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Solutions to the Cauchy problem for differential equations in Banach spaces with fractional order[☆]

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ABSTRACT

In this paper, we use the monotone iterative technique combined with cone theory to investigate the existence of solutions to the Cauchy problem for Caputo fractional differential equations in Banach spaces. New existence theorems are obtained for the case of a cone P being normal and fully regular respectively. Moreover, two examples are given to illustrate the abstract results.

1. Introduction

Fractional differential equations have been studied extensively by various methods (see, for instance, [1–13]). In recent years, the monotone iterative technique is also used to deal with fractional differential equations (see, e.g., [14]–[15] and references therein) in the spaces of real functions. In this paper, we investigate the following Cauchy problem for differential equations with fractional order in a real Banach space E

$$ {}^cD^q u(t) = f(t, u(t)), \quad u(0) = u_0, \tag{1.1}$$

where ${}^cD^q$ is the standard Caputo's derivative of order $0 < q < 1$, $t \in J = [0, 1]$, $f \in C(J \times E, E)$.

Let P be a cone in E which defines a partial ordering in E by $x \le y$ if and only if $y - x \in P$. If $x \le y$ and $x \ne y$, we write $x < y$.

P is said to be normal if there exists a positive constant N such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$, where θ denotes the zero element of E, and P is said to be fully regular if

$$x_1 \le x_2 \le \cdots \le x_n \le \cdots, \qquad \sup_n \|x_n\| < \infty$$

implies

$$\|x_n - x\| \to 0, \quad \text{as } n \to \infty$$

for some $x \in E$.

Clearly, the full regularity of P implies the normality of P.

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Denote by $C(J,E)$ the Banach space of all continuous functions $x : J \to E$ with norm $\|x\|_C = \sup_{t\in J} \|x(t)\|$. Set

$$Q := \{x \in C(J,E),\ x(t) \geq \theta \text{ for } t \in J\}.$$

Then Q is a cone in space $C(J,E)$, and so, $C(J,E)$ is partially ordered by $Q:\ u \leq v$ if and only if $v - u \in Q$, i.e., $u(t) \leq v(t)$ for $t \in J$.

We write $R_0^+ = [0, +\infty)$.

A function $x \in C(J,E)$ is called a solution of (1.1) if it satisfies (1.1).

Definition 1.1 ([16,17]). Let S be a bounded set of a real Banach space E. Set

$$\begin{aligned}\alpha(S) = \inf\{&\delta > 0 : S \text{can be expressed as the union of a finite number of}\\ &\text{sets such that the diameter of each set does not exceed} \delta,\\ &\text{i.e.} S = U_{i=1}^m S_i \text{with } diam(S_i) \leq \delta, i = 1, 2, \ldots, m\}.\end{aligned}$$

Clearly, $0 \leq \alpha(S) < \infty$. $\alpha(S)$ is called the Kuratowski measure of noncompactness.

Lemma 1.1 ([17]). *$H \subset C(J,E)$ is relatively compact if and only if H is equicontinuous and, for any $t \in J$, $H(t)$ is a relatively compact set in E.*

Lemma 1.2 ([17]). *If $H \subset C(J,E)$ is bounded and equicontinuous, then*

(a) $\alpha_C(H) = \alpha(H(J))$,
(b) $\alpha(H(J)) = \max_{t\in J} \alpha(H(t))$, *where* $H(J) = \{x(t) : x \in H, t \in J\}$.

Lemma 1.3 ([18]). *Let $v, w \in C([0,T],R)$, $f \in C([0,T]\times R, R)$ and*

(i) $v(t) \leq v(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, v(s)) \mathrm{d}s$,
(ii) $w(t) \geq w(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, w(s)) \mathrm{d}s,\ 0 \leq t \leq T$,

one of the foregoing inequalities being strict. Suppose further that $f(t,x)$ is nondecreasing in x for each t and

$$v(0) < w(0).$$

Then we have

$$v(t) < w(t), \quad 0 \leq t \leq T.$$

Remark 1.1. If $v, w \in C(J,E)$, $f \in C(J \times E, E)$, Lemma 1.3 still holds. The proof is similar to that of Theorem 2.1 in [18].

