $D \cup \big\{\phi(c, c', K)\big\}$ determines the neighborhood basis of f_c. The crucial fact is that this can be done in a Borel way.

Now we claim that

$$
\begin{aligned}
L \in \mathcal{L}_{\mathbf{f}} \Leftrightarrow (L \in \mathcal{L}_{\mathcal{D}'}) \vee \Big[& (\forall x \in D \; \big(f_n(x)\big)_{n \in L} \text{ converges}) \; \wedge \\
& \big\{ \exists s \in S \text{ with } s = (c_1, c_2, K) \text{ such that} \\
& [\forall x \in D \; f_{c_1}(x) = \lim_{n \in L} f_n(x)] \; \wedge \\
& [\big(f_n(\phi(s))\big)_{n \in L} \text{ converges}] \; \wedge \\
& [f_{c_1}(\phi(s)) = \lim_{n \in L} f_n(\phi(s))] \big\} \Big].
\end{aligned}
$$

Grating this, the proof is completed as the above expression gives a $\mathbf{\Sigma}_1^1$ definition of $\mathcal{L}_{\mathbf{f}}$. As $\mathcal{L}_{\mathbf{f}}$ is also $\mathbf{\Pi}_1^1$, this implies that $\mathcal{L}_{\mathbf{f}}$ is Borel, as desired.

It remains to prove the above equivalence. First assume that $L \in \mathcal{L}_{\mathbf{f}}$. We need to show that L satisfies the expression on the right. If $L \in \mathcal{L}_{\mathcal{D}'}$ this is clearly true. If $L \notin \mathcal{L}_{\mathcal{D}'}$, then pick a code $c \in C \setminus \mathcal{L}_{\mathcal{D}'}$ such that $f_c = f_L$. By the above claim and the remarks of the previous paragraph we can easily verify that again L satisfies the expression on the right. Conversely, let L fulfill the right side of the equivalence. If $L \in \mathcal{L}_{\mathcal{D}'}$ we are done. If not, then by the Bourgain–Fremlin–Talagrand theorem, it suffices to show that all convergent subsequences of $(f_n)_{n \in L}$ have the same limit. The first two conjuncts enclosed in the square brackets on the right side of the equivalence guarantee that each such convergent subsequence of $(f_n)_{n \in L}$ converges either to f_{c_1} or to f_{c_2}. The last two conjuncts guarantee that it is not f_{c_2}, so it is always f_{c_1}. Thus $L \in \mathcal{L}_{\mathbf{f}}$ and the proof is completed. $\square$

Remark 5. (1) Let $\mathcal{K}$ be a pre-metric compactum of degree at most two and let $D \subseteq X$ countable such that at most two functions in $\mathcal{K}$ coincide on D. Notice that the set C of codes of $\mathcal{K}$ is naturally divided into two parts, namely

$$C_2 = \{c \in C : \exists c' \in C \text{ with } f_c \neq f_{c'} \text{ and } f_c(x) = f_{c'}(x) \; \forall x \in D\}$$

and its complement $C_1 = C \setminus C_2$. The assumption that $\mathcal{K}$ is pre-metric of degree exactly two simply reduces to the assumption that the functions coded by C_1 are at most countable. We could say that C_1 is the set of metrizable codes, as it is immediate that the set $\{f_c : c \in C_1\}$ is a metrizable subspace of $\mathcal{K}$. It is easy to check, using the set S defined in the proof of Theorem 16, that C_2 is always $\mathbf{\Sigma}_1^1$. As we shall see, it might happen that C_1 is $\mathbf{\Pi}_1^1$-true. However, if the set C_1 of metrizable codes is Borel, or equivalently if C_2 is Borel, then the set $\mathcal{L}_\mathbf{f}$ is Borel too. Indeed, let Φ be the second part of the disjunction of the expression in the proof of Theorem 16. Then it is easy to see, using the same arguments as in the proof of Theorem 16, that

$$L \in \mathcal{L}_\mathbf{f} \Leftrightarrow (L \in \Phi) \vee \Big(\exists c \in C_1 \ \forall x \in D \ f_c(x) = \lim_{n \in L} f_n(x) \Big).$$

Clearly the above formula gives a $\mathbf{\Sigma}_1^1$ definition of $\mathcal{L}_\mathbf{f}$, provided that C_1 is Borel.

(2) Besides the split interval, there exists another important example of a separable Rosenthal compact which is pre-metric of degree exactly two. This is the separable companion of the Alexandroff duplicate of the Cantor set $D(2^{\mathbb{N}})$ (see [25] for more details). An interesting feature of this compact is that it is *not* hereditarily separable.

