Discrete Mathematics 309 (2009) 1528–1537

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

# Roman domination in regular graphs☆

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ARTICLE INFO

*Article history:*
Received 26 October 2006
Received in revised form 11 March 2008
Accepted 11 March 2008
Available online 24 April 2008

*Keywords:*
Regular graph
Roman domination number
Roman graph

ABSTRACT

A *Roman domination function* on a graph $G = (V(G), E(G))$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u\in V(G)} f(u)$. The minimum weight of a Roman dominating function on a graph $G$ is called the *Roman domination number* of $G$. Cockayne et al. [E. J. Cockayne et al. Roman domination in graphs, Discrete Mathematics 278 (2004) 11–22] showed that $\gamma(G) \le \gamma_R(G) \le 2\gamma(G)$ and defined a graph $G$ to be *Roman* if $\gamma_R(G) = 2\gamma(G)$. In this article, the authors gave several classes of Roman graphs: $P_{3k}, P_{3k+2}, C_{3k}, C_{3k+2}$ for $k \ge 1$, $K_{m,n}$ for $\min\{m, n\} \ne 2$, and any graph $G$ with $\gamma(G) = 1$; In this paper, we research on regular Roman graphs and prove that: (1) the *circulant graphs* $C(n; \{1, 3\})(n \ge 7,\ n \not\equiv 4 \pmod 5)$ and $C(n; \{1, 2, \ldots, k\})$ $(k \le \lfloor \frac{n}{2} \rfloor)$, $n \not\equiv 1 \pmod{2k+1}$, $(n \ne 2k)$ are Roman graphs, (2) the generalized *Petersen* graphs $P(n, 2k+1)(n \ne 4k+2,\ n \equiv 0 \pmod 4$ and $0 \le k \le \lfloor \frac{n}{2} \rfloor)$, $P(n, 1)$ $(n \not\equiv 2 \pmod 4)$, $P(n, 3)$ $(n \ge 7,\ n \not\equiv 3 \pmod 4)$ and $P(11, 3)$ are Roman graphs, and (3) the *Cartesian product* graphs $C_{5m}\Box C_{5n}(m \ge 1,\ n \ge 1)$ are Roman graphs.



## 1. Introduction

For notation and graph theory terminology in general we follow [4,5]. Throughout this paper, we only consider finite, simple undirected graphs without isolated vertices. A graph $G = (V(G), E(G))$ is a set $V(G)$ of vertices and a subset $E(G)$ of the unordered pairs of vertices, called edges. The open neighborhood and the closed neighborhood of a vertex $v \in V$ are denoted by $N(v) = \{u \in V(G) : vu \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$, respectively. For a set $S \subseteq V(G)$, $N(S) = \cup_{v\in S} N(v)$ and $N[S] = \cup_{v\in S} N[v]$. The maximum degree of any vertex in $V(G)$ is denoted by $\Delta(G)$. When $H \subseteq V(G)$, the induced subgraph $G[H]$ consists of $H$ and all edges whose endpoints are contained in $H$.

A set $S \subseteq V(G)$ is a dominating set if for each $v \in V(G)$ either $v \in S$ or $v$ is adjacent to some $w \in S$. That is, $S$ is a dominating set if and only if $N[S] = V(G)$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$, and a dominating set $S$ of minimum cardinality is called a *$\gamma$-set* of $G$.

For a graph $G$, let $f : V \to \{0, 1, 2\}$, and let $(V_0; V_1; V_2)$ be the ordered partition of $V$ induced by $f$, where $V_i = \{v \in V(G) | f(v) = i\}$ and $|V_i| = n_i$, for $i = 0, 1, 2$. Note that there exists a 1-1 correspondence between the functions $f : V(G) \to \{0, 1, 2\}$ and the ordered partitions $(V_0; V_1; V_2)$ of $V(G)$. So we will write $f = (V_0; V_1; V_2)$.

A function $f : V(G) \to \{0, 1, 2\}$ is a Roman dominating function (*RDF*) if $V_2$ dominates $V_0$, i.e. $V_0 \subseteq N[V_2]$. The *weight* of $f$ is $f(V(G)) = \sum_{v\in V(G)} f(v) = 2n_2 + n_1$. The minimum weight of an *RDF* of $G$ is called the *Roman domination number* of $G$, denoted

☆ The research is supported by Chinese Natural Science Foundations (60373096, 60573022) and by Specialized Research Fund for the Doctoral Program of Higher Education (20030141003).

