Remarks on the decay of the local energy for semilinear wave equation

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In this note, we prove the global well posedness and the local energy decay for semilinear wave equation with small data.

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1. Introduction and position of the problem

The aim of this note is to study the following nonlinear wave equation,

\[
\begin{align*}
\Box u + \lambda u |u|^{p-1} &= 0, \quad \text{in} \, \mathbb{R} \times \mathbb{R}^3, \\
u(0, x) &= f(x) \in C^1(\mathbb{R}^3) \quad \text{and} \quad \partial_t u(0, x) = g(x) \in C^0(\mathbb{R}^3),
\end{align*}
\]

(1.1)

where \( \Box = \partial^2_t - \Delta \) and \( \lambda \in \mathbb{R} \).

We assume that \( |f(x)| \leq \frac{\varepsilon}{(1+|x|)^{p-1}} \) and \( |g(x)| + |\nabla f(x)| \leq \frac{\varepsilon}{(1+|x|)^{p}} \), for some \( \varepsilon > 0 \). We notice that a large amount of works have been devoted to the existence and uniqueness of solutions to systems of type (1.1) In addition to the works of Pecher [1] and John [2], we have the result of Li [3] who obtained in particular the existence and uniqueness for system (1.1) with \( p > \frac{8}{3} \).

The main purpose of this note is to study the local energy decay of the solutions of (1.1).

The global energy of \( u \) at time \( t \) is defined by

\[
E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^3} \left( |\partial_t u(t, x)|^2 + |\nabla u(t, x)|^2 \right) \, dx + \frac{\lambda}{p+1} \int_{\mathbb{R}^3} |u(t, x)|^{p+1} \, dx,
\]

(1.2)

which is time independent.

We also define the local energy by

\[
E_\rho(u(t)) = \frac{1}{2} \int_{B_\rho} \left( |\partial_t u(t, x)|^2 + |\nabla u(t, x)|^2 \right) \, dx + \frac{\lambda}{p+1} \int_{B_\rho} |u(t, x)|^{p+1} \, dx,
\]

(1.3)

where \( B_\rho = B_{\rho/2}(0, \rho) \).

For the literature we essentially quote the result of Lin [4] who obtained a super-exponential decay of solutions of a semilinear wave equation when the initial data decays sufficiently rapid at infinity and \( 1 < p < 5 \). We also mention the results of C. Morawetz and Strauss [5,6] who obtained a various rates of decay (from polynomial to exponential) in free space.

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Now, we define the following functional space which is inspired from the space introduced in [1].

\[
X^p_{\delta, R} = \left\{ u \in C^0(\mathbb{R} \times \mathbb{R}^3), 0 \leq l \leq 1, \left\| u \right\|_{V^p} \leq \delta \right\},
\]

where we denoted

\[
\left\| u \right\|_{V^p} = \sup_{x \in \mathbb{R}^3} \left( 1 + \left| \left| x \right| \right| + \left| \left| t \right| \right| \right) \left( 1 + \left| \left| x \right| \right| - \left| \left| t \right| \right| \right)^{p-2} \left| u(t, x) \right|.
\]

We then prove the global well posedness and the local energy decay for (1.1).

**Theorem 1.1.** Assume that \( p > 1 + \sqrt{2} \). Then there exist \( \varepsilon_0 > 0, \delta \) and \( R > 0 \) such that, for every \( \varepsilon \in [0, \varepsilon_0] \) the system (1.1) admits a unique solution in the space \( X^p_{\delta, R} \). Moreover, there exists a constant \( C = C(\rho, \varepsilon_0) > 0 \) such that following inequality

\[
E_p(u)(t) = \frac{1}{2} \int_{B^p} \left( |\partial_t u(t, x)|^2 + |\nabla_x u(t, x)|^2 \right) dx + \frac{\lambda}{p-1} \int_{B^p} |u(t, x)|^{p+1} \leq \frac{C}{(1 + t)^{2/p-2}},
\]

holds for every \( u \) solution of (1.1).

**Remark 1.1.**

(1) The results in Theorem 1.1 complete the work of Pecher [1] who proves the global well posedness and scattering for \( p \in [1 + \sqrt{2}, 3] \).

