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Generalizations of P_0- and P-Properties; Extended Vertical and Horizontal Linear Complementarity Problems

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ABSTRACT

Generalizing the concept of $\mathscr{W}_0$-pair of Willson, we introduce the notions of column (row) $\mathscr{W}_0$- and column (row) $\mathscr{W}$-properties for a set of $k + 1$ square matrices $\{M_0, M_1, \ldots, M_k\}$ (of the same dimension), where $k \geqslant 1$. When $k = 1$ and $M_0 = I$, these reduce to the familiar $\mathbf{P}_0$- and $\mathbf{P}$-properties of a square matrix. We show that these notions are related to the extended vertical and horizontal LCPs. Specifically, we show that these notions appear in certain feasible/infeasible interior point algorithms and that the column (row) $\mathscr{W}$-property is characterized by the unique solvability in extended horizontal (vertical) LCPs. As a by-product of our analysis, we show that a monotone horizontal LCP is equivalent to a (standard) LCP and that for a monotone horizontal LCP, feasibility implies solvability.

1. INTRODUCTION

A matrix $M \in R^{n \times n}$ is called a $\mathbf{P}_0$- ($\mathbf{P}$-) matrix if every principal minor of M is nonnegative (respectively, positive). These notions, introduced by Fiedler and Pták [8, 9], have numerous applications in diverse fields; see [1] for details.

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In order to see how these concepts can be generalized, we introduce some notation. Let

$$\mathscr{M} = \{M_0, M_1, \dots, M_k\} \tag{1}$$

be a set of $k + 1$ matrices in $R^{n\times n}$ with $k \geqslant 1$. For any matrix $A \in R^{n\times n}$, let $A_{\cdot j}$ ($A_{j\cdot}$) denote the jth column (row) of A.

Given $\mathscr{M}$ as in (1), a matrix $R \in R^{n\times n}$ is called a *column representative* of $\mathscr{M}$ if

$$R_{\cdot j} \in \{(M_0)_{\cdot j}, (M_1)_{\cdot j}, \dots, (M_k)_{\cdot j}\} \qquad (j = 1, \dots, n),$$

and a *row representative* of $\mathscr{M}$ if

$$R_{j\cdot} \in \{(M_0)_{j\cdot}, (M_1)_{j\cdot}, \dots, (M_k)_{j\cdot}\} \qquad (j = 1, \dots, n).$$

It is clear that any principal minor of a matrix M is the determinant of a suitable column/row representative matrix of $\{I, M\}$ where I is the $n \times n$ identity matrix. This means that a matrix is a $\mathbf{P}_0$- ($\mathbf{P}$-) matrix if and only if the determinants of all column/row representative matrices of $\{I, M\}$ are nonnegative (positive). This leads to the notions of the column (row) $\mathscr{W}_0$-property and the column (row) $\mathscr{W}$-property.

DEFINITION 1. Consider $\mathscr{M}$ given by (1). We say that $\mathscr{M}$ has the

(a) column (row) $\mathscr{W}_0$-property if one of the following holds:

(i) determinants of all column (row) representative matrices of $\mathscr{M}$ are nonnegative and there is at least one such determinant which is positive;

(ii) all column (row) representative matrices of $\mathscr{M}$ are nonpositive and there is at least one such determinant which is negative;

(b) column (row) $\mathscr{W}$-property if one of the following holds:

(i) determinants of all column (row) representative matrices of $\mathscr{M}$ are positive;

(ii) determinants of all column (row) representative matrices of $\mathscr{M}$ are negative.

For $k = 1$, the definition of column $\mathscr{W}_0$-property reduces to that of $\mathscr{W}_0$-pair introduced by Willson [29], who used this concept to study certain equations arising in nonlinear networks; see [30] for more details.

Our motivation for introducing the above concepts comes from the *extended vertical* and *horizontal linear complementarity problems*, which are generalizations of the (standard) linear complementarity problem [3].

Given a block matrix

$$\mathbf{B} = [B_0, B_1, \dots, B_k] \tag{2}$$

and a block vector $\mathbf{b} = [b_0, b_1, \dots, b_k]$ where each $B_j \in R^{n\times n}$ and $b_j \in R^n$, the *extended vertical* LCP($\mathbf{B}, \mathbf{b}$) is to find a vector $x \in R^n$ such that

$$(B_0 x + b_0) \wedge (B_1 x + b_1) \wedge \cdots \wedge (B_k x + b_k) = 0 \tag{3}$$

where "$\wedge$" denotes the componentwise minimum.

If $B_0 = I$ and $b_0 = 0$, the above problem is called a *vertical* LCP (for short, VLCP) [3]. This problem, first studied by Cottle and Dantzig [2], arises in control theory [24, 25], hydrodynamic lubrication [20], nonlinear networks [10, 11], etc. The article [13] by Gowda and Sznajder contains a comprehensive analysis of this problem as well as other pertinent references. In that paper it is shown that the extended vertical LCP($\mathbf{B}, \mathbf{b}$) has a unique solution for all $\mathbf{b}$ if and only if $\{B_0, B_1, \dots, B_k\}$ has the row $\mathscr{W}$-property. We shall say more about this in Section 3.