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doi:10.1016/j.disc.2008.03.006

by $\gamma_R(G)$. And we say that a function $f = (V_0; V_1; V_2)$ is a $\gamma_R$-*function* if it is an *RDF* and $f(V) = \gamma_R(G)$. A graph $G$ is a *Roman graph* (or *Roman*) if $\gamma_R(G) = 2\gamma(G)$.

In 2004, Cockayne et al. [2] studied the graph theoretic properties of this variant of the domination number of a graph and proved:

**Proposition 1.1** (*[2]*). *For any graph $G$ of order $n$, $\frac{2n}{\Delta(G)+1} \leq \gamma_R(G)$.*

**Proposition 1.2** (*[2]*). *For any graph $G$, $\gamma(G) = \gamma_R(G)$ if and only if $G = \overline{K_n}$.*

**Proposition 1.3** (*[2]*). *For any graph $G$ of order $n$, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.*

**Proposition 1.4** (*[2]*). *A graph $G$ is Roman if and only if it has a $\gamma_R$-function $f = (V_0; V_1; V_2)$ with $n_1 = 0$.*

**Proposition 1.5** (*[2]*). *Let $f = (V_0; V_1; V_2)$ be any $\gamma_R$-function. Then*

(a) *$G[V_1]$, the subgraph induced by $V_1$, has maximum degree 1.*
(b) *No edge of $G$ joins $V_1$ and $V_2$.*
(c) *Each vertex of $V_0$ is adjacent to at most two vertices of $V_1$.*
(d) *$V_2$ is a $\gamma$-set of $G[V_0 \cup V_2]$.*
(e) *Let $H = G[V_0 \cup V_2]$. Then each vertex $v \in V_2$ has at least two $H$-pn's (i.e. private neighbours relative to $V_2$ in the graph $H$).*
(f) *If $v$ is isolated in $G[V_2]$ and has precisely one external $H$-pn, say $w \in V_0$, then $N(w) \cap V_1 = \emptyset$.*
(g) *Let $k_1$ equal to the number of non-isolated vertices in $G[V_2]$, let $C = \{v \cap V_0 : |N(v) \cap V_2| \geq 2\}$, and let $|C| = c$. Then $n_0 \geq n_2 + k_1 + c$.*

In [2], the following classes of graphs were found to be Roman graphs: $P_{3k}$, $P_{3k+2}$, $C_{3k}$, $C_{3k+2}$ for $k \geq 1$, $K_{m,n}$ for $\min\{m, n\} \neq 2$, and any graph $G$ with $\Delta(G) = n - 1$ (that is any graph with $\gamma(G) = 1$). In [6], a characterization of Roman trees was given.

For more references and other Roman dominating problems, we can refer to [1,6–9].

The generalized *Petersen* graph $P(n, k)$ is defined to be a graph on $2n$ vertices with $V(P(n, k)) = \{v_i, u_i : 0 \leq i \leq n-1\}$ and $E(P(n, k)) = \{v_iv_{i+1}, v_iu_i, u_iu_{i+k} : 0 \leq i \leq n-1$, where subscripts are taken modulo $n\}$.

In 2007, Yang, Fu and Jiang [3] studied the generalized Petersen graph $P(n, 3)$ and proved

**Theorem 1.1** (*[3]*). $\gamma(P(n, 3)) = n - 2\lfloor \frac{n}{4} \rfloor$ $(n \neq 11)$.

The *circulant graph* $C(n; S_c)$ is the graph with the vertex set $V(C(n; S_c)) = \{v_i | 0 \leq i \leq n-1\}$ and the edge set $E(C(n; S_c)) = \{v_iv_j | 0 \leq i, j \leq n-1, (i-j) \bmod n \in S_c\}$, $S_c \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor$, where subscripts are taken modulo $n\}$.

The *Cartesian product* $G \square H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$, in which the vertex $(a, b)$ is adjacent to the vertex $(c, d)$ whenever $a = c$ and $b$ is adjacent to $d$, or $b = d$ and $a$ is adjacent to $c$.