Example 1. We proceed to give examples of pre-metric compacta of degree at most two for which Theorem 16 is not valid. Let us recall first the split Cantor set $S(2^{\mathbb{N}})$, which is simply the combinatorial analogue of the split interval. In what follows by $\leqslant$ we shall denote the lexicographical ordering on $2^{\mathbb{N}}$ and by $<$ its strict part. For every $x \in 2^{\mathbb{N}}$ let $f_x^+ = \chi_{\{y:x\leqslant y\}}$ and $f_x^- = \chi_{\{y:x<y\}}$. The split Cantor set $S(2^{\mathbb{N}})$ is $\{f_x^+ : x \in 2^{\mathbb{N}}\} \cup \{f_x^- : x \in 2^{\mathbb{N}}\}$. Clearly $S(2^{\mathbb{N}})$ is a hereditarily separable Rosenthal compact and it is a fundamental example of a pre-metric compactum of degree at most two (see [25]). There is a canonical dense sequence in $S(2^{\mathbb{N}})$ defined as follows. Fix a bijection $h : 2^{<\mathbb{N}} \to \mathbb{N}$ such that $h(s) < h(t)$ if $|s| < |t|$ and enumerate the nodes of Cantor tree as $(s_n)_n$ according to h. For every $s \in 2^{<\mathbb{N}}$ let $x_s^0 = s^\frown 0^\infty \in 2^{\mathbb{N}}$ and $x_s^1 = s^\frown 1^\infty \in 2^{\mathbb{N}}$. For every $n \in \mathbb{N}$ let $f_{4n} = f_{x_{s_n}^0}^+$, $f_{4n+1} = f_{x_{s_n}^1}^+$, $f_{4n+2} = f_{x_{s_n}^0}^-$ and $f_{4n+3} = f_{x_{s_n}^1}^-$. The sequence $(f_n)_{n\in\mathbb{N}}$ is a dense sequence in $S(2^{\mathbb{N}})$.

Let A be a subset of $2^{\mathbb{N}}$ such that A does not contain the eventually constant sequences. To every such A one associates naturally a subset of $\mathbb{R}^A$, which we will denote by $S(A)$, by simply restricting every function of $S(2^{\mathbb{N}})$ on A. Clearly if A is $\mathbf{\Sigma}_1^1$, then $S(A)$ is again a hereditarily separable Rosenthal compact. Notice however that if $2^{\mathbb{N}} \setminus A$ is uncountable, then $S(A)$ is not of degree exactly two. The dense sequence $(f_n)_n$ of $S(2^{\mathbb{N}})$ still remains a dense sequence in $S(A)$. Viewing $(f_n)_n$ as a dense sequence in $S(A)$, we let

$$\mathcal{L}_A = \{L \in [\mathbb{N}] : (f_n|_A)_{n\in L} \text{ is pointwise convergent on } A\}.$$

Under the above notations we have the following.

Proposition 17. *Let $A \subseteq 2^{\mathbb{N}}$ be $\mathbf{\Sigma}_1^1$ and such that A does not contain the eventually constant sequences. Then $2^{\mathbb{N}} \setminus A$ is Wadge reducible to $\mathcal{L}_A$. In particular, if A is $\mathbf{\Sigma}_1^1$-complete, then $\mathcal{L}_A$ is $\mathbf{\Pi}_1^1$-complete. Moreover, if A is Borel, then $\mathcal{L}_A$ is Borel too.*

Proof. Consider the map $\Phi : 2^{\mathbb{N}} \to 2^{2^{<\mathbb{N}}}$, defined by

$$\Phi(x) = \Big\{ x(0) + 1, x(0)^\frown(x(1)+1), x(0)^\frown x(1)^\frown(x(2)+1), \dots \Big\}$$

where the above addition is taken modulo 2. Clearly Φ is continuous. Let $h : 2^{<\mathbb{N}} \to \mathbb{N}$ be the fixed enumeration of the nodes of the Cantor tree. For every $x \in 2^{\mathbb{N}}$ we put $L_x = \{h(t) : t \in \Phi(x)\} \in [\mathbb{N}]$ and also for every $t \in \Phi(x)$ let $x_t^0 = t^\frown 0^\infty \in 2^{\mathbb{N}}$. Notice that $(t_n)_{n\in L_x}$ is the enumeration of $\Phi(x)$ according to h and moreover that x is the limit of the sequence $(x_{t_n}^0)_{n\in L_x}$. However, as one easily observes, if x is not eventually constant, then there exist infinitely many $n \in L_x$ such that $x_{t_n}^0 < x$ and infinitely many $n \in L_x$ such that $x < x_{t_n}^0$.