Lemma 1.4 ([7]). *Let H be a countable set of strongly measurable function $x : J \to E$ such that there exists an $M \in L(J, R_0^+)$ such that $\|x(t)\| \leq M(t)$ a.e. $t \in J$ for all $x \in H$. Then $\alpha(H(t)) \in L(J, R_0^+)$ and*

$$\alpha\left(\left\{\int_J x(t)\mathrm{d}t : x \in H\right\}\right) \leq 2\int_J \alpha(H(t))\mathrm{d}t.$$

Lemma 1.5 (Mönch ([16])). *Let D be a closed and convex subset of E and $u_0 \in D$. Assume that the continuous operator $A : D \to D$ has the following property: $C \subset D$ is countable, $C \subset \overline{co}(\{u_0\} \cup A(C)) \to C$ is relatively compact. Then A has a fixed point in D.*

2. Results

Theorem 2.1. *Let cone P be normal. Assume that*

(H_1) *$t \in J$, $u_1 \leq u_2$ implies $f(t,u_1) \leq f(t,u_2)$.*
(H_2) *For any $t \in J$ and $r > 0$, $f(t, B_r) = \{f(t,u) : u \in B_r\}$ is relatively compact in E, where $B_r = \{u \in C(J,E),\ \|u\|_C \leq r\}$.*

Then the Cauchy problem (1.1) has minimal and maximal solutions in $C(J,E)$.

Proof. Write

$$\mathcal{B}_R = \{u \in C(J, E), \|u\|_C \le R\},$$

where

$$R > \|u_0\| + \frac{M_1 + \delta}{\Gamma(q+1)} + \delta,$$

$\delta > 0$ is a constant. We will show that the Cauchy problem (1.1) has minimal and maximal solutions in $\mathcal{B}_R$ in three steps.
Step 1. We prove that the Cauchy problem (1.1) has at least one solution in $\mathcal{B}_R$.

Define

$$(Au)(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))\mathrm{d}s, \quad \text{for any } u \in \mathcal{B}_R. \tag{2.1}$$

Since (H_2) implies that $f(t, B_r)$ is bounded for any $r > 0$, i.e., there is a positive constant M_1 such that

$$\|f(t, u)\| \le M_1, \quad \text{for any } t \in J,\ u \in B_r,$$

we have, for every $u \in \mathcal{B}_R$ and $t \in J$,

$$\begin{aligned} \|(Au)(t)\| &\le \|u_0\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{\alpha-1} \|f(s, u(s))\|\mathrm{d}s \\ &\le \|u_0\| + \frac{M_1}{\Gamma(q+1)} \\ &< R, \end{aligned}$$

that is,

$$\|Au\|_C \le R.$$

So $A : \mathcal{B}_R \to \mathcal{B}_R$.

Moreover, let $u_n, \overline{u} \in C(J, E)$, $\|u_n - \overline{u}\|_C \to 0$ $(n \to \infty)$. Then for each $t \in J$,

$$\|(Au_n)(t) - (A\overline{u})(t)\| \le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} \|f(s, u_n(s)) - f(s, \overline{u}(s))\|\mathrm{d}s.$$

It is easy to see that

$$f(t, u_n(t)) \to f(t, \overline{u}(t)), \quad \text{as } n \to \infty,\ t \in J \tag{2.2}$$

and

$$(t-s)^{q-1} \|f(s, u_n(s)) - f(s, \overline{u}(s))\| \le 2M_1 (t-s)^{q-1} \in L^1(J, R_0^+) \quad \text{for } s \in J. \tag{2.3}$$

It follows from (2.2) and (2.3) and the Lebesgue dominated convergence theorem that

$$\|(Au_n) - (A\overline{u})\|_C \to 0, \quad \text{as } n \to \infty,$$

that is, A is continuous.

