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Uniqueness of meromorphic functions sharing a small function and its applications

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ABSTRACT

In this paper, we shall study the uniqueness problems of meromorphic functions sharing a small function. Our results improve or generalize many previous results on value sharing of meromorphic functions.

1. Introduction and main results

Let $\mathbb{C}$ denote the complex plane and $f(z)$ be a non-constant meromorphic function on $\mathbb{C}$. We assume the reader is familiar with the standard notion used in the Nevanlinna value distribution theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$, and $S(r,f)$ denotes any quantity that satisfies the condition $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside of a possible exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$, provided that $T(r,a) = S(r,f)$.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let a be a small function with respect to f and g. We say that $f(z), g(z)$ share $a(z)$ CM (counting multiplicities) if $f(z) - a(z)$, $g(z) - a(z)$ have the same zeros with the same multiplicities and we say that $f(z), g(z)$ share $a(z)$ IM (ignoring multiplicities) if we do not consider the multiplicities. $N_k(r,f)$ denotes the truncated counting function bounded by k. Moreover, $GCD(n_1, n_2, \ldots, n_k)$ denotes the greatest common divisor of positive integers $n_1, n_2, \ldots, n_k$.

We say that a finite value z_0 is called a fixed point of f if $f(z_0) = z_0$ or z_0 is a zero of $f(z) - z$.

For the sake of simplicity, we also use the notion $m^* := \chi_\mu m$, where

$$\chi_\mu = \begin{cases} 0, & \mu = 0, \\ 1, & \mu \neq 0. \end{cases}$$

The following theorem in the value distribution theory is well-known [1,2].

Theorem A. *Let $f(z)$ be a transcendental meromorphic function, $n \geq 1$ a positive integer. Then $f^n f' = 1$ has infinitely many solutions.*

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Fang and Hua [3], Yang and Hua [4] obtained a unicity theorem respectively corresponding to Theorem A.

Theorem B. *Let f and g be two non-constant entire (meromorphic) functions and $n \geq 6$ ($n \geq 11$) be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share 1 CM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

Note that $f^n(z)f'(z) = \frac{1}{n+1}(f^{n+1}(z))'$, Fang [5] considered the case of the kth derivative and proved

Theorem C. *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k + 4$. If $(f^n(z))^{(k)}$ and $(g^n(z))^{(k)}$ share 1 CM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.*

Theorem D. *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k + 8$. If $(f^n(z)(f(z) - 1))^{(k)}$ and $(g^n(z)(g(z) - 1))^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.*

Zhang and Lin [6,7] generalized Theorems C and D as follows.

Theorem E. *Let f and g be two non-constant entire functions, and let n, m and k be three positive integers with $n > 2k+m^*+4$, and λ, μ be constants such that $|\lambda| + |\mu| \neq 0$. If $(f^n(z)(\mu f^m(z) + \lambda))^{(k)}$ and $(g^n(z)(\mu g^m(z) + \lambda))^{(k)}$ share 1 CM, then*

(i) *when $\lambda\mu \neq 0$, $f^d \equiv g^d$, $d = GCD(m, n)$; especially, when $d = 1$, $f \equiv g$.*
(ii) *when $\lambda\mu = 0$, either $f(z) \equiv tg(z)$, where t is a constant satisfying $t^{n+m^*} = 1$, or $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k\lambda^2(c_1c_2)^{n+m^*}[(n + m^*)c]^{2k} = 1$ or $(-1)^k\mu^2(c_1c_2)^{n+m^*}[(n + m^*)c]^{2k} = 1$.*

Theorem F. *Let f and g be two non-constant entire functions, and let n, m, k be three positive integers with $n > 2k + m + 4$. If $(f^n(z)(f(z) - 1)^m)^{(k)}$ and $(g^n(z)(g(z) - 1)^m)^{(k)}$ share 1 CM, then either $f(z) \equiv g(z)$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m - \omega_2^n(\omega_2 - 1)^m$.*

Moreover, Zhang et al. [8] considered some more general differential polynomials. They obtained

Theorem G. *Let f and g be two nonconstant entire functions. Let n, k, and m be three positive integers with $n \geq 3m + 2k + 5$ and let $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$ or $P(w) \equiv c_0$, where $a_0 \neq 0, a_1, \ldots, a_{m-1}, a_m \neq 0$, $c_0 \neq 0$ are complex constants. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share 1 CM, then*