Now define $H : 2^{\mathbb{N}} \to [\mathbb{N}]$ by

$$H(x) = \{4h(t) : t \in \Phi(x)\} = \{4n : n \in L_x\}.$$

Clearly H is continuous. We claim that

$$x \notin A \Leftrightarrow H(x) \in \mathcal{L}_A.$$

Indeed, first assume that $x \notin A$. As we have remarked before, we have that $x = \lim_{n \in L_x} x_{t_n}^0$. Notice that the sequence $(f_n)_{n \in H(x)}$ is simply the sequence $\left(f_{x_t^0}^+\right)_{t \in \Phi(x)}$. Observe that for every $y \neq x$ the sequence $\left(f_n(y)\right)_{n \in H(x)}$ converges to 0 if $y < x$ and to 1 if $x < y$. As $x \notin A$, this implies that $(f_n)_{n \in H(x)}$ is pointwise convergent, and so $H(x) \in \mathcal{L}_A$. Conversely, assume that $x \in A$. As A does not contain the eventually constant sequences, by the remarks after the definition of Φ, we get that there exist infinitely many $n \in H(x)$ such that $f_n(x) = 0$ and infinitely many $n \in H(x)$ such that $f_n(x) = 1$. Hence the sequence $\left(f_n(x)\right)_{n \in H(x)}$ does not converge, and as $x \in A$, we conclude that $H(x) \notin \mathcal{L}_A$. As H is continuous, this completes the proof that $2^{\mathbb{N}} \setminus A$ is Wadge reducible to $\mathcal{L}_A$. Finally, the fact that if A is Borel, then $\mathcal{L}_A$ is Borel too follows by straightforward descriptive set-theoretic computation and we prefer not to bother the reader with it. □

Remark 6. Besides the fact that Theorem 16 cannot be lifted to all pre-metric compacta of degree at most two, Proposition 17 has another consequence. Namely, that we cannot bound the Borel complexity of $\mathcal{L}_{\mathbf{f}}$ for a dense sequence $\mathbf{f} = (f_n)_n$ in $\mathcal{K}$. This is in contrast with the situation with $\mathcal{L}_{\mathbf{f}, f}$ for some $f \in \mathcal{K}$, which when it is Borel (equivalently when f is a G_δ point), it is always $\mathbf{\Pi}_3^0$.

Concerning the class of not first countable separable Rosenthal compacta we have the following.

Proposition 18. *Let $\mathcal{K}$ be a separable Rosenthal compact. If there exists a non-G_δ point f in $\mathcal{K}$, then for every dense sequence $\mathbf{f} = (f_n)_n$ in $\mathcal{K}$ the set $\mathcal{L}_{\mathbf{f}}$ is $\mathbf{\Pi}_1^1$-complete.*

The proof of Proposition 18 is essentially based on a result of A. Krawczyk from [11]. To state it, we need to recall some pieces of notation and few definitions. For every $a, b \in [\mathbb{N}]$ we write $a \subseteq^* b$ if $a \setminus b$ is finite, while we write $a \perp b$ if the set $a \cap b$ is finite. If $\mathcal{A}$ is a subset of $[\mathbb{N}]$, we let $\mathcal{A}^\perp = \{b : b \perp a \; \forall a \in \mathcal{A}\}$ and $\mathcal{A}^* = \{\mathbb{N} \setminus a : a \in \mathcal{A}\}$. For every $\mathcal{A}, \mathcal{B} \subseteq [\mathbb{N}]$ we say that $\mathcal{A}$ is countably $\mathcal{B}$-generated if there exists $\{b_n : n \in \mathbb{N}\} \subseteq \mathcal{B}$ such that for every $a \in \mathcal{A}$ there exists $k \in \mathbb{N}$ with $a \subseteq b_0 \cup \cdots \cup b_k$. An ideal $\mathcal{I}$ on $\mathbb{N}$ is said to be bi-sequential if for every $p \in \beta\mathbb{N}$ with $\mathcal{I} \subseteq p^*$, $\mathcal{I}$ is countably p^*-generated. Finally, for every $t \in \mathbb{N}^{<\mathbb{N}}$ let $\hat{t} = \{s : t \sqsubset s\}$. We will use the following result of Krawczyk (see [11], Lemma 2).

Proposition 19. *Let $\mathcal{I}$ be $\mathbf{\Sigma}_1^1$, bi-sequential and not countably $\mathcal{I}$-generated ideal on $\mathbb{N}$. Then there exists a 1–1 map $\psi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ such that, setting $\mathcal{J} = \{\psi^{-1}(a) : a \in \mathcal{I}\}$, the following hold.*

(P1) *For every $\sigma \in \mathbb{N}^{\mathbb{N}}$, $\{\sigma|n : n \in \mathbb{N}\} \in \mathcal{J}$.*
(P2) *For every $b \in \mathcal{J}$ and every $n \in \mathbb{N}$, there exist $t_0, \ldots, t_k \in \mathbb{N}^n$ with $b \subseteq^* \hat{t}_0 \cup \cdots \cup \hat{t}_k$.*

We continue with the proof of Proposition 18.