To define an *extended horizontal* LCP, we consider a block matrix

$$\mathbf{C} = [C_0, C_1, \dots, C_k] \tag{4}$$

where $C_j \in R^{n\times n}$. Let $\mathbf{c}$ be a block vector defined as q for $k = 1$ and as $[q, d_1, \dots, d_{k-1}]$ for $k \geqslant 2$, where $q \in R^n$ and $0 < d_j \in R^n$ for $j = 1, 2, \dots, k-1$. Then the *extended horizontal* LCP($\mathbf{C}, \mathbf{c}$) is to find vectors $x_0, x_1, \dots, x_k$ in R^n such that

$$C_0 x_0 = q + \sum_{j=1}^{k} C_j x_j,$$

$$x_0 \wedge x_1 = 0, \qquad (d_j - x_j) \wedge x_{j+1} = 0 \quad (j = 1, \dots, k-1) \tag{5}$$

(where, of course, only the first complementarity condition is considered when $k = 1$). In [15], Kaneko considers the above problem by assuming $C_0 = I$ and cites applications to mathematical programming and to structural mechanics. (See [22] for applications to mechanics, inventory theory, and statistics.) By slightly modifying a result of Kaneko [15], we show in Section 3 that the extended horizontal LCP has a unique solution for all $q \in R^n$ and all $d_j > 0$ if and only if $\{C_0, C_1, \dots, C_k\}$ has the column $\mathscr{W}$-property.

Concerning the extended horizontal LCP, the case $k = 1$ seems to have become important in the study of feasible/infeasible interior point algorithms for linear and convex quadratic programming problems. In this case, we shall drop the word "extended" and call the problem a *horizontal* LCP (for short, HLCP), a term coined in [3] and used by Zhang [31]. Thus, given a block matrix $[A, B]$ where $A, B \in R^{n\times n}$ and a vector $q \in R^n$, HLCP($[A, B], q$) is to find vectors x and y in R^n such that

$$\begin{aligned} Ax - By &= q, \\ x \wedge y &= 0 \end{aligned} \tag{6}$$

This problem, which reduces to the standard LCP when $B = I$, covers convex quadratic programming problems and plays an important role in electrical networks [28]. For an equivalence of HLCP and piecewise linear systems, see Eaves and Lemke [4]. As pointed out by Zhang [31] and Güler [14], the HLCP formulation of a convex quadratic programming problem is better suited for computational aspects than the LCP formulation. Under the condition

$$Ax - By = 0 \quad \Rightarrow \quad x^T y \geqslant 0, \tag{7}$$

which we shall call *column monotonicity*, Zhang [31] describes an infeasible interior point algorithm to solve HLCP($[A, B], q$). The algorithm depends crucially on the nonsingularity of the matrix

$$\begin{bmatrix} A & -B \\ X & Y \end{bmatrix} \tag{8}$$

where X and Y are positive diagonal matrices. As Zhang shows, the nonsingularity of the above matrix follows from the column monotonicity of $\{A, B\}$.

We show in Section 4 that for an arbitrary $[A, B]$ the matrix (8) is nonsingular for all positive diagonal matrices X and Y if and only if $\{A, B\}$ has the column $\mathscr{W}_0$-property. Generalizing this, we show that the nonsingularity of certain Jacobians in feasible/infeasible interior point algorithms for solving extended horizontal (vertical) LCPs naturally lead to the column (row) $\mathscr{W}_0$-property.

We show in Section 6 that when $\{A, B\}$ is column monotone, HLCP($[A, B], q$) is equivalent to a monotone LCP. As a by-product, we deduce that when $\{A, B\}$ is column monotone, the feasibility of the problem HLCP($\{A, B\}, q$) implies its solvability, a result first noted by Güler using maximal monotone operator theory.

We also show that when $\{A, B\}$ has the row monotonicity property, the feasibility of the vertical LCP($[A, B], [a, b]$) implies its solvability.

2. PRELIMINARIES

Given two vectors x and y in R^n, $x^T y$ denotes the usual inner product between x and y.

We list below few matrix-theoretic results needed in this paper.

1. For $A, B, X, Y \in R^{n \times n}$, X, Y commuting, the following formula holds [21]:

$$\det\begin{bmatrix} A & -B \\ X & Y \end{bmatrix} = \det(AY + BX). \tag{9}$$

2. We have the Schur determinantal formula [21]:

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \det(D - CA^{-1}B) \tag{10}$$

where A and D are square and A is invertible.

3. For $M_0, M_1, \ldots, M_k, X_0, X_1, \ldots, X_k \in R^{n \times n}$ where each X_j is diagonal, the following holds:

$$\begin{aligned} \det(M_0 X_0 + \cdots + M_k X_k) &= \sum \det Z \det K \\ &= \sum (z_1 z_2 \cdots z_n) \det K, \end{aligned} \tag{11}$$

where K is a column representative of $\{M_0, \ldots, M_k\}$, Z is a column representative of $\{X_0, \ldots, X_k\}$, and the same indexed columns are selected to form K and Z. Here $z_1, z_2, \ldots, z_n$ are the diagonal entries of Z, and the summation is over all column representatives of $\{M_0, M_1, \ldots, M_k\}$. This follows directly from multilinearity of the determinant function.

4. For $D, M \in R^{n\times n}$ where D is diagonal, we have

$$\det(D + M) = \sum_{\alpha} \det M_{\alpha\alpha} \det D_{\bar{\alpha}\bar{\alpha}}, \tag{12}$$

where $\alpha \subseteq \{1,\dots,n\}$, $\bar{\alpha}$ is the complement of α in $\{1,\dots,n\}$, and $M_{\alpha\alpha}$ is the principal submatrix corresponding to the index pair (α, α). This formula follows from (11).