(i) *when $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$, either $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD(n + m, \ldots, n + m - i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0)$;*
(ii) *when $P(w) \equiv c_0$, either $f(z) = c_1/\sqrt[n]{c_0}e^{cz}$, $g(z) = c_2/\sqrt[n]{c_0}e^{-cz}$, where c_1, c_2, and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem H. *Let f and g be two nonconstant meromorphic functions. Let n and m be two positive integers with $n > \max\{m + 10, 3m + 3\}$, and let $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$, where $a_0 \neq 0, a_1, \ldots, a_{m-1}, a_m \neq 0$ are complex constants. If $f^nP(f)f'$ and $g^nP(g)g'$ share 1 CM, then either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = GCD(n + m + 1, \ldots, n + m + 1 - i, \ldots, n + 1)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^{n+1}\left(\frac{a_m\omega_1^m}{n+m+1} + \frac{a_{m-1}\omega_1^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1}\right) - \omega_2^n\left(\frac{a_m\omega_2^m}{n+m+1} + \frac{a_{m-1}\omega_2^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1}\right)$.*

Related to Theorem A, Fang [9] proved that a meromorphic function f^nf' has infinitely many fixed points when f is transcendental and n is a positive integer. Then Fang and Qiu [10] obtained the following uniqueness theorem.

Theorem I. *Let f and g be two non-constant entire functions and $n \geq 6$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share z CM, then either $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

Lin and Yi [11] obtained:

Theorem J. *Let f and g be two non-constant entire functions, and let $n \geq 7$ be a positive integer. If $f^n(f - 1)f'$ and $g^n(g - 1)g'$ share z CM, then $f \equiv g$.*

Zhang [12] extended Theorems I and J as follows.

Theorem K. *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k + 4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, then either*

(1) *$k = 1$, $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^n(nc)^2 = -1$, or*
(2) *$f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.*

Theorem L. *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k+6$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share z CM, then $f \equiv g$.*

Regarding Theorems K and L, Xu et al. [13] considered the case of meromorphic functions. They obtained

Theorem M. *Let f and g be two non-constant meromorphic functions, and let n, k be two positive integers with $n > 3k+10$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^n(nc)^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.*

Theorem N. *Let f and g be two non-constant meromorphic functions satisfying $\Theta(\infty, f) > 2/n$, and let n, k be two positive integers with $n > 3k+12$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share z CM, f and g share ∞ IM, then $f \equiv g$.*

For more results in such directions, see [14–17]. The purpose of this paper is to study the uniqueness theorem for general differential polynomials $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ sharing a small function and its applications. Now we state our results.

Theorem 1.1. *Let f and g be two non-constant meromorphic functions, and $a(z)(\not\equiv 0, \infty)$ be a small function with respect to f. Let n, k, and m be three positive integers with $n > 3k+m+8$ and $P(w)$ be defined as in Theorem G. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share a CM, then*

(I) *when $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$, one of the following three cases holds:*
  (I1) *$f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$,*
  (I2) *f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0)$,*
  (I3) *$[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} = a^2$;*

(II) *when $P(w) \equiv c_0$, one of the following two cases holds:*
  (II1) *$f \equiv tg$ for a constant t such that $t^n = 1$,*
  (II2) *$c_0^2[f^n]^{(k)}[g^n]^{(k)} = a^2$.*

Remark 1.1. In Theorem 1.1, one cannot easily get the representation of $f(z)$ and $g(z)$ like in Theorems B and C from (I3) or (II2). Wang and Gao [18, Remark 3.1, Examples 3.2–3.4] gave some examples at the end of their paper to discuss the problem.

Now we give some applications of Theorem 1.1. The following theorem improves or generalizes Theorems D, F, L and N.