Proof of Proposition 18. Let $\mathbf{f} = (f_n)_n$ be a dense sequence in $\mathcal{K}$ and let $f \in \mathcal{K}$ be a non-G_δ point. Consider the ideal

$$\mathcal{I} = \{L \in [\mathbb{N}] : f \notin \overline{\{f_n\}}_{n \in L}^p\}.$$

In [11], page 1099, it is shown that $\mathcal{I}$ is a $\mathbf{\Sigma}_1^1$, bi-sequential ideal on $\mathbb{N}$ which is not countably $\mathcal{I}$-generated (the bi-sequentiality of $\mathcal{I}$ can be derived either by a result of Pol [20], or by the non-effective version of Debs' theorem [3]). Also, by the Bourgain–Fremlin–Talagrand theorem, we have that $\mathcal{I}^\perp = \mathcal{L}_{\mathbf{f}, f}$. We apply Proposition 19 and we get a 1–1 map $\psi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ satisfying (P1) and (P2).

CLAIM. *For every $T \in$ WF infinite, $T \in \mathcal{J}^\perp$.*

Proof of the claim. Assume not. Then there exist $T \in$ WF infinite and $b \in \mathcal{J}$ with $b \subseteq T$. For every $s \in T$ let $T_s = \{t \in T : s \sqsubseteq t\}$. We let $S = \{s \in T : T_s \cap b \text{ is infinite}\}$. Then S is downwards closed subtree of T. Moreover, by (P2) in Proposition 19, we see that S is finitely splitting. Finally, notice that for every $s \in S$ there exists $n \in \mathbb{N}$ with $s^\frown n \in S$. Indeed, let $s \in S$ and put $b_s = T_s \cap b \in \mathcal{J}$. Let $N_s = \{n \in \mathbb{N} : s^\frown n \in T\}$ and observe that $b_s \setminus \{s\} = \bigcup_{n \in N_s}(T_{s^\frown n} \cap b_s)$. By (P2) in Proposition 19 again, we get that there exists $n_0 \in N_s$ with $T_{s^\frown n_0} \cap b_s$ infinite. Thus $s^\frown n_0 \in S$. It follows that S is a finitely splitting, infinite tree. By König's Lemma, we see that $S \notin$ WF. But then $T \notin$ WF, a contradiction. The claim is proved. ◊

Fix $T_0 \in$ WF infinite. The map $\Psi : \mathrm{Tr} \to [\mathbb{N}]$ defined by $\Psi(T) = \{\psi(t) : t \in T \cup T_0\}$ is clearly continuous. If $T \in$ WF, then $T \cup T_0 \in$ WF. By the above claim, we see that $T \cup T_0 \in \mathcal{J}^{\perp}$, and so, $\Psi(T) \in \mathcal{I}^{\perp} = \mathcal{L}_{\mathbf{f},f} \subseteq \mathcal{L}_{\mathbf{f}}$. On the other hand, if $T \notin$ WF, then by (P1) in Proposition 19 and the above claim, we get that there exist $L \in \mathcal{L}_{\mathbf{f}} \setminus \mathcal{L}_{\mathbf{f},f}$ and $M \in \mathcal{L}_{\mathbf{f},f}$ with $L \cup M \subseteq \Psi(T)$. Hence $\Psi(T) \notin \mathcal{L}_{\mathbf{f}}$. It follows that WF is Wadge reducible to $\mathcal{L}_{\mathbf{f}}$ and the proof is completed. $\square$

The last part of this section is devoted to the construction of canonical $\mathbf{\Pi}_1^1$-ranks on the sets $\mathcal{L}_{\mathbf{f}}$ and $\mathcal{L}_{\mathbf{f},f}$. So let X be a Polish space, $\mathbf{f} = (f_n)_n$ a sequence relatively compact in $\mathcal{B}_1(X)$ and f an accumulation point of $(f_n)_n$. As the sets $\mathcal{L}_{\mathbf{f}}$ and $\mathcal{L}_{\mathbf{f},f}$ do not depend on the topology on X, we may assume, by enlarging the topology of X if necessary, that the functions $(f_n)_n$ and the function f are continuous (see [9]). We need to deal with decreasing sequences of closed subsets of X à la Cantor. We fix a countable dense subset D of X. Let $(B_n)_n$ be an enumeration of all closed balls in X (taken with respect to some compatible complete metric) with centers in D and rational radii. If X happens to be locally compact, we will assume that every ball B_n is compact. We will say that a finite sequence $w = (l_0, \ldots, l_k) \in \mathbb{N}^{<\mathbb{N}}$ is acceptable if

(i) $B_{l_0} \supseteq B_{l_1} \supseteq \cdots \supseteq B_{l_k}$, and
(ii) $\mathrm{diam}(B_{l_i}) \leq \frac{1}{i+1}$ for all $i = 0, \ldots, k$.