(2) Let also indicate that optimality of the decay rate is still an open problem.

(3) The proof of Theorem 1.1 is based on a fixed point process and uses in crucial way the properties of the fundamental solution of the wave operator on \( \mathbb{R}^3 \).

2. Fundamental lemmas

In this section, we give some preliminary lemmas.

**Lemma 2.1 ([1]).** If \( h \) is a continuous function and \( r = \left| x \right| \) then

\[
\int_{\left| y - x \right| = t} h(y) dS_y = \frac{2\pi t}{r} \int_{\left| r - t \right|}^{t+r} \sigma h(\sigma) d\sigma.
\]

**Lemma 2.2.** Assume \( p > 1 + \sqrt{2} \) and define

\[
g(\sigma, s) = \frac{\sigma}{(1 + \sigma + s)^p (1 + |s - \sigma|)^{p-2}}.
\]

Then for some \( C = C(p) \) the following inequality holds

\[
\int_0^t \left( \int_{|r-t|}^{r+t-s} g(\sigma, s) d\sigma \right) ds \leq \frac{C_0}{(1 + r + t)(1 + |r - t|)^{p-2}} = N(r, t) \text{ for } r \geq 0, t \in \mathbb{R}_+.
\]

**Proof.** The region of integration is divided into three parts as follows:

\[
0 \leq r \leq t - 1, \quad t - 1 \leq r \leq t + 1 \quad \text{and} \quad r \geq t + 1.
\]

We just treat the first case and we note that the other cases can be treated in the same way.

We substitute \( \gamma = s + \sigma, \beta = s - \sigma \)

\[
\int_0^t \left( \int_{|r-t|}^{r+t-s} g(\sigma, s) d\sigma \right) ds \leq \int_0^t \left( \int_{|r-t|}^{r+t-s} \frac{d\sigma}{(1 + \sigma + s)^p (1 + |s - \sigma|)^{p-2}} \right) ds
\]

\[
\leq \int_{t-r}^{t+r} \frac{dy}{(1 + y)^{p-1}} \int_{-\infty}^{r-t} \frac{d\beta}{(1 + |\beta|)^{p-2}}
\]

\[
\leq C \int_{t-r}^{t+r} \frac{dy}{(1 + y)^{p-1}}.
\]

If \( 1 + t - r \geq \frac{1 + t + r}{2} \), i.e \( 1 + t \geq 3r \) one can estimate

\[
\int_{t-r}^{t+r} \frac{dy}{(1 + y)^{p-1}} \leq \frac{2r}{(1 + t - r)^{p-1}} \leq \frac{4r}{(1 + t + r)(1 + t - r)^{p-2}}.
\]
Whereas in the case $1 + t - r \geq \frac{1+\alpha^2}{2}$ i.e $1 + t \leq 3r$ one estimates by

$$\int_{t-r}^{t+r} \frac{dy}{(1 + y)^{p-1}} \leq \frac{1}{p-2} \left[ \frac{1}{c} \left( \frac{1}{(1 + t - r)^{p-2}} - \frac{1}{(1 + t + r)^{p-2}} \right) \right] \leq \frac{1}{(1 + t - r)^{p-2}} \leq \frac{c}{(1 + t + r)(1 + t - r)^{p-2}}.$$

This completes the proof of Lemma 2.2. □

**Remark 2.1.** As a direct consequence of Lemma 2.2 we define

$$V_p = \left\{ u \in C^0(\mathbb{R} \times \mathbb{R}^3) \mid \|u\|_{V_p} < \infty \right\}.$$

Note that $\| \cdot \|_{V_p}$ is an algebra norm.