Theorem 1.2. *Let f and g be two nonconstant meromorphic functions, $a(z)(\not\equiv 0, \infty)$ be a small function with respect to f with finitely many zeros and poles. Let n, k and m be three positive integers with $n > 3k+m+7$, $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$, where $a_0 \neq 0, a_1, \ldots, a_{m-1}, a_m \neq 0$, are complex constants. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share a CM, f and g share ∞ IM, then one of the following two cases holds:*

(1) *$f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$;*
(2) *f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0)$.*

Many authors have considered uniqueness theorems concerning fixed points, such as Theorems J–N and I. Here we do further consideration and replace z by a general polynomial $p(z)$ with $\deg(p) \leq 5$, we get

Theorem 1.3. *Let f and g be two transcendental meromorphic functions, let $p(z)$ be a nonzero polynomial with $\deg(p) = l \leq 5$, n, k and m be three positive integers with $n > 3k+m+7$. Let $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$ be a nonzero polynomial. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share p CM, f and g share ∞ IM, then one of the following three cases holds:*

(1) *$f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$;*
(2) *f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0)$;*
(3) *$P(z)$ is reduced to a nonzero monomial, namely, $P(z) = a_iz^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$; if $p(z)$ is not a constant, then $f = c_1e^{cQ(z)}$, $g = c_2e^{-cQ(z)}$, where $Q(z) = \int_0^z p(z)\,dz$, c_1, c_2 and c are constants such that $a_i^2(c_1c_2)^{n+i}[(n+i)c]^2 = -1$,*

if $p(z)$ is a nonzero constant b, then $f = c_3e^{cz}$, $g = c_4e^{-cz}$, where c_3, c_4 and c are constants such that $(-1)^ka_i^2(c_3c_4)^{n+i}[(n+i)c]^{2k} = b^2$.

Note that $n > 2k+m+4 \geq k+6$. It is easy to obtain from Theorem 1.3 that

Corollary 1.4. *Let f and g be two transcendental entire functions, $p(z)$ be a nonzero polynomial with* $\deg p = l \le 3$, n, k, *and m be three positive integers with $n > 2k + m + 4$, let $P(w)$ be defined as in Theorem 1.3. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share p CM. Then the conclusions of Theorem 1.3 hold.*

Remark 1.2. From the proof of Theorem 1.3, we can see that the computation will be very complicated when $\deg(p)$ becomes large, so we are not sure whether Theorem 1.3 holds for the general polynomial $p(z)$. Nevertheless, Theorem 1.3 and Corollary 1.4 improve or generalize the previous results such as Theorems B–F, K–N, I and G.

The following theorem generalizes Theorems H and J.

Theorem 1.5. *Let f and g be two non-constant meromorphic functions and $a(z)(\not\equiv 0, \infty)$ be a small function of f. Let n and m be two positive integers with $n > \max\{m + 10, 3m + 3\}$, and let $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$, where $a_0 \ne 0, a_1, \ldots, a_{m-1}, a_m \ne 0$ are complex constants. If $f^nP(f)f'$ and $g^nP(g)g'$ share a CM, then either $f \equiv tg$ for a constant t such that $t^d = 1$ where $d = GCD(n + m + 1, \ldots, n + m + 1 - i, \ldots, n + 1)$, $a_{m-i} \ne 0$ for some $i \in \{0, 1, \ldots, m\}$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^{n+1}\left(\frac{a_m\omega_1^m}{n+m+1} + \frac{a_{m-1}\omega_1^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1}\right) - \omega_2^{n+1}\left(\frac{a_m\omega_2^m}{n+m+1} + \frac{a_{m-1}\omega_2^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1}\right)$.*

2. Preliminary lemmas and a main proposition

Lemma 2.1 ([19]). *Let $f(z)$ be a non-constant meromorphic function and let $a_0(z), a_1(z), \ldots, a_n(z)(\not\equiv 0)$ be small functions with respect to f. Then*

$$T(r, a_nf^n + a_{n-1}f^{n-1} + \cdots + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2 ([20–22]). *Let $f(z)$ be a non-constant meromorphic function. Let k be a positive integer, and let c be a nonzero finite complex number. Then*

$$\begin{aligned} T(r,f) &\le \overline{N}(r,f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - c}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &\le \overline{N}(r,f) + N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f), \end{aligned}$$

where $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \ne 0$.

Lemma 2.3 ([23]). *Let $f(z)$ be a non-constant meromorphic function and s, k be two positive integers. Then*

$$N_s\left(r, \frac{1}{f^{(k)}}\right) \le T(r, f^{(k)}) - T(r,f) + N_{s+k}\left(r, \frac{1}{f}\right) + S(r,f),$$
$$N_s\left(r, \frac{1}{f^{(k)}}\right) \le k\overline{N}(r,f) + N_{s+k}\left(r, \frac{1}{f}\right) + S(r,f).$$

Lemma 2.4 ([21]). *Let $f(z)$ be a non-constant meromorphic function, and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \le N\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f).$$

Lemma 2.5 ([4]). *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and n, k be two positive integers, a be a finite nonzero constant. If f and g share a CM, then one of the following cases holds:*

(i) $T(r,f) \le N_2(r, 1/f) + N_2(r, 1/g) + N_2(r,f) + N_2(r,g) + S(r,f) + S(r,g)$. *The same inequality holding for $T(r,g)$;*
(ii) $fg \equiv a^2$;
(iii) $f \equiv g$.