By convention $(\varnothing)$ is acceptable. Notice that if $w_1 \sqsubset w_2$ and w_2 is acceptable, then w_1 is acceptable too. We will also need the following notations.

Notation 1. By Fin we denote the set of all finite subsets of $\mathbb{N}$. For every $F, G \in$ Fin we write $F < G$ if $\max\{n : n \in F\} < \min\{n : n \in G\}$. For every $L \in [\mathbb{N}]$, by $\mathrm{Fin}(L)$ we denote the set of all finite subsets of L. Finally, by $[\mathrm{Fin}(L)]^{<\mathbb{N}}$ we denote the set of all finite sequences $t = (F_0, \ldots, F_k) \in (\mathrm{Fin}(L))^{<\mathbb{N}}$ which are increasing, i.e. $F_0 < F_1 < \cdots < F_k$.

The construction of the $\mathbf{\Pi}_1^1$-ranks on $\mathcal{L}_{\mathbf{f}}$ and $\mathcal{L}_{\mathbf{f},f}$ will be done by finding appropriate reductions of the sets in question to well-founded trees. In particular, we shall construct the following.

(C1) A continuous map $[\mathbb{N}] \ni L \mapsto T_L \in \mathrm{Tr}(\mathbb{N} \times \mathrm{Fin} \times \mathbb{N})$, and
(C2) a continuous map $[\mathbb{N}] \ni L \mapsto S_L \in \mathrm{Tr}(\mathbb{N} \times \mathbb{N})$

such that

(C3) $L \in \mathcal{L}_{\mathbf{f}}$ if and only if $T_L \in \mathrm{WF}(\mathbb{N} \times \mathrm{Fin} \times \mathbb{N})$, and
(C4) $L \in \mathcal{L}_{\mathbf{f},f}$ if and only if $S_L \in \mathrm{WF}(\mathbb{N} \times \mathbb{N})$.

It follows, by (C1)–(C4) above, that the maps $L \to o(T_L)$ and $L \to o(S_L)$ are $\mathbf{\Pi}_1^1$-ranks on $\mathcal{L}_{\mathbf{f}}$ and $\mathcal{L}_{\mathbf{f},f}$.

1. *The reduction of $\mathcal{L}_{\mathbf{f}}$ to* $\mathrm{WF}(\mathbb{N} \times \mathrm{Fin} \times \mathbb{N})$. Let $d \in \mathbb{N}$. For every $L \in [\mathbb{N}]$ we associate a tree $T_L^d \in \mathrm{Tr}(\mathbb{N} \times \mathrm{Fin} \times \mathbb{N})$ as follows. We let

$$T_L^d = \left\{ (s,t,w) \ : \ \exists k \text{ with } |s| = |t| = |w| = k, \right.$$
$$s = (n_0 < \cdots < n_{k-1}) \in [L]^{<\mathbb{N}},$$
$$t = (F_0 < \cdots < F_{k-1}) \in [\mathrm{Fin}(L)]^{<\mathbb{N}},$$
$$w = (l_0, \ldots, l_{k-1}) \in \mathbb{N}^{<\mathbb{N}} \text{ is acceptable and}$$
$$\forall 0 \leq i \leq k-1 \ \forall z \in B_{l_i} \text{ there exists } m_i \in F_i \text{ with}$$
$$\left. |f_{n_i}(z) - f_{m_i}(z)| > \frac{1}{d+1} \right\}.$$

Next we glue the sequence of trees $(T_L^d)_{d \in \mathbb{N}}$ in a natural way and we build a tree $T_L \in \mathrm{Tr}(\mathbb{N} \times \mathrm{Fin} \times \mathbb{N})$ defined by the rule

$$(s,t,w) \in T_L \Leftrightarrow \exists d \ \exists (s',t',w') \text{ such that } (s',t',w') \in T_L^d \text{ and}$$
$$s = d^\frown s', t = \{d\}^\frown t', w = d^\frown w'.$$

It is clear that the map $[\mathbb{N}] \ni L \mapsto T_L \in \mathrm{Tr}(\mathbb{N} \times \mathrm{Fin} \times \mathbb{N})$ is continuous. Moreover the following holds.