3. THE COLUMN AND ROW $\mathscr{W}$-PROPERTIES

THEOREM 2. *For $\mathscr{M}$ given by* (1), *the following are equivalent:*

(a) *$\mathscr{M}$ has the column $\mathscr{W}$-property.*

(b) *For arbitrary nonnegative diagonal matrices $X_0, X_1, \dots, X_k \in R^{n\times n}$ with* $\operatorname{diag}(X_0 + X_1 + \cdots + X_k) > 0$,

$$\det(M_0 X_0 + M_1 X_1 + \cdots + M_k X_k) \neq 0.$$

(c) *M_0 is invertible, and $\tilde{\mathscr{M}} = \{I, M_0^{-1}M_1, \dots, M_0^{-1}M_k\}$ has the column $\mathscr{W}$-property.*

(d) *For all q and $d_j > 0$ in R^n, the extended horizontal* LCP($\mathbf{C}, \mathbf{c}$) *has a unique solution where $\mathbf{C} = [M_0, \dots, M_k]$ and $\mathbf{c}$ is q when $k = 1$, and $\mathbf{c} = [q, d_1, \dots, d_{k-1}]$ when $k \geqslant 2$.*

Proof. (a) $\Rightarrow$ (b): Suppose (a) holds, and assume without loss of generality that the determinant of every column representative K of $\mathscr{M}$ is positive. Let $X_0, X_1, \dots, X_k$ be nonnegative diagonal matrices with $\operatorname{diag}(X_0 + X_1 + \cdots + X_k) > 0$. Then in (11), every term in the summation is nonnegative. Moreover, for appropriate Z, the product $z_1 z_2 \cdots z_n$ is positive. It follows that the left side of (11) is positive and hence nonzero.

(b) $\Rightarrow$ (a): Suppose that (b) holds. By continuity, we can assume that $\operatorname{sgn}\det(M_0 X_0 + M_1 X_1 + \cdots + M_k X_k)$ is the same, say $+1$, for all nonnegative diagonal matrices $X_0, X_1, \dots, X_k$ with $\operatorname{diag}(X_0 + X_1 + \cdots + X_k) > 0$. Since every column representative of $\mathscr{M}$ is of the form $(M_0 X_0 + M_1 X_1 + \cdots + M_k X_k)$, we have (a).

(a) $\Rightarrow$ (c): Suppose (a) holds, so that $\det M_0 \neq 0$. Now any column representative matrix of $\tilde{\mathscr{M}}$ is of the form $M_0^{-1}R$ where R is some column representative matrix of $\mathscr{M}$. Thus, from (a), all column representative matrices of $\tilde{\mathscr{M}}$ have the same (positive) determinantal sign, i.e., $\tilde{\mathscr{M}}$ has the column $\mathscr{W}$-property. The implication (c) $\Rightarrow$ (a) follows from reversing the above argument. Finally, the equivalence (c) $\Leftrightarrow$ (d) follows from a result of Kaneko

[15, Theorem A.1] and the observation that if $M_0 u = 0$ for some nonzero u, then with $q := M_0 u^+ = M_0 u^-$, and $d_j > 0$ arbitrary, the extended horizontal LCP in (d) has two distinct solutions, namely, $(x_0, x_1, \dots, x_k) = (u^+, 0, \dots, 0)$ and $(x_0, x_1, \dots, x_k) = (u^-, 0, \dots, 0)$. ■

A few remarks are in order. First, we consider the equivalence of (a) and (d) for $k = 1$. It says that when A and B are in $R^{n \times n}$, the set $\{A, B\}$ has the column $\mathscr{W}$-property if and only if HLCP($[A, B], q$) has a unique solution for all $q \in R^n$. This result has been noted by Kuhn and Löwen [18] in connection with piecewise affine bijections of R^n. However, a simple proof of this result can be given using the linear complementarity theory based on the observations that (i) if $Au = 0$, $u \neq 0$, then $(x, y) = (u^+, 0)$ and $(x, y) = (u^-, 0)$ are two distinct solutions of HLCP($[A, B], Au^+$), and (ii) HLCP($[A, B], q$) has a unique solution for all q if and only if A is invertible and $A^{-1}B$ is a $\mathbf{P}$-matrix.

THEOREM 3. *For $\mathscr{M}$ given by* (1), *the following are equivalent*:

(a) *$\mathscr{M}$ has the row $\mathscr{W}$-property.*

(b) *For arbitrary nonnegative diagonal matrices $X_0, X_1, \dots, X_k \in R^{n \times n}$ with* $\operatorname{diag}(X_0 + X_1 + \cdots + X_k) > 0$,

$$\det(X_0 M_0 + \cdots + X_k M_k) \neq 0.$$

(c) *M_0 is invertible, and $\hat{\mathscr{M}} = \{I, M_1 M_0^{-1}, \dots, M_k M_0^{-1}\}$ has the row $\mathscr{W}$-property.*

(d) *For all $\mathbf{b} \in R^{n \times (k+1)}$, the extended vertical* LCP($\mathbf{M}, \mathbf{b}$) *has a unique solution where $\mathbf{M} = [M_0, M_1, \dots, M_k]$ and $\mathbf{b} = [b_0, b_1, \dots, b_k]$ with $b_j \in R^n$, $j = 0, 1, \dots, n$.*

The proof of this theorem is very similar to that of Theorem 2 except that for proving the equivalence of (a) and (d), we quote a result of Gowda and Sznajder [13, Theorem 17].

At this particular stage, one might wonder whether the row and the column $\mathscr{W}$-properties are different. To exhibit the difference, we give the following example.

EXAMPLE 4. Consider

$$A = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix},$$

$$BA^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad A^{-1}B = \begin{bmatrix} 3 & 1 \\ -4 & -1 \end{bmatrix}.$$

Then BA^{-1} is a **P**-matrix, while $A^{-1}B$ is not a **P**-matrix. In view of Theorems 2 and 3, $\{A, B\}$ has the row $\mathscr{W}$-property but not the column $\mathscr{W}$-property.