By using a similar method to Yang and Hua [4], we can prove the following lemma.

Lemma 2.6. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, let n, k be two positive integers and a be a finite nonzero constant. If f and g share a CM and ∞ IM, then one of the following cases holds:*

(i) $T(r,f) \le N_2(r, 1/f) + N_2(r, 1/g) + 3\overline{N}(r,f) + S(r,f) + S(r,g)$. *The same inequality holding for $T(r,g)$;*
(ii) $fg \equiv a^2$;
(iii) $f \equiv g$.

Lemma 2.7. *Let f, g be non-constant meromorphic functions, let n, k be two positive integers with $n > k + 2$, and let $P(w)$ be defined as in Theorem G. Let $a(z)(\not\equiv 0, \infty)$ be a small function of f. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share a IM, then $T(r, f) = O(T(r, g))$, $T(r, g) = O(T(r, f))$.*

Proof. Let $F = f^nP(f)$. By the second fundamental theorem for small functions (see [24]), we have

$$T(r, F^{(k)}) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{F^{(k)}}\right) + \overline{N}\left(r, \frac{1}{F^{(k)} - a}\right) + (\varepsilon + o(1))T(r, F) \tag{2.1}$$

for all $\varepsilon > 0$.

By (2.1), Lemmas 2.1 and 2.3 with $s = 1$ applying to F, we have

$$\begin{aligned}(n + m)T(r, f) &\leq N_{k+1}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F^{(k)} - a}\right) + \overline{N}(r, f) + (\varepsilon + o(1))T(r, f)\\ &\leq (k + 1)\overline{N}\left(r, \frac{1}{f}\right) + N_{k+1}\left(r, \frac{1}{P(f)}\right) + \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{[f^nP(f)]^{(k)} - a}\right) + (\varepsilon + o(1))T(r, f)\\ &\leq (k + 2 + m)T(r, f) + \overline{N}\left(r, \frac{1}{[g^nP(g)]^{(k)} - a}\right) + (\varepsilon + o(1))T(r, f),\end{aligned}$$

namely

$$\begin{aligned}(n - k - 2)T(r, f) &\leq \overline{N}\left(r, \frac{1}{[g^nP(g)]^{(k)} - a}\right) + (\varepsilon + o(1))T(r, f)\\ &\leq (n + m)(k + 1)T(r, g) + (\varepsilon + o(1))T(r, f).\end{aligned}$$

Since $n > k + 2$, take $\varepsilon < 1$ and we have $T(r, f) = O(T(r, g))$. Similarly we have $T(r, g) = O(T(r, f))$. This completes the proof of Lemma 2.7. □

By the similar arguments to the proof of Lemma 2.7, we get the following proposition.

Proposition 2.1. *Let f be a transcendental meromorphic function. Let n, k be two positive integers with $n > k+2$, and let $P(w)$ be defined as in Theorem G, $a(z)(\not\equiv 0, \infty)$ be a small function with respect to f. Then $[f^nP(f)]^{(k)} - a(z)$ has infinitely many zeros.*

By the same reason as in Lemma 5 of [13], we obtain the following lemma.

Lemma 2.8. *Let f and g be two non-constant meromorphic functions. Let $P(w)$ be defined as in Theorem G, and $k, m, n > 2k + m + 1$ be three positive integers. If $[f^nP(f)]^{(k)} = [g^nP(g)]^{(k)}$, then $f^nP(f) = g^nP(g)$.*

Lemma 2.9. *Let f, g be non-constant meromorphic functions, let n, k be two positive integers with $n > k + 2$, and let $P(w)$ be defined as in Theorem G. Let $a(z)(\not\equiv 0, \infty)$ be a small function with respect to f with finitely many zeros and poles. If $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} = a^2$, f and g share ∞ IM, then $P(w)$ is reduced to a nonzero monomial, namely, $P(w) = a_iw^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$.*

Proof. If $P(w)$ is not reduced to a nonzero monomial, then, without loss of generality, we assume that $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$, where $a_0 \neq 0, a_1, \ldots, a_{m-1}, a_m \neq 0$ are complex constants.