Lemma 20. *Let $L \in [\mathbb{N}]$. Then $L \in \mathcal{L}_{\mathbf{f}}$ if and only if $T_L \in \mathrm{WF}(\mathbb{N} \times \mathrm{Fin} \times \mathbb{N})$.*

Proof. First, notice that if $L \notin \mathcal{L}_{\mathbf{f}}$, then there exist $L_1, L_2 \in [L]$ such that $L_1 \cap L_2 = \varnothing$, $L_1, L_2 \in \mathcal{L}_{\mathbf{f}}$ and $f_{L_1} \neq f_{L_2}$, where as usual f_{L_1} and f_{L_2} are the pointwise limits of the sequences $(f_n)_{n \in L_1}$ and $(f_n)_{n \in L_2}$ respectively. Pick $x \in X$ and $d \in \mathbb{N}$ such that $|f_{L_1}(x) - f_{L_2}(x)| > \frac{1}{d+1}$. Clearly we may assume that $|f_n(x) - f_m(x)| > \frac{1}{d+1}$ for every $n \in L_1$ and every $m \in L_2$. Let $L_1 = \{n_0 < n_1 < \cdots\}$ and $L_2 = \{m_0 < m_1 < \cdots\}$ be the increasing enumerations of L_1 and L_2. Using the continuity of the functions $(f_n)_n$, we find $w = (l_0, \ldots, l_k, \ldots) \in \mathbb{N}^{\mathbb{N}}$ such that $w|k$ is acceptable for all $k \in \mathbb{N}$, $\bigcap_k B_{l_k} = \{x\}$ and $|f_{n_k}(z) - f_{m_k}(z)| > \frac{1}{d+1}$ for all $k \in \mathbb{N}$ and $z \in B_{l_k}$. Then $\big((n_0, \ldots, n_k), (\{m_0\}, \ldots, \{m_k\}), w|k\big) \in T_L^d$ for all $k \in \mathbb{N}$, which shows that $T_L \notin \mathrm{WF}(\mathbb{N} \times \mathrm{Fin} \times \mathbb{N})$.

Conversely assume that T_L is not well-founded. There exists $d \in \mathbb{N}$ such that T_L^d is not well-founded too. Let $\big((s_k, t_k, w_k)\big)_k$ be an infinite branch of T_L^d. Let $N = \bigcup_k s_k = \{n_0 < \cdots < n_k < \cdots\} \in [L]$, $\mathcal{F} = \bigcup_k t_k = (F_0 < \cdots < F_k < \cdots) \in \mathrm{Fin}(L)^{\mathbb{N}}$ and $w = \bigcup_k w_k = (l_0, \ldots, l_k, \ldots) \in \mathbb{N}^{\mathbb{N}}$. By the definition of T_L^d, we get that $\bigcap_k B_{l_k} = \{x\} \in X$ and that for every $k \in \mathbb{N}$ there exists $m_k \in F_k \subseteq L$ with $|f_{n_k}(x) - f_{m_k}(x)| > \frac{1}{d+1}$. As $F_i < F_j$ for all $i < j$, we see that $m_i < m_j$ if $i < j$. It follows that $M = \{m_0 < \cdots < m_k < \cdots\} \in [L]$. Thus, the sequence $\big(f_n(x)\big)_{n \in L}$ is not Cauchy and so $L \notin \mathcal{L}_{\mathbf{f}}$, as desired. $\quad\square$

By Lemma 20, the reduction of $\mathcal{L}_{\mathbf{f}}$ to $\mathrm{WF}(\mathbb{N} \times \mathrm{Fin} \times \mathbb{N})$ is constructed. Notice that for every $L \in \mathcal{L}_{\mathbf{f}}$ and every $d_1 \leq d_2$ we have $o(T_L^{d_1}) \leq o(T_L^{d_2})$ and moreover $o(T_L) = \sup\{o(T_L^d) : d \in \mathbb{N}\} + 1$.

Remark 7. We should point out that the reason why in the definition of T_L^d the node t is a finite sequence of finite sets rather than natural numbers, is to get the estimate in Proposition 22 below. Having natural numbers instead of finite sets would also lead to a canonical rank.

2. *The reduction of $\mathcal{L}_{\mathbf{f}, f}$ to $\mathrm{WF}(\mathbb{N} \times \mathbb{N})$.* The reduction is similar to that of the previous step, and so, we shall indicate only the necessary changes. Let $d \in \mathbb{N}$. As before, for every $L \in [\mathbb{N}]$ we associate a tree $S_L^d \in \mathrm{Tr}(\mathbb{N} \times \mathbb{N})$ as follows. We let

$$S_L^d = \Bigg\{ (s, w) \; : \; \exists k \in \mathbb{N} \text{ with } |s| = |w| = k,$$
$$s = (n_0 < \cdots < n_{k-1}) \in [L]^{<\mathbb{N}},$$
$$w = (l_0, \ldots, l_{k-1}) \in \mathbb{N}^{<\mathbb{N}} \text{ is acceptable and}$$
$$\forall 0 \leq i \leq k - 1 \; \forall z \in B_{l_i} \text{ we have } |f_{n_i}(z) - f(z)| > \frac{1}{d+1} \Bigg\}.$$

Next we glue the sequence of trees $(S_L^d)_{d \in \mathbb{N}}$ as we did with the sequence $(T_L^d)_{d \in \mathbb{N}}$ and we build a tree $S_L \in \mathrm{Tr}(\mathbb{N} \times \mathbb{N})$ defined by the rule

$$(s, w) \in S_L \; \Leftrightarrow \; \exists d \; \exists (s', w') \text{ such that } (s', w') \in T_L^d \text{ and}$$
$$s = d^\frown s', w = d^\frown w'.$$

Again it is easy to check that the map $[\mathbb{N}] \ni L \mapsto S_L \in \mathrm{Tr}(\mathbb{N} \times \mathbb{N})$ is continuous. Moreover we have the following analogue of Lemma 20. The proof is identical and is left to the reader.