It is easy to construct examples of $\mathscr{M}$ with the column (row) $\mathscr{W}$-property. Suppose every M_i in $\mathscr{M}$ [as given by (1)] is a strictly column (row) diagonally dominant **Z**-matrix with positive diagonal. Then every column (row) representative of $\mathscr{M}$ is a **P**-matrix [3], and $\mathscr{M}$ has the column (row) $\mathscr{W}$-property.

We end this section by noting that conditions weaker than the row $\mathscr{W}$-property have been considered in the works of Fujisawa and Kuh; Fujisawa, Kuh, and Ohtsuki; Rheinboldt and Vandergraft; Kojima and Saigal; Schramm; and Kuhn and Löwen. There the bijectivity of a piecewise linear function on R^n is described in terms of the signs of determinants of certain matrices induced by the function. See Ralph [23] for detailed references.

4. THE COLUMN AND ROW $\mathscr{W}_0$-PROPERTIES

Referring to the definition of column (row) $\mathscr{W}_0$-property, we note the existence of at least one representative whose determinant is nonzero. We shall see below that this nonzero condition, while obvious when $\mathscr{M} = \{I, M\}$, also holds in the (important) monotone case. It is also crucial in certain feasible/infeasible interior point algorithms. First we present a result that is similar to Theorem 2.

THEOREM 5. *For $\mathscr{M}$ given by* (1), *the following are equivalent*:

(i) *$\mathscr{M}$ has the column $\mathscr{W}_0$-property.*
(ii) *For arbitrary positive diagonal matrices $X_0, X_1, \dots, X_k \in R^{n \times n}$,*

$$\det(M_0 X_0 + M_1 X_1 + \cdots + M_k X_k) \neq 0.$$

Proof. The proof that (i) $\Rightarrow$ (ii) is similar to that of (a) $\Rightarrow$ (b) in Theorem 2. Now suppose that (ii) holds, so that $\det(M_0 X_0 + M_1 X_1 + \cdots + M_k X_k)$ is either positive for all $X_0, X_1, \dots, X_k$ or negative for all $X_0, X_1, \dots, X_k$, where each X_j is a positive diagonal matrix. By continuity, $\det(M_0 Y_0 + M_1 Y_1 + \cdots + M_k Y_k)$ is either nonnegative for all nonnegative diagonal matrices $Y_0, Y_1, \dots, Y_k$ or nonpositive for all nonnegative diagonal matrices $Y_0, Y_1, \dots, Y_k$. Since each column representative matrix of $\mathscr{M}$ can be written as $M_0 Y_0 + M_1 Y_1 + \cdots + M_k Y_k$ for appropriate $Y_0, Y_1, \dots, Y_k$, we see that all column representative matrices of $\mathscr{M}$ have determinants which are simultaneously nonpositive or nonnegative. Now, to show that there is a column representative matrix of $\mathscr{M}$ whose determinant is nonzero, we use the formula (11) and note that since the left hand side is nonzero, one of the terms must be nonzero, which means that the determinant of some column representative matrix of $\mathscr{M}$ is nonzero. Thus (i) holds. ■

The following result is now obvious.

THEOREM 6. *For $\mathscr{M}$ given by* (1), *the following are equivalent*:

(1) *$\mathscr{M}$ has the row $\mathscr{W}_0$-property.*
(2) *For arbitrary positive diagonal matrices $X_0, \dots, X_k \in R^n$,*

$$\det(X_0 M_0 + \cdots + X_k M_k) \neq 0.$$

REMARK 7. When $k = 1$, (9) and condition (ii) in Theorem 5 imply that $\{A, B\}$ has the column $\mathscr{W}_0$-property if and only if for arbitrary positive diagonal matrices X and Y,

$$\det\begin{bmatrix} A & -B \\ X & Y \end{bmatrix} \neq 0. \tag{13}$$

The special case of this equivalence, for $A = I$, was proved by Kojima et al. [17], who were interested in developing a unified interior point algorithm for $\mathbf{P}_0$-matrices.

As in Theorems 2 and 3, when M_0 is invertible, $\mathscr{M}$ has the column (row) $\mathscr{W}_0$-property if and only if $\{I, M_0^{-1}M_1, \dots, M_0^{-1}M_k\}$ ($\{I, M_1M_0^{-1}, \dots, M_kM_0^{-1}\}$) has the column (row) $\mathscr{W}_0$-property.

4.1. *The Monotone Case*

For A and B in $R^{n \times n}$, we shall say that $\{A, B\}$ is *column monotone* if

$$Ax - By = 0 \quad \Rightarrow \quad x^T y \geqslant 0,$$

(or equivalently, if $As + Bt = 0 \Rightarrow s^T t \leqslant 0$), and *row monotone* if $\{A^T, B^T\}$ is column monotone.

We shall say that $\{C, D\}$ is a column rearrangement of $\{A, B\}$ if for each index i, either $C_i = A_i$ and $D_i = B_i$ or else $C_i = B_i$ and $D_i = A_i$. With this definition, we easily verify the following statements.

(i) If $\{A, B\}$ is column monotone, then so is any column rearrangement of $\{A, B\}$.

(ii) If $\{A, B\}$ is column monotone and $\{C, -D\}$ is a column rearrangement of $\{A, -B\}$, then $\{C, D\}$ is column monotone.