Under the conditions of Lemma 2.9, by Lemma 2.7, we know that either f and g are both transcendental meromorphic functions or they are both rational functions. Since f and g share ∞ IM, the poles of f and g can only come from the poles of a, whose number is finite. Thus both f and g have only finitely many poles. If z_0 is a zero of f, then z_0 is a zero of a, the number of whose zeros is finite, hence f has finitely many zeros and so does g.

Case 1. If f and g are transcendental meromorphic functions. Let $f = he^{\alpha}$, where α is a non-constant entire function and h is a nonzero rational function. Thus, by induction we have

$$[a_if^{i+n}]^{(k)} = P_i(\alpha', \alpha'', \ldots, \alpha^{(k)}, h, h', \ldots, h^{(k)})e^{(i+n)\alpha}, \tag{2.2}$$

where P_i $(i = 1, 2, \ldots, m)$ are differential polynomials with coefficients which are rational functions in h or its derivatives. Obviously, $P_0 \not\equiv 0, \ldots, P_m \not\equiv 0$, where if $a_i \neq 0$ for some $i \in \{0, 1, \ldots, m - 1\}$, then $P_i \not\equiv 0$.

$T(r, P_i) = S(r, f)$, $N\left(r, \frac{1}{P_me^{m\alpha}+\cdots+P_0}\right) = S(r, f)$. By the second fundamental theorem for small functions (see [24]), we have

$$\begin{aligned}mT(r, f) &= T(r, P_me^{m\alpha} + \cdots + P_1e^{\alpha}) + S(r, f)\\ &\leq \overline{N}\left(r, \frac{1}{P_me^{m\alpha} + \cdots + P_1e^{\alpha}}\right) + \overline{N}\left(r, \frac{1}{P_me^{m\alpha} + \cdots + P_0}\right) + \overline{N}(r, P_me^{m\alpha} + \cdots + P_1e^{\alpha})\\ &\quad + (\varepsilon + o(1))T(r, f)\end{aligned}$$

$$\leq \overline{N}\left(r, \frac{1}{P_m e^{(m-1)\alpha} + \cdots + P_2 e^{\alpha} + P_1}\right) + (\varepsilon + o(1))T(r,f)$$
$$\leq (m-1)T(r,f) + (\varepsilon + o(1))T(r,f),$$

for all $\varepsilon > 0$. Take $\varepsilon < 1$ and we obtain a contradiction.

Case 2. If f and g are rational functions, then a is a nonzero constant, thus f and g have no zeros and no poles, which is impossible since f and g are not constants.

The above two cases imply that $P(w)$ is reduced to a nonzero monomial, namely, $P(w) = a_i w^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$. □

Lemma 2.10 ([20, Theorem 3.10]). *Suppose that f is a non-constant meromorphic function, $k \geq 2$ is an integer. If*

$$N(r,f) + N(r, 1/f) + N(r, 1/f^{(k)}) = S(r, f'/f),$$

then $f = e^{az+b}$, where $a \neq 0$, b are constants.

Lemma 2.11. *Let $p(z)$, $q(z)$, $r(z)$ be three polynomials satisfying*

$$p^2(z) - q^2(z) = r^2(z). \tag{2.3}$$

If $\deg(p) = \deg(r) > 2\deg(q)$, *then* $q(z) \equiv 0$.

Proof. Suppose that $q(z) \not\equiv 0$, then $p^2(z) \not\equiv r^2(z)$, namely, $p(z) + r(z) \not\equiv 0$ and $p(z) - r(z) \not\equiv 0$. Rewrite (2.3) as

$$q^2(z) = p^2(z) - r^2(z) = (p(z) + r(z))(p(z) - r(z)). \tag{2.4}$$

It is easy to obtain from (2.4) that $2\deg(q) = \deg(q^2) \geq \deg(p) > 2\deg(q)$, which is a contradiction. This completes the proof of Lemma 2.11. □

3. Proof of Theorem 1.1

Let $F = [f^n P(f)]^{(k)}$, $G = [g^n P(g)]^{(k)}$, $F^* = f^n P(f)$, $G^* = g^n P(g)$, $F_1 = F/a$, $G_1 = G/a$, then F_1 and G_1 share 1 CM.