On the other hand, for $t_1, t_2 \in J$, $t_1 < t_2$ and $u \in \mathcal{B}_R$, we deduce that

$$\begin{aligned} \|(Au)(t_2) - (Au)(t_1)\| &= \frac{1}{\Gamma(q)} \left\| \int_0^{t_2} (t_2 - s)^{q-1} f(s, u(s))\mathrm{d}s - \int_0^{t_1} (t_1 - s)^{q-1} f(s, u(s))\mathrm{d}s \right\| \\ &\le \frac{1}{\Gamma(q)} \int_0^{t_1} |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}| \|f(s, u(s))\|\mathrm{d}s \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \|f(s, u(s))\|\mathrm{d}s \\ &\le \left[\int_0^{t_1} |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}|\mathrm{d}s + \int_{t_1}^{t_2} (t_2 - s)^{q-1}\mathrm{d}s \right] \frac{M_1}{\Gamma(q)} \\ &\le \left[2(t_2 - t_1)^{\alpha} + (t_2^{\alpha} - t_1^{\alpha}) \right] \frac{M_1}{\Gamma(q+1)}. \end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

Thus, for any

$$B \subset \mathcal{B}_R, \qquad B(t) = \{u(t) : u \in B\},$$

by Lemma 1.2, we have

$$\alpha_C(AB) = \max_{t \in J} \alpha(AB(t)). \tag{2.4}$$

So, $A(\mathcal{B}_R)$ is equicontinuous.

Now, let

$$V = \{u_m : m = 1, 2, \ldots\} \subset \mathcal{B}_R$$

with

$$V \subset \overline{co}(\{x_0\} \cup AV)$$

for some $x_0 \in \mathcal{B}_R$. Then V is relatively compact.

Actually, it is obvious that

$$(t-s)^{q-1}\|f(s, u_m(s))\| \leq M_1(t-s)^{q-1} \in L^1(J, R_0^+), \quad \text{for } s \in J. \tag{2.5}$$

By (2.1), (2.5) and Lemma 1.4, we obtain

$$\begin{aligned} \alpha((AV)(t)) &\leq \frac{1}{\Gamma(q)} \alpha\left(\int_0^t (t-s)^{q-1}(f(s, V(s)))\mathrm{d}s\right) \\ &\leq \frac{2}{\Gamma(q)} \int_0^t (t-s)^{q-1}(\alpha(f(s, V(s))))\mathrm{d}s. \\ &= 0. \end{aligned} \tag{2.6}$$

It follows from (2.6) that

$$\alpha((AV)(t)) = 0 \quad \text{for } t \in J.$$

This, together with (2.4), yields that $\alpha_C(AV) = 0$. Moreover, we have

$$\alpha_C(V) \leq \alpha_C\{\{x_0\} + (AV)\} = \alpha_C(AV).$$

Thus, $\alpha_C(V) = 0$, and V is relatively compact.

Finally, Mönch's fixed point theorem guarantees that A has a fixed point in $\mathcal{B}_R$.

Step 2. We prove that the Cauchy problem (1.1) has a maximal solution in $\mathcal{B}_R$.

Let

$$\theta < \cdots < \varepsilon_n < \varepsilon_{n-1} < \cdots < \varepsilon_2 < \varepsilon_1, \; n = 1, 2, \ldots,$$

where

$$\|\varepsilon_n\| < \delta \quad \text{and} \quad \lim_{n \to \infty} \|\varepsilon_n\| = 0.$$

We consider the following fractional differential equation

$${}^cD^q u(t) = f(t, u(t)) + \varepsilon_n, \quad u(0) = u_0 + \varepsilon_n, \tag{2.7}$$

which is equivalent to the following fractional integral equation

$$u(t) = u_0 + \varepsilon_n + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[f(s, u(s)) + \varepsilon_n]\mathrm{d}s. \tag{2.8}$$