The next theorem and its corollary for the column case have been noted earlier by Sandberg and Willson [30, p. 84] and Willson [29, Theorem 2], who use the term "passive pair" to describe the column monotonicity property. For the sake of completeness, we have included the proofs.

THEOREM 8. *Assume that* $\{A, B\}$ *is column (row) monotone. Then* $\{A, B\}$ *has the column (row)* $\mathscr{W}_0$*-property.*

Proof. We deal only with the column property. (The row property is deduced by working with the transposes.) Let X, Y be arbitrary diagonal positive matrices. In view of Theorem 5, we need to show that $AX + BY$ is nonsingular. Assume that $(AX + BY)u = 0$. By the column monotonicity of $\{A, B\}$, $(Xu)^T Yu \leqslant 0$. Since $X^T Y$ is a positive definite diagonal matrix, we must have $u = 0$. ∎

As a consequence of Theorems 5, 6, and 8, we have the following

COROLLARY 9. *If* $\{A, B\}$ *is column (row) monotone, then there exists a column (row) representative matrix of* $\{A, B\}$ *which is nonsingular.*

The two concepts, namely, the column and row $\mathscr{W}_0$-properties for $\mathscr{M} = \{M_0, M_1, \dots, M_k\}$, coincide when $M_0 = I$ or when each M_j is symmetric. The case $M_0 = I$ was considered by Ebiefung and Kostreva [7], who refer to the above property simply as the $\mathbf{P}_0$-property. The following example demonstrates that the column and row $\mathscr{W}_0$-properties are indeed different and that the column monotone property need not imply the row $\mathscr{W}_0$-property.

EXAMPLE 10. Let

$$C = \begin{bmatrix} 1.5 & -0.5 \\ 0.5 & -0.5 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$C^{-1}D = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad DC^{-1} = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix}.$$

Since $C^{-1}D$ is a positive semidefinite matrix, $\{C, D\}$ is column monotone and hence $\{C, D\}$ has the column $\mathscr{W}_0$-property. However, DC^{-1} is not a $\mathbf{P}_0$-matrix, that is, $\{C, D\}$ does not have the row $\mathscr{W}_0$-property.

We end this subsection by noting an equivalent formulation of the column (row) monotonicity property. This observation is due to Peiyi Liu and (implicitly) Güler [14].

THEOREM 11. *The pair $\{A, B\}$ is column (row) monotone if and only if $A + B$ is nonsingular and AB^T (A^TB) is positive semidefinite.*

Proof. We deal only with the column monotonicity property. Assume that $\{A, B\}$ is column monotone. Let $(A + B)u = 0$. By the monotonicity we get $-\|u\|^2 \geqslant 0$, which gives $u = 0$, so the matrix $A + B$ is nonsingular. Let $\{C, D\}$ be a column rearrangement of $\{A, B\}$, so that C is nonsingular and $C^{-1}D$ is positive semidefinite. Straightforward verification shows that for any fixed $x \in R^n$, $x^TAB^Tx = x^TCD^Tx$. Fix x and let $y = C^Tx$. Then

$$x^T(CD^T)x = (C^Tx)^TD^Tx = y^T(C^{-1}D)^Ty \geqslant 0.$$

Hence, $x^TAB^Tx \geqslant 0$, i.e., AB^T is positive semidefinite. In the other direction, we first note that when $A + B$ is nonsingular, there is a column rearrangement $\{C, D\}$ of $\{A, B\}$ where C is nonsingular. (This can be seen by expanding the determinant of $A + B$.) Reversing our argument above, we prove the remaining part of the theorem. ∎

REMARK 12. When $A + B = I$, it turns out that the column monotonicity property is the same as the row monotonicity property for $\{A, B\}$. This can be seen as follows. If $A + B = I$ then there exists a column rearrangement $\{C, D\}$ of $\{A, B\}$ such that $C + D = I$ with C invertible. Assume $\{A, B\}$ is column monotone, so that $\{C, D\}$ is also column monotone. But then the matrix $C^{-1}D = DC^{-1}$ is positive semidefinite. Then $\{C^T, D^T\}$ is column monotone, whence $\{A^T, B^T\}$ is column monotone. This means that $\{A, B\}$ is row monotone.

4.2. *Feasible / Infeasible Interior Point Algorithms*

In this subsection, we show how the column and row $\mathscr{W}_0$-properties arise in the feasible/infeasible interior point algorithms for solving extended horizontal LCPs and extended vertical LCPs. However, we shall not be concerned with the actual convergence of these algorithms.

First consider an extended horizontal LCP. For ease of notation and exposition, we consider the following problem; the general case can be dealt with similarly. Given matrices A, B, and C in $R^{n\times n}$ and vectors q and $d > 0$ in R^n, the problem is to find (x, y, z) such that

$$Ax = q + By + Cz,$$
$$x \wedge y = 0,$$
$$(d - y) \wedge z = 0.$$

We define the *feasible set* and the *nonnegative set* corresponding to the above problem by

$$\mathscr{F} = \{(x, y, z): Ax = q + By + Cz, x \wedge y \geqslant 0, (d - y) \wedge z \geqslant 0\}$$

and

$$\mathscr{O} = \{(x, y, z): x \wedge y \geqslant 0, (d - y) \wedge z \geqslant 0\}.$$

Note that $x \wedge y \geqslant 0$ simply means that both x and y are nonnegative. By a *feasible* [*infeasible*] *interior point algorithm* for the above problem, we mean an algorithm where the initial point (x^0, y^0, z^0) and all the subsequent points (x^k, y^k, z^k) generated by the algorithm are in $\mathrm{int}(\mathscr{O}) \cap \mathscr{F}[\mathrm{int}(\mathscr{O})]$. Here "int" refers to the topological interior.