(I) $P(w) = a_m w^m + a_{m-1} w^{m-1} + \cdots + a_1 w + a_0$. Since a is a small function with respect to f. By Lemma 2.7, a is a small function with respect to g. Thus by Lemma 2.5, one of the following cases holds:

(i) $T(r, F_1) \leq N_2(r, 1/F_1) + N_2(r, 1/G_1) + N_2(r, F_1) + N_2(r, G_1) + S(r, F_1) + S(r, G_1)$, the same inequality holding for $T(r, G_1)$;
(ii) $FG \equiv a^2$;
(iii) $F \equiv G$.

For Case (i), we have

$$T(r,F) \leq N_2(r, 1/F) + N_2(r, 1/G) + N_2(r, F) + N_2(r, G) + S(r, F) + S(r, G). \tag{3.1}$$

By Lemma 2.3 with $s = 2$, we obtain

$$T(r, F^*) \leq T(r, F) - N_2(r, 1/F) + N_{k+2}(r, 1/F^*) + S(r, F), \tag{3.2}$$

and

$$N_2(r, 1/G) \leq N_{k+2}(r, 1/G^*) + k\overline{N}(r, G) + S(r, G). \tag{3.3}$$

Combining (3.1)–(3.3) gives

$$\begin{aligned} T(r, F^*) &\leq N_{k+2}(r, 1/F^*) + N_{k+2}(r, 1/G^*) + (k+2)\overline{N}(r, g) + 2\overline{N}(r, f) + S(r, f) + S(r, g) \\ &\leq (k+2)\overline{N}(r, 1/f) + N(r, 1/P(f)) + (k+2)\overline{N}(r, 1/g) \\ &\quad + N(r, 1/P(g)) + (k+2)\overline{N}(r, g) + 2\overline{N}(r, f) + S(r, f) + S(r, g) \\ &\leq (2k+m+4)T(r, g) + (k+m+4)T(r, f) + S(r, f) + S(r, g). \end{aligned}$$

It follows from Lemma 2.1 and the above inequality that

$$(n+m)T(r,f) \leq (2k+m+4)T(r,g) + (k+m+4)T(r,f) + S(r,f) + S(r,g). \tag{3.4}$$

Similarly we have

$$(n+m)T(r,g) \leq (2k+m+4)T(r,f) + (k+m+4)T(r,g) + S(r,f) + S(r,g). \tag{3.5}$$

From (3.4) and (3.5) we deduce that

$$(n - 3k - m - 8)(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g), \tag{3.6}$$

which is a contradiction since $n > 3k + m + 8$.

For Case (ii), we have $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} = a^2$.

For Case (iii), we have $[f^nP(f)]^{(k)} \equiv [g^nP(g)]^{(k)}$. By Lemma 2.8, we get $f^nP(f) \equiv g^nP(g)$. Similar to the proof in Theorem G, we can obtain the desired results.

(II) $P(w) \equiv c_0$. The case can be dealt with as in part of the proof of Case (I).

This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.3

By Lemma 2.6, and similar method to the proof of Theorem 1.1, we only need to consider

$$[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} = p^2. \tag{4.1}$$

Note that $n > 3k + m + 7 > k + 2$. By Lemma 2.9, $P(w)$ is reduced to a nonzero monomial, namely, $P(w) = a_iw^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$. Thus we have

$$a_i^2[f^{n+i}]^{(k)}[g^{n+i}]^{(k)} = p^2. \tag{4.2}$$

Let $s = n+i$. Since f and g share ∞ IM, we get that f and g have no poles, thus they are both transcendental entire functions. Since $l \le 5$ and $s \ge n > 3k + m + 7 \ge k + 10 \ge k + 2l$, by computation we deduce that f and g have no zeros. Thus

$$f = e^{\alpha}, \qquad g = e^{\beta}, \tag{4.3}$$

where $\alpha(z)$, $\beta(z)$ are two non-constant entire functions.

We claim that $\alpha + \beta \equiv C$, where C is a constant.