Lemma 21. *Let $L \in [\mathbb{N}]$. Then $L \in \mathcal{L}_{\mathbf{f}, f}$ if and only if $S_L \in \mathrm{WF}(\mathbb{N} \times \mathbb{N})$.*

This gives us the reduction of $\mathcal{L}_{\mathbf{f}, f}$ to $\mathrm{WF}(\mathbb{N} \times \mathbb{N})$. As before we have $o(S_L) = \sup\{o(S_L^d) : d \in \mathbb{N}\} + 1$ for every $L \in \mathcal{L}_{\mathbf{f}, f}$.

We proceed now to discuss the question whether for a given $L \in \mathcal{L}_{\mathbf{f}, f}$ we can bound the order of the tree S_L by the order of T_L. The following example shows that this is not in general possible.

Example 2. Let $A(2^{\mathbb{N}}) = \{\delta_\sigma \; : \; \sigma \in 2^{\mathbb{N}}\} \cup \{0\}$ be the one point compactification of $2^{\mathbb{N}}$. This is not a separable Rosenthal compact, but it can be supplemented to one in a standard way (see [18,13,25]). Specifically, let $(s_n)_n$ be the enumeration of the Cantor tree $2^{<\mathbb{N}}$ as in Example 1. For every $n \in \mathbb{N}$, let $f_n = \chi_{V_{s_n}}$, where $V_{s_n} = \{\sigma \in 2^{\mathbb{N}} : s_n \sqsubset \sigma\}$. Then $A(2^{\mathbb{N}}) \cup \{f_n\}_n$ is a separable Rosenthal compact. Now let A be a $\mathbf{\Sigma}_1^1$ non-Borel subset of $2^{\mathbb{N}}$. Following [19]

(see also [13]), let $\mathcal{K}_A$ be the separable Rosenthal compact obtained by restricting every function in $A(2^{\mathbb{N}}) \cup \{f_n\}_n$ on A. The sequence $\mathbf{f}_A = (f_n|_A)_n$ is a countable dense subset of $\mathcal{K}_A$ consisting of continuous functions and $0 \in \mathcal{K}_A$ is a non-G_δ point (and obviously continuous). Consider the sets

$$\mathcal{L}_{\mathbf{f}}^A = \{L \in [\mathbb{N}] : (f_n|_A)_{n \in L} \text{ is pointwise convergent on } A\}$$

and

$$\mathcal{L}_{\mathbf{f},0}^A = \{L \in [\mathbb{N}] : (f_n|_A)_{n \in L} \text{ is pointwise convergent to } 0 \text{ on } A\}.$$

Let ϕ be a $\mathbf{\Pi}_1^1$-rank on $\mathcal{L}_{\mathbf{f}}^A$ and ψ a $\mathbf{\Pi}_1^1$-rank on $\mathcal{L}_{\mathbf{f},0}^A$. We claim that there does not exist a map $\Phi : \omega_1 \to \omega_1$ such that $\psi(L) \leq \Phi(\phi(L))$ for all $L \in \mathcal{L}_{\mathbf{f},0}^A$. Assume not. Let

$$R = \{L \in [\mathbb{N}] : \exists \sigma \in 2^{\mathbb{N}} \text{ with } s_n \sqsubset \sigma \ \forall n \in L\}.$$

Then R is a closed subset of $\mathcal{L}_{\mathbf{f}}^A$. For every $L \in R$, let $\sigma_L = \bigcup_{n \in L} s_n \in 2^{\mathbb{N}}$. The map $R \ni L \mapsto \sigma_L \in 2^{\mathbb{N}}$ is clearly continuous. Observe that for every $L \in R$ we have that $L \in \mathcal{L}_{\mathbf{f},0}^A$ if and only if $\sigma_L \notin A$. As R is a Borel subset of $\mathcal{L}_{\mathbf{f}}^A$, by boundedness we get that $\sup\{\phi(L) : L \in R\} = \xi < \omega_1$. Let $\zeta = \sup\{\Phi(\lambda) : \lambda \leq \xi\}$. The set $B = R \cap \{L \in \mathcal{L}_{\mathbf{f},0}^A : \psi(L) \leq \zeta\}$ is Borel and $B = R \cap \mathcal{L}_{\mathbf{f},0}^A$. Hence, the set $\Sigma_B = \{\sigma_L : L \in B\}$ is an analytic subset of $2^{\mathbb{N}} \setminus A$. As $2^{\mathbb{N}} \setminus A$ is $\mathbf{\Pi}_1^1$-true, there exists $\sigma_0 \in 2^{\mathbb{N}} \setminus A$ with $\sigma_0 \notin \Sigma_B$. Pick $L \in R$ with $\sigma_L = \sigma_0$. Then $L \in B$ yet $\sigma_L \notin \Sigma_B$, a contradiction.