Now define the function $F: R^n \times R^n \times R^n \to R^n \times R^n \times R^n$ by

$$F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} Ax - By - Cz - q \\ x * y \\ (d - y) * z \end{bmatrix},$$

where $x * y$ denotes the Hadamard product of vectors x and y defined by $(x * y)_i = x_i y_i$ $(i = 1, 2, \dots, n)$. The algorithm consists in starting with a point (x^0, y^0, z^0) in the interior of $\mathscr{O}$ and generating a sequence (x^k, y^k, z^k) in the following way.

At any stage, given (x^k, y^k, z^k) and a vector $(u^k, v^k, w^k) \in R^n \times R^n \times R^n$ [which depends on (x^k, y^k, z^k)], define $(x^{k+1}, y^{k+1}, z^{k+1})$ by

$$(x^{k+1}, y^{k+1}, z^{k+1}) = (x^k, y^k, z^k) + \alpha_k(\Delta x, \Delta y, \Delta z)$$

where α_k is a step size and $(\Delta x, \Delta y, \Delta z)$ is a solution of

$$F_k' \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} u^k \\ v^k \\ w^k \end{bmatrix},$$

F_k' being the derivative of F at (x^k, y^k, z^k). The choice of the initial point (x^0, y^0, z^0) and (u^k, v^k, w^k) determines the nature of the algorithm—whether it is a feasible interior point or an infeasible interior point algorithm. Now, in order to solve for $(\Delta x, \Delta y, \Delta z)$ uniquely, we demand that F_k' be nonsingular. Clearly,

$$F_k' = \begin{bmatrix} A & -B & -C \\ Y^k & X^k & 0 \\ 0 & -Z^k & D - Y^k \end{bmatrix},$$

where X^k is a diagonal matrix whose diagonal is x^k etc. Writing $L^k = D - Y^k$ and using the Schur determinantal formula, we see that

$$\det F_k' = \det(AX^kL^k + BY^kL^k + CZ^kY^k).$$

So in order to make F_k' nonsingular, we demand that $\det(AXL + BYL + CZY)$ be nonzero for all positive diagonal matrices $X, Y,$ and Z with $L + Y = D$. But this amounts to saying that for all positive diagonal matrices $X, Y,$ and Z of small norm, $\det(AX + BY + CZ)$ is nonzero, i.e., $\{A, B, C\}$ has the column $\mathscr{W}_0$-property. The gist of the above discussion is that the nonsingularity of the Jacobian in the "Newton step" of the above algorithm is tied to the column $\mathscr{W}_0$-property of $\{A, B, C\}$. In other words, while designing algorithms of above type, one cannot have arbitrary matrices A, B, and C.

Remark 13. In the analysis above, the difference $d - y$ appears in the set $\mathscr{O}$ as well as in the definition of F. This results in expressions of the form $D - Y^k$ in F_k'. Such expressions can be avoided by letting $u = d - y$ and defining

$$\mathscr{F} = \{(x, y, z, u): Ax = q + By + Cz,\ y + u = d,\ x \wedge y \geqslant 0, u \wedge z \geqslant 0\},$$

$$\mathscr{O} = \{(x, y, z, u): x \wedge y \geqslant 0, u \wedge z \geqslant 0\},$$

and

$$F\left(\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}\right) = \begin{bmatrix} x * y \\ u * z \\ u + y - d \\ Ax - By - Cz - q \end{bmatrix}$$

This setup also leads to the column $\mathscr{W}_0$-property of $\{A, B, C\}$. We omit the details.

Now consider an extended vertical LCP corresponding to $[A, B, C]$ and the vector $[a, b, c]$. The problem is to find an x such that $(Ax + a) \wedge (Bx + b) \wedge (Cx + c) = 0$. Writing $y = Ax + a$, $z = Bx + b$, and $w = Cx + c$, we can write the above problem as a system of equations

$$\begin{aligned} y \wedge z \wedge w &= 0, \\ y - Ax - a &= 0, \\ z - Bx - b &= 0, \\ w - Cx - c &= 0. \end{aligned}$$

For this problem, the *feasible* and *nonnegative* sets are given by

$$\begin{aligned} \mathscr{F} = \{(x, y, z, w) : y - Ax - a = 0, z - Bx - b = 0, \\ w - Cx - c = 0, y \wedge z \wedge w \geqslant 0\} \end{aligned}$$

and

$$\mathscr{O} = \{(x, y, z, w) : y \wedge z \wedge w \geqslant 0\}.$$

The feasible/infeasible interior point algorithm for solving this problem is as in the extended horizontal LCP case, except that instead of the function F, we use the function G defined by

$$G\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} y * z * w \\ y - Ax - a \\ z - Bx - b \\ w - Cx - c \end{bmatrix}.$$

As before, the algorithm depends, at any stage, on the invertibility of the derivative of G. The derivative of G at (x, y, z, w) is given by

$$G' = \begin{bmatrix} 0 & ZW & YW & YZ \\ -A & I & 0 & 0 \\ -B & 0 & I & 0 \\ -C & 0 & 0 & I \end{bmatrix},$$

and (by the Schur formula)

$$\det G' = \det(ZWA + YWB + YZC).$$

As before, we demand that this determinant be nonzero for all positive diagonal matrices Y, Z, and W. This leads to the row $\mathscr{W}_0$-property of $\{A, B, C\}$.