It follows from (2.8) that

$$\begin{aligned} \|u(t)\| &\leq \|u_0\| + \|\varepsilon_n\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{\alpha-1}\|f(s, u(s)) + \varepsilon_n\|\mathrm{d}s \\ &< \|u_0\| + \delta + \frac{M_1 + \delta}{\Gamma(q+1)} \\ &< R. \end{aligned}$$

Thus,

$$\|u\|_C \leq R.$$

From Step 1, we know that the Cauchy problem (2.7) has a solution $u(t, \varepsilon_n)$ in $\mathcal{B}_R$.

By (2.8), we know that

$$u(t, \varepsilon_n) = u(0, \varepsilon_n) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[f(s, u(s, \varepsilon_n)) + \varepsilon_n]\mathrm{d}s, \tag{2.9}$$

where

$$u(0, \varepsilon_n) = u_0 + \varepsilon_n.$$

This yields

$$\begin{aligned}
&u(0, \varepsilon_2) < u(0, \varepsilon_1),\\
&u(t, \varepsilon_2) = u(0, \varepsilon_2) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[f(s, u(s, \varepsilon_2)) + \varepsilon_2]\mathrm{d}s,\\
&u(t, \varepsilon_1) \ = \ u(0, \varepsilon_1) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[f(s, u(s, \varepsilon_1)) + \varepsilon_1]\mathrm{d}s\\
&\qquad\qquad > \ u(0, \varepsilon_1) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[f(s, u(s, \varepsilon_1)) + \varepsilon_2]\mathrm{d}s.
\end{aligned}$$

Combining (H_1) with Remark 1.1, we have

$$u(t, \varepsilon_2) < u(t, \varepsilon_1), \quad t \in J.$$

Hence,

$$\cdots < u(t, \varepsilon_n) < u(t, \varepsilon_{n-1}) < \cdots < u(t, \varepsilon_2) < u(t, \varepsilon_1), \quad n = 1, 2, \ldots. \tag{2.10}$$

Let

$$V_1(t) = \{u(t, \varepsilon_n) : n = 1, 2, \ldots\}, \quad t \in J.$$

From the above proof of the fact that $A(\mathcal{B}_R)$ is equicontinuous, we know that V_1 is equicontinuous.

On the other hand, we know that

$$(t-s)^{q-1}\|f(s, u(s, \varepsilon_n)) + \varepsilon_n\| \le (M_1 + \delta)(t-s)^{q-1} \in L^1(J, R_0^+), \quad \text{for } s \in J. \tag{2.11}$$

Thus, a combination of (2.9), (2.11) and Lemma 1.4 gives that

$$\begin{aligned}
\alpha(V_1(t)) &\le \frac{1}{\Gamma(q)}\alpha\left(\int_0^t (t-s)^{q-1}(f(s, V_1(s)) + \varepsilon_n)\mathrm{d}s\right)\\
&\le \frac{2}{\Gamma(q)} \int_0^t (t-s)^{q-1}(\alpha(f(s, V_1(s))))\mathrm{d}s.\\
&= 0.
\end{aligned} \tag{2.12}$$

It follows from (2.12) that

$$\alpha(V_1(t)) = 0 \quad \text{for } t \in J.$$

This, together with Lemma 1.2, yields that

$$\alpha_C(V_1) = 0. \tag{2.13}$$

It follows from (2.13) that V_1 is relatively compact in $C(J, E)$, and so, there exists a subsequence of $\{u(t, \varepsilon_n)\}$ which converges uniformly on J to some $u^* \in C(J, E)$. In view of (2.10), we see that $\{u(t, \varepsilon_n)\}$ is non-increasing. Let $\{u(t, \varepsilon_{n_i})\}$ converge to u^*, for $i > j$, we have $u(t, \varepsilon_{n_i}) < u(t, \varepsilon_{n_j})$, which implies that $u^* \le u(t, \varepsilon_{n_j})$. Similarly, we obtain $u^* \le u(t, \varepsilon_n)$. For $\forall\, \varepsilon > 0$, there exists k such that