**Lemma 2.3.** Let $u_0$ be the solution of the following linear wave equation

$$\left\{ \begin{array}{l}
\partial_t^2 u_0 - \Delta u_0 = 0, \\
u_0(x, 0) = f(x) \in C^1(\mathbb{R}^3), \ \partial_t u_0(x, 0) = g(x) \in C^0(\mathbb{R}^3),
\end{array} \right.$$

and take $\varepsilon > 0$ and $k > 2$ such that

$$|f(x)| \leq \frac{\varepsilon}{(1 + |x|)^{k-1}} \quad \text{and} \quad |g(x)| + |\nabla f(x)| \leq \frac{\varepsilon}{(1 + |x|)^k}, \quad \text{for all} \ x \in \mathbb{R}^3.$$

Then

$$|u_0(x, t)| \leq \frac{C \varepsilon}{(1 + |x| + t)(1 + |x| - t)^{k-2}}, \quad \text{for} \ x \in \mathbb{R}^3 \ \text{and} \ t \in \mathbb{R}^+, \ i.e. u_0 \in V_k.$$

**Proof.** According to the classical representation formula, we have

$$u_0(x, t) = \frac{t}{4\pi} \int_{|y|=1} g(x + ty) dS_y + \frac{\partial}{\partial t} \left( \frac{t}{4\pi} \int_{|y|=1} f(x + ty) dS_y \right)$$

$$= \frac{t}{4\pi} \int_{|y|=1} g(x + ty) dS_y + \frac{t}{4\pi} \int_{|y|=1} f(x + ty) dS_y + \frac{t}{4\pi} \int_{|y|=1} (\nabla_y f(x + ty), \xi) dS_y$$

$$= \frac{1}{4\pi t} \int_{|y| \leq t} g(y) dS_y + \frac{1}{4\pi t^2} \int_{|y| = t} f(y) dS_y + \frac{t}{4\pi} \int_{|y| = 1} (\nabla_y f(x + ty), y) dS_y$$

$$= I_1 + I_2 + I_3.$$

We treat the first term as follows

$$|I_1| = \frac{C}{t} \int_{|y| \leq t} |g(y)| dS_y \leq \frac{C \varepsilon}{t} \int_{|y| \leq t} \frac{dS_y}{(1 + |y|)^k} \leq \frac{2\pi C \varepsilon}{r} \frac{\sigma d\sigma}{(1 + \sigma)^k}.$$

If $r \geq 1$ and $r \geq \frac{1}{2}$ we estimate

$$\frac{1}{r} \int_{|r-t|}^{r+t} \frac{\sigma d\sigma}{(1 + \sigma)^k} \leq \frac{c}{r} \int_{|r-t|}^{\infty} \frac{d\sigma}{(1 + \sigma)^{k-1}} \leq \frac{c}{(1 + r + t)(1 + |r-t|)^{k-2}}.$$

If $r \leq \frac{1}{2}$ or $\frac{1}{2} \leq r \leq 1$ we have

$$\frac{1}{r} \int_{|r-t|}^{r+t} \frac{\sigma d\sigma}{(1 + \sigma)^k} \leq \frac{1}{r} \int_{|r-t|}^{r+t} \frac{d\sigma}{(1 + |r-t|)^{k-1}} \leq \frac{2}{c} \frac{1}{(1 + |r-t|)^{k-2}}.$$

Finally the second and third terms can be handled in the same way.

The proof of Lemma 2.3 is achieved. □
3. Proof of Theorem 1.1: Existence and decay of the local energy

We denote by

$$\mathcal{E}(u)(t, x) = \frac{\lambda}{4\pi} \int_0^t \frac{1}{t - \tau} \left( \int_{|y - x| = t - \tau} u^p(\tau, y) dS_y \right) d\tau,$$

where $u$ satisfies (1.1).

In order to run a fixed point theorem we estimate for $u \in V_p$

$$|\mathcal{E}(u)(t, x)| \leq C \frac{1}{4\pi} \int_0^t \frac{1}{t - \tau} \left( \int_{|y - x| = t - \tau} |u^p(\tau, y)| dS_y \right) d\tau$$

$$\leq C \frac{1}{4\pi} \int_0^t \frac{1}{t - \tau} \left( \int_{|y - x| = t - \tau} \frac{dS_y}{(1 + |y - x|)^{p(1 + \nu^2)(p - 2)}} \right) d\tau \|u\|_{V_p}^p$$