5. A PERTURBATION RESULT

In this section we extend a classical result of Fiedler and Pták [9] that M is a $\mathbf{P}_0$-matrix if and only if $\forall \varepsilon > 0$, $M + \varepsilon I$ is a $\mathbf{P}$-matrix. The extension given below should (undoubtedly) be useful—as in [13]—in the stability aspects of extended vertical and horizontal LCPs.

THEOREM 14. *$\mathscr{M}$ given by* (1) *has the column $\mathscr{W}_0$-property if and only if there exists $\mathscr{N} = \{N_0, N_1, \dots, N_k\}$ such that for every $\varepsilon > 0$, $\mathscr{M} + \varepsilon\mathscr{N} := \{M_0 + \varepsilon N_0, M_1 + \varepsilon N_1, \dots, M_k + \varepsilon N_k\}$ has the column $\mathscr{W}$-property.*

Proof. Assume that $\mathscr{M}$ has the column $\mathscr{W}_0$-property and (i) of Definition 1 holds. [For (ii) the proof is the same.] Let $\overline{M}_0$ be a given nonsingular column representative matrix of $\mathscr{M}$. Create new matrices $\overline{M}_1, \dots, \overline{M}_k$ in the following way. For $l = 1, \dots, k$, let

$$(\overline{M}_l)_{\cdot j} = \begin{cases} (M_l)_{\cdot j} & \text{if } (M_l)_{\cdot j} \text{ is not a column of } \overline{M}_0, \\ (M_0)_{\cdot j} & \text{if } (M_l)_{\cdot j} \text{ is a column of } \overline{M}_0. \end{cases}$$

This is a rearrangement of columns of matrices in $\mathscr{M}$. According to this construction, $\overline{\mathscr{M}} = \{\overline{M}_0, \dots, \overline{M}_k\}$ has the column $\mathscr{W}_0$-property with $\overline{M}_0$ nonsingular. As in the proof of Theorem 2, we see that $\mathscr{C} := \{I, \overline{M}_0^{-1}\overline{M}_1, \dots, \overline{M}_0^{-1}\overline{M}_k\}$ has the column $\mathscr{W}_0$-property. If C is any column representative matrix of $\mathscr{C}$, then C is a $\mathbf{P}_0$-matrix. By the result of Fiedler and Pták quoted above, $C + \varepsilon I$ is a $\mathbf{P}$-matrix for every $\varepsilon > 0$. It then follows that $\{I, \overline{M}_0^{-1}\overline{M}_1 + \varepsilon I, \dots, \overline{M}_0^{-1}\overline{M}_k + \varepsilon I\}$ has the column $\mathscr{W}$-property, or again, $\{\overline{M}_0, \overline{M}_1 + \varepsilon\overline{M}_0, \dots, \overline{M}_k + \varepsilon\overline{M}_0\}$ has the column $\mathscr{W}$-property for every $\varepsilon > 0$.

Hence, $\mathscr{M} + \varepsilon\mathscr{N}$ has the column $\mathscr{W}$-property, where $\mathscr{N} = \{N_0, N_1, \dots, N_k\}$ is defined, for $j = 1, \dots, n$ and $l = 1, \dots, k$, by

$$(N_0)_{\cdot j} = \begin{cases} 0 & \text{if } (M_0)_{\cdot j} \text{ is a column in } \overline{M}_0, \\ (\overline{M}_0)_{\cdot j} & \text{if } (M_0)_{\cdot j} \text{ is not a column in } \overline{M}_0, \end{cases}$$

$$(N_l)_{\cdot j} = \begin{cases} (\overline{M}_0)_{\cdot j} & \text{if } (M_l)_{\cdot j} \text{ is a column in } \overline{M}_l, \\ 0 & \text{if } (M_l)_{\cdot j} \text{ is not a column in } \overline{M}_l. \end{cases}$$

Now, assume that for some $\mathscr{N}$, $\mathscr{M} + \varepsilon\mathscr{N}$ has the column $\mathscr{W}$-property for every $\varepsilon > 0$. We verify condition (ii) in Theorem 5. Let $X_0, \dots, X_k$ be positive diagonal matrices. By Theorem 2 and (11), for any $\varepsilon > 0$,

$$0 \neq d(\varepsilon) := \det[(M_0 + \varepsilon N_0) X_0 + \cdots + (M_k + \varepsilon N_k) X_k]$$

$$= \sum z_1 \cdots z_n \det[C_1^\varepsilon \dots C_n^\varepsilon],$$

where $z_1, \dots, z_n > 0$, $C_j^\varepsilon \in \{(M_0 + \varepsilon N_0)_{\cdot j}, \dots, (M_k + \varepsilon N_k)_{\cdot j}\}$. Again, applying (11) to $\det[C_1^\varepsilon \ \dots \ C_n^\varepsilon]$, we see that $d(\varepsilon)$ is a polynomial in ε whose coefficients are linear combinations of determinants of column representative matrices of $\mathscr{M}$. Since d is nontrivial, at least one representative matrix of $\mathscr{M}$ should be nonsingular.

On the other hand, the intermediate-value theorem enables us to claim that if $\mathscr{M} + \varepsilon\mathscr{N}$ has the column $\mathscr{W}$-property, then all column representative matrices $[C_1^\varepsilon \ \dots \ C_n^\varepsilon]$ have the same nonzero determinantal sign. By letting $\varepsilon \to 0^+$, we see that determinants of various column representative matrices of $\mathscr{M}$ are either all nonnegative or all nonpositive. Since we have shown that at least one of the above is nonzero, $\mathscr{M}$ has the column $\mathscr{W}_0$-property. ■

REMARK 15. For $M_0 = I$, the above result was noted by Ebiefung and Kostreva in [7]. In their paper, each N_j is the identity matrix. A result similar to Theorem 14 can be stated for the row $\mathscr{W}_0$-property.