Although we cannot, in general, bound the order of the tree S_L by that of T_L, the following proposition shows that this is possible for an important special case.

Proposition 22. *Let X be locally compact, $\mathcal{K}$ a separable Rosenthal compact on X, $\mathbf{f} = (f_n)_n$ a dense sequence in $\mathcal{K}$ consisting of continuous functions and $f \in \mathcal{K}$. If f is continuous, then $o(S_L) \leq o(T_L)$ for all $L \in \mathcal{L}_{\mathbf{f},f}$.*

In particular, there exists a $\mathbf{\Pi}_1^1$-rank ϕ on $\mathcal{L}_{\mathbf{f}}$ and a $\mathbf{\Pi}_1^1$-rank ψ on $\mathcal{L}_{\mathbf{f},f}$ with $\psi(L) \leq \phi(L)$ for all $L \in \mathcal{L}_{\mathbf{f},f}$.

Proof. We will show that for every $d \in \mathbb{N}$ we have $o(S_L^d) \leq o(T_L^d)$ for every $L \in \mathcal{L}_{\mathbf{f},f}$. This clearly completes the proof. So fix $d \in \mathbb{N}$ and $L \in \mathcal{L}_{\mathbf{f},f}$. We shall construct a monotone map $\Phi : S_L^d \to [\text{Fin}(L)]^{<\mathbb{N}}$ such that for every $(s, w) \in S_L^d$ the following hold.

(i) $|(s, w)| = |\Phi((s, w))|$.
(ii) If $s = (n_0 < \cdots < n_k)$, $w = (l_0, \ldots, l_k)$ and $\Phi((s, w)) = (F_0 < \cdots < F_k)$, then for every $i \in \{0, \ldots, k\}$ and every $z \in B_{l_i}$ there exists $m_i \in F_i$ with $|f_{n_i}(z) - f_{m_i}(z)| > \frac{1}{d+1}$.

Assuming that Φ has been constructed, let $M : S_L^d \to T_L^d$ be defined by

$$M((s, w)) = (s, \Phi((s, w)), w).$$

Then it is easy to see that M is a well-defined monotone map, and so, $o(S_L^d) \leq o(T_L^d)$ as desired.

We proceed to the construction of Φ. It will be constructed by recursion on the length of (s, w). We set $\Phi((\varnothing, \varnothing)) = (\varnothing)$. Let $k \in \mathbb{N}$ and assume that $\Phi((s, w))$ has been defined for every $(s, w) \in S_L^d$ with $|(s, w)| \leq k$. Let $(s', w') = (s ^\frown n_k, w ^\frown l_k) \in S_L^d$ with $|s'| = |w'| = k+1$. By the definition of S_L^d, we have that $|f_{n_k}(z) - f(z)| > \frac{1}{d}$ for every $z \in B_{l_k}$. Put $p = \max\{n : n \in F \text{ and } F \in \Phi((s, w))\} \in \mathbb{N}$. For every $z \in B_{l_k}$ we may select $m_z \in L$ with $m_z > p$ and such that $|f_{n_k}(z) - f_{m_z}(z)| > \frac{1}{d+1}$. As the functions $(f_n)_n$ are continuous, we pick an open neighborhood U_z of z such that $|f_{n_k}(y) - f_{m_z}(y)| > \frac{1}{d+1}$ for all $y \in U_z$. By the compactness of B_{l_k}, there exists $z_0, \ldots, z_{j_k} \in B_{l_k}$ such that $U_{z_0} \cup \cdots \cup U_{z_{j_k}} \supseteq B_{l_k}$. Let $F_k = \{m_{z_i} : i = 0, \ldots, j_k\} \in \text{Fin}(L)$ and notice that $F \leq p < F_k$ for every $F \in \Phi((s, w))$. We set

$$\Phi((s', w')) = \Phi((s, w)) ^\frown F_k \in [\text{Fin}(L)]^{<\mathbb{N}}.$$

It is easy to check that $\Phi((s', w'))$ satisfies (i) and (ii) above. The proof is completed. $\quad\square$

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