$$\|u(t, \varepsilon_{n_k}) - u^*\| < \frac{\varepsilon}{N}.$$

Thus, for $n \ge n_k$, we get $u^* \le u(t, \varepsilon_n) \le u(t, \varepsilon_{n_k})$. This, together with the normality of P, yields that

$$\|u(t, \varepsilon_n) - u^*\| \le N\|u(t, \varepsilon_{n_k}) - u^*\| < \varepsilon,$$

which implies that $\{u(t, \varepsilon_n)\}$ itself converges to u^* uniformly on J. So, we have

$$f(t, u(t, \varepsilon_n)) + \varepsilon_n \to f(t, u^*(t)) \quad \text{as } n \to \infty,\ t \in J. \tag{2.14}$$

It follows from (2.5), (2.9), (2.11) and (2.14) and the Lebesgue dominated convergence theorem that

$$u^* = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u^*(s)) \mathrm{d}s.$$

Consequently, u^* is a solution to the Cauchy problem (1.1).

Let $u(t)$ be any solution of the Cauchy problem (1.1). It is obvious that

$$\begin{aligned} & u(0) = u_0 < u_0 + \varepsilon_n = u(0, \varepsilon_n), \\ & u(t) < u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(s, u(s)) + \varepsilon_n] \mathrm{d}s, \\ & u(t, \varepsilon_n) = u_0 + \varepsilon_n + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [f(s, u(s, \varepsilon_n)) + \varepsilon_n] \mathrm{d}s. \end{aligned}$$

By (H_1) and Remark 1.1, we deduce that

$$u(t) < u(t, \varepsilon_n).$$

Let $n \to \infty$, we have

$$u(t) \le u^*(t).$$

Thus, $u^*(t)$ is a maximal solution of the Cauchy problem (1.1).

Step 3. We prove that the Cauchy problem (1.1) has a minimal solution in $\mathcal{B}_R$.

Since it is similar to the proof of Step 2, we omit it.

The proof is complete. □

Theorem 2.2. *Let cone P be fully regular and* (H_1) *in* Theorem 2.1 *hold. Assume that*

(H_3) *$f \in C(J \times P, P)$, $u_0 > \theta$, and there exist $a, b \in C(J, R_0^+)$ such that*

$$\begin{aligned} & \|f(t, u)\| \le a(t) + b(t)\|u\|, \\ & a^* := \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} a(s) \mathrm{d}s < \infty, \qquad b^* := \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} b(s) \mathrm{d}s < 1, \ t \in J. \end{aligned}$$

Then problem (1.1) *has a positive solution $\overline{u}$ in $C(J, P)$, which is a minimal positive solution of problem* (1.1).

Proof. Let

$$\begin{aligned} & v_0(t) = \theta, \\ & v_m(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, v_{m-1}(s)) \mathrm{d}s. \end{aligned}$$

From (H_3), we see that

$$\begin{aligned} \|v_m(t)\| & \le \|u_0\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s, v_{m-1}(s))\| \mathrm{d}s \\ & \le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} b(s) \mathrm{d}s \|v_{m-1}\|_C + \|u_0\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} a(s) \mathrm{d}s. \end{aligned}$$

Therefore,

$$\begin{aligned} \|v_m\|_C & \le b^* \|v_{m-1}\|_C + k_1 \\ & \le b^*(b^* \|v_{m-2}\|_C + k_1) + k_1 \\ & \le \cdots \\ & \le (b^*)^m \|v_0\|_C + ((b^*)^{m-1} + \cdots + b^* + 1) k_1 \\ & = \frac{1 - (b^*)^m}{1 - b^*} k_1 \\ & \le \frac{k_1}{1 - b^*}, \end{aligned} \tag{2.15}$$

where

$$k_1 = \|u_0\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} a(s) \mathrm{d}s.$$