In this paper, we study Roman domination in regular graphs and give the following new classes of Roman graphs: (1) the circulant graphs $C(n; \{1, 3\})$ $(n \geq 7,\ n \not\equiv 4 \pmod 5)$ and $C(n; \{1, 2, \ldots, k\})$ $(k \leq \lfloor \frac{n}{2} \rfloor, n \not\equiv 1 \pmod{2k+1}), n \neq 2k)$, (2) the generalized Petersen graphs $P(n, 2k+1)(n \neq 4k+2,\ n \equiv 0 \pmod 4$ and $0 \leq k \leq \lfloor \frac{n}{2} \rfloor)$, $P(n, 1)$ $(n \not\equiv 2 \pmod 4))$, $P(n, 3)$ $(n \geq 7,\ n \not\equiv 3 \bmod 4)$ and $P(11, 3)$, and (3) the Cartesian product graphs $C_{5m} \square C_{5n}$ $(m \geq 1,\ n \geq 1)$.

## 2. Basic properties

Let $G$ be an $r$-regular graph with order $n$ $(r \geq 1)$, $m = \lfloor \frac{n}{r+1} \rfloor$, $t = n \bmod (r+1)$, then $n = (r+1)m + t$, $0 \leq t \leq r$.

Let $S$ be an arbitrary dominating set of $G$, then for each vertex $v \in V(G)$, $N[v] \cap S \neq \emptyset$, and $v$ is being dominated $|N[v] \cap S| \geq 1$ times. We define a function $rd$ counting the times $v$ is re-dominated as follows:

$$rd(v) = |N[v] \cap S| - 1.$$

For a vertex set $V' \subseteq V(G)$, let $rd(V') = \sum_{v \in V'} rd(v)$. Then, by Proposition 1.5(d), $V_2$ is a $\gamma$-set of $G[V_0 \cup V_2]$, and this gives us

**Lemma 2.1.** $rd(V(G[V_0 \cup V_2])) = (r+1)n_2 - (n - n_1)$.

**Lemma 2.2.** *If $f = (V_0; V_1; V_2)$ is any $\gamma_R$-function of $G$, then*

(1) $n_2 \geq \lceil \frac{n-n_1}{r+1} \rceil$.
(2) $f(V(G)) \geq 2m + \lceil \frac{2t+(r-1)n_1}{r+1} \rceil \geq \frac{2n+(r-1)n_1}{r+1}$.
(3) $f(V(G)) \geq 2m$ *for* $t = 0$.
(4) $f(V(G)) \geq 2m + 2$ *for* $t \geq 1$ *and* $(t, n_1) \neq (1, 1)$.

**Proof.** (1) By Proposition 1.5(d), $V_2$ is a $\gamma$-set of $G[V_0 \cup V_2]$, hence $(r+1)n_2 \geq n - n_1$. So $n_2 \geq \lceil \frac{n-n_1}{r+1} \rceil$.

(2) Since $f(V(G)) = 2n_2 + n_1$, we have

$$\begin{aligned}(r+1)f(V(G)) &= 2(r+1)n_2 + (r+1)n_1,\\ &\geq 2n - 2n_1 + (r+1)n_1,\\ &= 2(r+1)m + 2t + (r-1)n_1.\end{aligned}$$

Hence $f(V(G)) \geq 2m + \lceil \frac{2t+(r-1)n_1}{r+1} \rceil \geq 2m + \frac{2t+(r-1)n_1}{r+1} = \frac{2n+(r-1)n_1}{r+1}$.

(3) Suppose $t = 0$, then by (2), $f(V(G)) \geq 2m + \lceil \frac{2t+(r-1)n_1}{r+1} \rceil \geq 2m$.

(4) Suppose $t \geq 1$.

*Case* 1. Suppose $n_1 = 0$, then by (1), $n_2 \geq \lceil \frac{n-n_1}{r+1} \rceil = \lceil \frac{(r+1)m+t}{r+1} \rceil = m+1$. Hence $f(V(G)) = 2n_2 + n_1 = 2n_2 \geq 2m+2$.

*Case* 2. Suppose $n_1 = 1$ and $t \geq 2$, then by (2), $f(V(G)) \geq 2m + \lceil \frac{2t+(r-1)n_1}{r+1} \rceil \geq 2m + \lceil \frac{4+r-1}{r+1} \rceil = 2m+2$.