$$\leq \frac{C_1}{r} \int_0^t \left( \int_{|y - x| = t - \tau} (1 + \sigma + |y - x|)^p \frac{\sigma d\sigma}{(1 + \sigma + |y - x|)^{p(1 + \nu^2)(p - 2)}} \right) d\tau \|u\|_{V_p}^p$$

$$\leq \frac{C_2}{(1 + r + t)(1 + |r - t|)^{p(1 - \nu^2)}} \|u\|_{V_p}^p$$

which gives

$$\|\mathcal{E}(u)\|_{V_p} \leq C \|u\|_{V_p}^p. \quad (3.1)$$

One can easily verify that $\partial_{k_n} \mathcal{E}(u) = \mathcal{E}(\partial_{k_n} u^p)$.

Consequently one proves

$$\|\partial_{k_n} \mathcal{E}(u)\|_{V_p} \leq C \|u\|_{V_p}^{p-1} \|\partial_{k_n} u\|_{V_p}. \quad (3.2)$$

On the other hand

$$|(\mathcal{E}(u) - \mathcal{E}(v))(t, x)| \leq C \frac{1}{4\pi} \int_0^t \frac{1}{t - \tau} \left( \int_{|y - x| = t - \tau} |u^p - v^p| (\tau, y) dS_y \right) d\tau$$

$$\leq C \frac{1}{4\pi} \int_0^t \frac{1}{t - \tau} \left( \int_{|y - x| = t - \tau} (u(v + 1))^{p-1} (\tau, y) dS_y \right) d\tau.$$

Thus

$$\|\mathcal{E}(u) - \mathcal{E}(v)\|_{V_p} \leq C \|u - v\|_{V_p} \|u\|_{V_p}^{p-1} \|u\|_{V_p} \|v\|_{V_p} \|u - v\|_{V_p}. \quad (3.3)$$

and one easily verifies

$$\|\partial_{k_n} \mathcal{E}(u) - \partial_{k_n} \mathcal{E}(v)\|_{V_p} \leq C \|u - v\|_{V_p} \|u\|_{V_p}^{p-2} \|\partial_{k_n} u\|_{V_p} + \|\partial_{k_n} u - \partial_{k_n} v\|_{V_p} \|u\|_{V_p}^{p-1} \|v\|_{V_p} \|u - v\|_{V_p} \|u\|_{V_p}^{p-1}. \quad (3.4)$$

Then we write

$$\mathcal{E}(u)(t, x) = \frac{1}{4\pi} \int_0^t (t - \tau) \left( \int_{|y - x| = t} u^p(\tau, x + (t - \tau) y) dS_y \right) d\tau.$$

It is easy to check that

$$\partial_t \mathcal{E}(u)(t, x) = p \mathcal{E}(\partial_t u u^{p-1})(t, x) + \frac{1}{4\pi t} \int_{|y - x| = t} u^p(0, y) dS_y.$$

As $|u^p(0, y)| \leq \frac{C}{(1 + |y|^p)^{p-1}}$, we deduce that

$$|\partial_t \mathcal{E}(u)(t, x)| \leq p \left| \mathcal{E}(\partial_t u u^{p-1})(t, x) \right| + \frac{C}{t} \int_{|y - x| = t} \frac{dS_y}{(1 + |y|^{p-1})} \|u\|_{V_p}^p$$

$$\leq p \left| \mathcal{E}(\partial_t u u^{p-1})(t, x) \right| + \frac{C}{r} \int_{t - |t|}^{t + |t|} \frac{\sigma d\sigma}{(1 + \sigma)^{p+1}} \|u\|_{V_p}^p$$

since $p > 1 + \sqrt{2}$.