We deduce from (4.2) and (4.3) that either both α and β are transcendental entire functions or both α and β are polynomials. Moreover, we have

$$N(r, 1/(f^s)^{(k)}) \le N(r, 1/p^2(z)) = O(\log r).$$

From this and (4.3), we get

$$N(r, f^s) + N(r, 1/f^s) + N(r, 1/(f^s)^{(k)}) = O(\log r).$$

Let $k \ge 2$. Suppose that α is a transcendental entire function. Note that $S(r, s\alpha') = S\left(r, \frac{(f^s)'}{f^s}\right)$. Then

$$N(r, f^s) + N(r, 1/f^s) + N(r, 1/(f^s)^{(k)}) = S(r, s\alpha') = S\left(r, \frac{(f^s)'}{f^s}\right).$$

We deduce from Lemma 2.10 that α is a polynomial, which is a contradiction.

Thus α is a polynomial and so is β.

We deduce from (4.3) that $(f^s)^{(k)} = A[(\alpha')^k + P_{k-1}(\alpha')]e^{s\alpha}$, $(g^s)^{(k)} = B[(\beta')^k + Q_{k-1}(\beta')]e^{s\beta}$, where A, B are nonzero constants, $P_{k-1}(\alpha')$ and $Q_{k-1}(\beta')$ are differential polynomials in α' and β' of degree at most $k-1$, respectively. Thus we obtain

$$AB[(\alpha')^k + P_{k-1}(\alpha')][(\beta')^k + Q_{k-1}(\beta')]e^{s(\alpha+\beta)} = p^2(z). \tag{4.4}$$

We deduce from (4.4) that $\alpha(z) + \beta(z) \equiv C$ for a constant C.

Let $k = 1$, from $[f^nP(f)]'[g^nP(g)]' = p^2$ we get

$$AB\alpha'\beta'e^{s(\alpha+\beta)} = p^2(z). \tag{4.5}$$

Let $\alpha + \beta = \gamma$. If α and β are transcendental entire functions, obviously γ is not a constant, then (4.5) implies that

$$AB\alpha'(\gamma' - \alpha')e^{s\gamma} = p^2(z). \tag{4.6}$$

Since $T(r, \gamma') = m(r, \gamma') \le m\left(r, \frac{(e^{s\gamma})'}{e^{s\gamma}}\right) + O(1) = S(r, e^{s\gamma})$. Thus (4.6) implies that

$$\begin{aligned} T(r, e^{s\gamma}) &\le T\left(r, \frac{p^2}{\alpha'(\gamma' - \alpha')}\right) + O(1) \\ &\le (2 + o(1))T(r, \alpha') + S(r, e^{s\gamma}), \end{aligned}$$

which implies that

$$T(r, e^{s\gamma}) = O(T(r, \alpha')).$$

Thus $T(r,\gamma') = S(r, e^{s\gamma}) = S(r,\alpha')$. In view of (4.6) and by the second fundamental theorem for small functions (see [24]), we get

$$(1-\varepsilon)T(r,\alpha') \le \overline{N}\left(r,\frac{1}{\alpha'}\right) + \overline{N}\left(r,\frac{1}{\alpha'-\gamma'}\right) + o(T(r,\alpha'))$$
$$\le O(\log r) + o(T(r,\alpha')),$$

for all $\varepsilon > 0$. Take $\varepsilon < 1/2$ and we get that α' is a polynomial, which contradicts that α is a transcendental entire function. Thus α and β are both polynomials and $\alpha(z) + \beta(z) \equiv C$ for a constant C. Hence from (4.4) we get

$$C_1(\alpha')^{2k} = p^2 + \widetilde{P}_{2k-1}(\alpha'), \tag{4.7}$$

where C_1 is a nonzero constant and $\widetilde{P}_{2k-1}$ is a differential polynomial in α' of degree at most $2k-1$.

We shall divide our argument into two cases.

Case 1. $p(z)$ is not a constant. Then α' is a non-constant polynomial. For the sake of simplicity, let $C \equiv 0$, $a_i = 1$, where a_i is defined as in (4.2). If $k \ge 2$, we distinguish into five subcases below.

Subcase 1. $l = 1$. Since α' is not a constant, $\deg(\alpha') \ge 1$, by (4.7) we immediately get a contradiction.

Subcase 2. $l = 2$. Since $k \ge 2$, by (4.7) we get $\deg(\alpha') = 1$ and $k = 2$. Thus α'' is a nonzero constant. From (4.4) we get

$$[(s\alpha')^2 + s\alpha''][(s\beta')^2 + s\beta''] = p^2. \tag{4.8}$$

Note that $\alpha + \beta \equiv 0$. Then $\alpha' + \beta' \equiv 0$ and $\alpha'' + \beta'' \equiv 0$. From (4.8) we obtain

$$[(s\alpha')^2]^2 - (s\alpha'')^2 = p^2. \tag{4.9}$$

By Lemma 2.11, we derive $\alpha'' = 0$ from (4.9), which is a contradiction.