6. COMPLEMENTARITY RESULTS

In this section, we prove complementarity results for $\{A, B\}$ assuming column/row monotone properties.

THEOREM 16. *Suppose that* $\{A, B\}$ *is column monotone. Then for any* $p \in R^n$, HLCP($[A, B], p$) *is equivalent to a (standard) monotone LCP.*

Proof. Consider HLCP($[A, B], p$) which is to solve the system

$$Ax - By = p,$$

$$x \wedge y = 0.$$

By Corollary 9, there exists a nonsingular column representative of $\{A, B\}$. Hence $\{A, -B\}$ will have a nonsingular column representative, say, C. By rearranging the columns of $\{A, -B\}$ to form $\{C, -D\}$ and correspondingly the components of x and y, we can rewrite the above system as

$$Cu - Dv = p,$$

$$u \wedge v = 0.$$

We observe that $\{C, D\}$ is column monotone and hence $C^{-1}D$ is a monotone (that is, positive semidefinite) matrix. Clearly, HLCP($[C, D], p$) is equivalent to the monotone LCP($C^{-1}D, C^{-1}p$). ∎

REMARK 17. In view of the above result, we can say that the study of a horizontal LCP corresponding to a column monotone pair $\{A, B\}$ is similar to that of a monotone LCP. This might explain why the infeasible interior point algorithm as described in Zhang [31] works so well for horizontal LCPs corresponding to a column monotone pair. Certainly this explains, as we illustrate below, why the "feasibility implies solvability" result is valid for horizontal LCPs associated with a column monotone pair.

In the corollary below, the feasibility of HLCP($[A, B], p$) refers to the existence of $x \geqslant 0$ and $y \geqslant 0$ such that $Ax - By = p$.

COROLLARY 18. *Suppose that* $\{A, B\}$ *is column monotone. Then for each* $p \in R^n$, *the feasibility of* HLCP($[A, B], p$) *implies its solvability.*

Proof. If HLCP($[A, B], p$) is feasible, then LCP($C^{-1}D, C^{-1}p$) is feasible where $\{C, -D\}$ is a rearrangement of $\{A, -B\}$ (as in Theorem 16) with C invertible. Since $C^{-1}D$ is a monotone matrix, the feasibility of LCP($C^{-1}D, C^{-1}p$) implies its solvability, see [3]. ∎

Motivated by Theorem 16, we ask whether we can rewrite an HLCP equivalently as an LCP in other situations. The result given below answers this question.

THEOREM 19. *Suppose that* $\{A, B\}$ *is a pair such that for some open set* $\Omega \subset R^n$ *and for all* $p \in \Omega$, HLCP$([A, B], p)$ *has a solution. Then there is a column representative of* $\{A, B\}$ *which is nonsingular, and every* HLCP$([A, B], r)$ *is equivalent to an LCP.*

Proof. In view of the argument used in the proof of Theorem 16, it is enough to show that $\{A, B\}$ has a nonsingular column representative. Let us say that a vector q is degenerate for the HLCP if for some solution (x, y) satisfying (6) we have $x + y \not> 0$, i.e., for some index i, $x_i = y_i = 0$. Clearly, the set of degenerate vectors is contained in a finite union of subspaces of R^n of dimension less than n. Since an open set cannot be covered by these subspaces, we see that there is at least one nondegenerate vector, say p in Ω. Let (x, y) be a solution of HLCP$([A, B], p)$ so that $x + y > 0$. Let $\alpha := \{i : x_i > 0\}$ and $\beta := \{i : y_i > 0\}$. Then β is the complement of α in $\{1, 2, \ldots, n\}$. Let C be the matrix formed by the vectors $A_{\cdot j}$ and $B_{\cdot l}$ as j varies over the set α and l varies over the set β. Clearly, C is a column representative of $\{A, B\}$. We claim that C is nonsingular. Assume the contrary, so that for some nonzero vector z, $Cz = 0$. Define, for each λ, the vectors u and v by

$$u = \begin{bmatrix} x_\alpha + \lambda z_\alpha \\ 0 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 0 \\ y_\beta - \lambda z_\beta \end{bmatrix}.$$

When λ is small, (u, v) is a solution of HLCP$([A, B], p)$. Also, for an appropriate choice of λ, we can make $u + v \not> 0$. But this means that p is degenerate! Hence C is nonsingular, and the proof is complete. ■

The following result is obvious.

COROLLARY 20. *The conclusion of the above theorem is valid for* $\{A, B\}$ *when one of the following holds*:

(a) *The matrix* $[A, B]$ *has full row rank and for every* q, *the feasibility of* HLCP$([A, B], q)$ *implies its solvability.*

(b) HLCP$([A, B], q)$ *is solvable for all* $q \in R^n$.

We now state, without proof, the counterparts of the above results for a pair $\{A, B\}$ satisfying the row monotonicity property. Recall that the extended vertical LCP($[A, B]$, $[a, b]$) is to solve the system

$$(Ax + a) \wedge (Bx + b) = 0,$$

and that if $\{A, B\}$ is row monotone then so is any row rearrangement of $\{A, B\}$.

THEOREM 21. *Suppose that $\{A, B\}$ is row monotone. Then:*

(i) *There is a row representative of $\{A, B\}$ which is nonsingular.*
(ii) *For any $[a, b]$, the extended vertical* LCP($[A, B], [a, b]$) *is equivalent to a monotone LCP.*
(iii) *For the extended vertical* LCP($[A, B]$, $[a, b]$), *feasibility implies solvability.*

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