Similarly to the proof of Lemma 2.3 and in order to estimate the second term of the last inequality, we distinguish the two following cases:
If \( r \geq 1 \) and \( r \geq t/2 \) we obtain
\[
\frac{1}{r} \int \frac{\sigma \, d\sigma}{(1 + \sigma)^{p+1}} \leq \frac{1}{r} \int \frac{d\sigma}{(1 + |r-t|)^{p+1}} \leq \frac{c}{(1 + r + t)(1 + |r - t|)^{p-1}}.
\]

If \( r \leq \frac{t}{r} \) or \( \frac{t}{r} \leq r \leq 1 \) it follows that
\[
\frac{1}{r} \int \frac{\sigma \, d\sigma}{(1 + \sigma)^{p+1}} \leq \frac{1}{r} \int \frac{d\sigma}{(1 + |r-t|)^{p+1}} \leq \frac{2}{(1 + r + t)(1 + |r - t|)^{p-1}}.
\]

and we obtain
\[
\| \partial_t \mathcal{E}(u) \|_{V_p} \leq C \| u \|_{V_p}^{p-1} (\| \partial_t u \|_{V_p} + \| u \|_{V_p}).
\]

Finally we write
\[
(\partial_t \mathcal{E}(u) - \partial_t \mathcal{E}(v))(t, x) = p \mathcal{E}(\partial_t (u^{p-1} - v^{p-1}))(t, x) + p \mathcal{E}(v^{p-1}(\partial_t u - \partial_t v))(t, x)
\]
\[
+ \frac{1}{4\pi t} \int_{|x-y|=t} (u^p(0, y) - v^p(0, y)) dS_y.
\]

Consequently
\[
\| \partial_t \mathcal{E}(u) - \partial_t \mathcal{E}(v) \|_{V_p} \leq C \| \partial_t u \|_{V_p} \| u - v \|_{V_p} (\| u \|_{V_p}^{p-2} + \| v \|_{V_p}^{p-2})
\]
\[
+ \| \partial_t u - \partial_t v \|_{V_p} \| v \|_{V_p}^{p-1} + \| u - v \|_{V_p} (\| u \|_{V_p}^{p-1} + \| v \|_{V_p}^{p-1}).
\]

The rest of the proof is standard.

The estimates (3.1), (3.2) and (3.5) show that for an arbitrary given \( R \) one has
\[
\| \mathcal{E}(u) \|_{V_p} \leq C \delta^p, \quad \| \partial_q \mathcal{E}(u) \|_{V_p} \leq C \delta^{p-1} R \quad \text{and} \quad \| \partial_t (\mathcal{E}(u)) \|_{V_p} \leq C (\delta^p + \delta^{p-1} R).
\]

So \( u_0 + \mathcal{E}(u) \in X \) if \( u \in X^0_{\delta, R} \). Now we take \( \delta > 0 \) small enough, say,
\[
C \delta^{p-1} \leq \frac{1}{4} \quad \text{and} \quad C \delta^{p-2} R \leq \frac{1}{4},
\]

and we consider the sequence \( u_{n+1} = u_0 + \mathcal{E}(u_n), \ n \geq 0 \).

By (3.3), (3.5) and (3.6), we have
\[
\| u_{n+1} - u_n \|_{V_p} \leq \frac{1}{2} \| u_n - u_{n-1} \|_{V_p}.
\]

Consequently, we have
\[
\| u_{n+1} - u_n \|_{V_p} \leq \frac{c}{2n}, \quad \| \partial_q (u_{n+1} - u_n) \|_{V_p} \leq \frac{c}{2n} + \frac{1}{2} \| \partial_q (u_n - u_{n-1}) \|_{V_p}.
\]

Thus, we deduce that
\[
\| \partial_q (u_{n+1} - u_n) \|_{V_p} \leq \frac{c n}{2n} \quad \text{and} \quad \| \partial_t (u_{n+1} - u_n) \|_{V_p} \leq \frac{c n}{2n}.
\]

We then conclude that \( (u_n) \) converges in \( X^0_{\delta, R} \) to \( u \) which is the unique solution of the system (1.1).

Finally as \( u \in X^0_{\delta, R} \) then for \( t \geq 0 \) and \( x \in B(0, \rho) \) we have
\[
|\partial_t u(t, x)|^2 + |\nabla u(t, x)|^2 \leq \frac{C}{(1 + t)^{2p-2}},
\]
and then
\[
|u(t, x)|^{p+1} \leq \frac{C}{(1 + |x| + |t|)^{p+1}(1 + \| x - |t| \|)^{(p+1)(p-2)}}.
\]

This gives the energy decay.

References