Subcase 3. $l = 3$. Similarly as above, we get $\deg(\alpha') = 1$ and $k = 3$. Thus α'' is a nonzero constant. From (4.4) we get

$$[s^3(\alpha')^3 + 3s^2\alpha'\alpha''][s^3(\beta')^3 + 3s^2\beta'\beta''] = p^2. \tag{4.10}$$

Thus we have

$$(3s^2\alpha'\alpha'')^2 - ((s\alpha')^3)^2 = p^2. \tag{4.11}$$

By Lemma 2.11, we arrive at the same contradiction.

Subcase 4. $l = 4$. Similarly as above, we get either $\deg(\alpha') = 1$ and $k = 4$ or $\deg(\alpha') = 2$ and $k = 2$. If $\deg(\alpha') = 1$ and $k = 4$, then α'' is a nonzero constant. From (4.4) we get

$$[(s\alpha')^4 + 3(s\alpha'')^2]^2 - [6s^3(\alpha')^2\alpha'']^2 = p^2. \tag{4.12}$$

Without loss of generality, suppose that $\alpha' = z$, or else, we only need to do a transformation of $p(z)$. We deduce from (4.12) that

$$(sz)^8 - 30s^6z^4 + 9s^4 = p^2(z), \tag{4.13}$$

which implies $p^2(z) = p^2(-z)$. Thus $p(z) = p(-z)$ or $p(z) = -p(-z)$. Note that $l = 4$. Thus $p(z) = p(-z)$. Suppose that $p(z) = a_4z^4 + a_2z^2 + a_0$, where $a_4 \ne 0$, a_2, a_0 are constants. Comparing with the coefficients at both sides of (4.13), we get $a_2 = 0$, we derive a contradiction by calculation.

If $\deg(\alpha') = 2$ and $k = 2$, then we get (4.9). By Lemma 2.11, we arrive at a contradiction.

Subcase 5. $l = 5$. Similarly as above, we get $\deg(\alpha') = 1$ and $k = 5$.

From (4.4) we get

$$[10s^4(\alpha')^3\alpha'' + 12s^3\alpha'\alpha'']^2 - [(s\alpha')^5 + 3s^3\alpha'(\alpha'')^2]^2 = p^2. \tag{4.14}$$

By a similar argument to Subcase 4, we get a contradiction.

Hence $k = 1$. By induction we get

$$\alpha' + \beta' \equiv 0,$$
$$a_i^2(n+i)^2\alpha'\beta' = p^2(z).$$

By computation we get

$$\alpha' = cp(z), \qquad \beta' = -cp(z). \tag{4.15}$$

Hence

$$\alpha = cQ(z) + l_1, \qquad \beta = -cQ(z) + l_2, \tag{4.16}$$

where $Q(z)$ is defined as in Theorem 1.3, and l_1, l_2 are constants. Rewrite f and g as

$$f = c_1 e^{cQ(z)}, \qquad g = c_2 e^{-cQ(z)},$$

where c_1, c_2 and c are constants such that $a_i^2(c_1c_2)^{n+i}[(n+i)c]^2 = -1$.

Case 2. If $p(z)$ is a nonzero constant b, similar to the proof as above, we deduce that α' is a nonzero constant. Thus $\alpha = cz + l_3$, $\beta = -cz + l_4$. We can rewrite f and g as

$$f = c_3 e^{cz}, \qquad g = c_4 e^{-cz},$$

where c_3, c_4 and c are nonzero constants. In the end we deduce that $(-1)^k a_i^2 (c_3c_4)^{n+i}[(n+i)c]^{2k} = b^2$.

This completes the proof of Theorem 1.3. □

The proof of Theorem 1.2 is analogous to that of Theorem 1.1, by Lemma 2.9, the case of (I3) in Theorem 1.1 does not exist.

The proof of Theorem 1.5 is analogous to that of Theorem H, $a(z)$ has no influence on the result, thus we omit the details here.

5. Open problem

It is mentioned as in Remark 1.2, we pose the following

Problem 5.1. What happens to Theorem 1.3 if the condition "$l \leq 5$" is removed?

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