It follows from $\theta = v_0(t) \le v_1(t)$ and (H$_1$) that

$$v_1(t) \le v_2(t) \le \cdots \le \cdots, \quad \forall t \in J. \tag{2.16}$$

This, together with (2.15) and the full regularity of P, yields that

$$\lim_{m\to\infty} v_m(t) = \overline{u}(t). \tag{2.17}$$

Using a proof similar to that of Theorem 2.1, we deduce that $v_m(t)$ is equicontinuous on J. From this, (2.17) and Lemma 1.1, we know that V is relatively compact in $C(J, P)$, and so, there exists a subsequence of $\{v_n\}$ which converges uniformly on J to $\overline{u}$. Since P is also normal, we see from (2.16) that the entire sequence v_n converges to some $\overline{u}$.

Similarly as in the proof of Theorem 2.1, we can prove that

$$\overline{u}(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, \overline{u}(s))\mathrm{d}s.$$

It is obvious that $\overline{u} \in C(J, P)$ and $\overline{u}(t) > \theta$.

Let $u(t)$ be any positive solution of problem (1.1). Then $v_0(t) \le u(t)$ implies $v_1(t) \le u(t)$. Thus,

$$v_m(t) \le u(t) \quad (m = 1, 2, \ldots). \tag{2.18}$$

Now, taking limits as $m \to \infty$ in (2.18), we get $\overline{u}(t) \le u(t)$ *for* $t \in J$.

Hence, $\overline{u}(t)$ is minimal positive solution of Cauchy problem (1.1). This completes the proof of the theorem. □

3. Examples

Example 3.1. Let

$$E = c_0 = \{u = (u_1, \ldots, u_n, \ldots) : u_n \to 0\}$$

and

$$P = \{u = (u_1, \ldots, u_n, \ldots) \in c_0 : u_n \ge 0,\ n = 1, 2, 3 \cdots\}.$$

Thus, E is a Banach space with the norm $\|u\| = \sup_n |u_n|$ and P is a normal cone in E.

Let

$$q = \frac{1}{2}, \quad u = (u_1, \ldots, u_n, \ldots), \qquad f = (f_1, \ldots, f_n, \ldots),$$

$$f_n(t, u) = \frac{1}{100n}(t + u_n), \quad (n = 1, 2, 3 \cdots). \tag{3.1}$$

For every n, $n = 1, 2, 3 \cdots$, $t \in J$, we consider the following initial value problem

$$\begin{cases} {}^cD^q u_n = f_n(t, u), \\ u_n(0) = 0\,. \end{cases} \tag{3.2}$$

Conclusion. Problem (3.2) has minimal continuous solution $v_n^*(t)$ and maximal continuous solutions $w_n^*(t)$ satisfying $v_n^*(t) \to 0$, $w_n^*(t) \to 0$ as $n \to \infty$ for $t \in J$.

Proof. It is clear that $f \in C(J \times E, E)$, and $f \in C(J \times E, E)$ is nondecreasing. Thus, (H_1) is satisfied.

We now check (H$_2$). Let $t \in J$, $r > 0$ be given and $\{z^{(m)}\}$ be any sequence in $f(t, B)(B \subset B_r)$, where

$$z^{(m)} = (z_1^{(m)}, \ldots, z_n^{(m)}, \cdots), \quad z_n^{(m)} = f_n(t, u^{(m)}),$$
$$u^{(m)} = (u_1^{(m)}, \ldots, u_n^{(m)}, \cdots) \quad (n, m = 1, 2, 3, \ldots).$$

By (3.1), we have

$$|z_n^{(m)}| \le \frac{1+r}{100n}, \quad (n, m = 1, 2, 3, \ldots). \tag{3.3}$$

So, by the diagonal method, we can choose a subsequence $\{m_i\}$ of $\{m\}$ such that

$$z_n^{(m_i)} \to z_n \quad \text{as } i \to \infty,\ (n = 1, 2, 3, \ldots), \tag{3.4}$$

which yields

$$|z_n| \le \frac{1+r}{100n}. \tag{3.5}$$

Thus,

$$z = (z_1, \dots, z_n, \dots) \in c_0 = E.$$

For any $\varepsilon > 0$, (3.3) and (3.5) yield that we can choose a positive integer n_0 such that

$$|z_n^{(m_i)}| < \varepsilon,\ |z_n| < \varepsilon, \quad \forall\, n > n_0\ (i = 1, 2, 3, \dots). \tag{3.6}$$

By (3.4), we know that there is a positive i_0 such that

$$|z_n^{(m_i)} - z_n| < \varepsilon, \quad \forall\, i > i_0\ (n = 1, 2, 3, \dots, n_0). \tag{3.7}$$

From (3.6) and (3.7), it follows that

$$\|z^{(m_i)} - z\| = \sup_n |z_n^{(m_i)} - z_n| \le 3\varepsilon, \quad \forall i > i_0.$$

This means that

$$\|z^{(m_i)} - z\| \to 0 \quad \text{as } i \to \infty,$$

which implies that

$$\alpha(f(J, B)) = 0.$$

This yields that, for any $t \in J$ and $r > 0$, $f(t, B_r)$ is relatively compact in E. Thus, (H_2) is satisfied.

Hence, our conclusion follows from Theorem 2.1. □

Example 3.2. Let

$$E = l^1 = \left\{ u = (u_1, \dots, u_n, \dots) : \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

and

$$P = \{u = (u_1, \dots, u_n, \dots) \in l^1 : u_n \ge 0,\ n = 1, 2, 3, \dots\}.$$

Then, E is a Banach space with norm $\|u\| = \sum_{n=1}^{\infty} |u_n|$ and P is a normal cone in E.

Let

$$q = \frac{1}{2}, u = (u_1, \dots, u_n, \dots), \qquad f = (f_1, \dots, f_n, \cdots),$$

$$f_n(t, u) = \frac{1}{10n^3}(t + u_n), \quad (n = 1, 2, 3 \cdots).$$

For every n, $n = 1, 2, 3 \cdots$, $t \in J = [0, 1]$, we consider the following initial value problem

$$\begin{cases} {}^cD^q u_n = f_n(t, u), \\ u_n(0) = \dfrac{1}{n^2}. \end{cases} \tag{3.8}$$

Conclusion. Problem (3.8) has a minimal and positive continuous solution $u_n^*(t)$ satisfying

$$\sum_{n=1}^{\infty} |u_n^*(t)| < \infty \quad \text{for } t \in J.$$

Proof. Since l^1 is weakly complete, we see that P is regular (see Ref. [17], Remark 1.2.4). From the regularity of P and the definition of l^1, we see that P is fully regular.

In this situation, let

$$u_0 = \left(1, \frac{1}{2^2}, \dots, \frac{1}{n^2}, \dots\right).$$

It is clear that

$$f \in C(J \times P, P), \quad u_0 > \theta.$$

For $t \in J$, $u \in P$,

$$f_n(t, u) \le \frac{1}{10} + \frac{1}{10} u_n.$$

Thus,

$$\|f(t,u)\| \le \frac{1}{10} + \frac{1}{10}\|u\|.$$

Here,

$$a(t) = b(t) = \frac{1}{10},$$
$$a^* = b^* = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t \frac{1}{10}(t-s)^{\frac{1}{2}-1}\mathrm{d}s = \frac{1}{5\sqrt{\pi}}t^{\frac{1}{2}} \le \frac{1}{5\sqrt{\pi}} < 1.$$

This means that condition (H_3) holds. It is clear that (H_1) holds. Hence, our conclusion follows from Theorem 2.2. □

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