*Case* 3. Suppose $n_1 \geq 2$, then by (2), $f(V(G)) \geq 2m + \lceil \frac{2t+(r-1)n_1}{r+1} \rceil \geq 2m + \lceil \frac{2+2(r-1)}{r+1} \rceil = 2m + 1 + \lceil \frac{r-1}{r+1} \rceil = 2m+2$. □

In this paper, we will denote the vertices of $G$ as follows: black circles denote vertices in $V_2$, grey circles denote vertices in $V_1$ and white circles denote vertices in $V_0$.

## 3. Roman domination in circulant graphs

**Lemma 3.1.** *For 4-regular graph $C(n; \{1, 3\})$ $(n \geq 7)$,*

$$\gamma_R(C(n;\{1,3\})) = \begin{cases} 2m, & \text{if } t = 0;\\ 2m+2, & \text{if } t = 1, 2, 3;\\ 2m+3, & \text{if } t = 4.\end{cases}$$

**Proof.** Let

$$S_{1,2} = \begin{cases}\{v_{5i+2} : 0 \leq i \leq m-1\}, & \text{if } t = 0;\\ \{v_{5i+2} : 0 \leq i \leq m\}, & \text{if } t \neq 0.\end{cases}$$

$$S_{1,1} = \begin{cases}\{v_0\}, & \text{if } t = 4;\\ \emptyset, & \text{if } t \neq 4.\end{cases}$$

$$S_{1,0} = N(S_{1,2}).$$

Then $N[S_{1,2}] \cup S_{1,1} = V(C(n; (1, 3)))$, and $f = (V_0; V_1; V_2) = (S_{1,0}; S_{1,1}; S_{1,2})$ is a Roman dominating function of $C(n; \{1, 3\})$ with

$$f(V(C(n;(1,3)))) = \begin{cases} 2m, & \text{if } t = 0;\\ 2m+2, & \text{if } t = 1, 2, 3;\\ 2m+3, & \text{if } t = 4.\end{cases}$$

Hence we have

$$\gamma_R(C(n;\{1,3\})) \leq \begin{cases} 2m, & \text{if } t = 0;\\ 2m+2, & \text{if } t = 1, 2, 3;\\ 2m+3, & \text{if } t = 4.\end{cases}$$

In the following part of this proof, we will prove that

$$\gamma_R(C(n;\{1,3\})) \geq \begin{cases} 2m, & \text{if } t = 0;\\ 2m+2, & \text{if } t = 1, 2, 3;\\ 2m+3, & \text{if } t = 4.\end{cases}$$

*Case* 1. $t = 0$. By Lemma 2.2(3), $\gamma_R(C(n; \{1, 3\})) \geq 2m$.

*Case* 2. $t = 1, 2, 3$ and $(t, n_1) \neq (1, 1)$. By Lemma 2.2(4), $\gamma_R(C(n; \{1, 3\})) \geq 2m+2$.

*Case* 3. $(t, n_1) = (1, 1)$. By Lemma 2.2(1), $n_2 \geq \lceil \frac{n-n_1}{5} \rceil = \lceil \frac{5m+1-1}{5} \rceil = m$. Hence $\gamma_R(C(n; \{1, 3\})) = 2n_2 + n_1 \geq 2m+1$. Assume that $\gamma_R(C(n; \{1, 3\})) = 2m+1$. Then by Lemma 2.1, $rd(V(G[V_0 \cup V_2])) = (r+1)n_2 - (n-n_1) \geq 5m - (5m+1-1) = 0$. Without loss of generality, let $v_{5m} \in V_1$. By Proposition 1.5(b), we have $v_0 \in V_0$. By the definition of Roman dominating function, $N(v_0) \cap V_2 \neq \emptyset$, we have $\{v_1, v_3, v_{5m-2}\} \cap V_2 \neq \emptyset$.

*Case* 3.1. Suppose $v_1 \in V_2$. Let $v_i \in V_2$ be the vertex dominating $v_{5m-2}$, then since $rd(V(G[V_0 \cup V_2])) = 0$, we have $v_i \notin \{v_{5m-2}, v_{5m-1}, v_0\}$. By Proposition 1.5(b), we have $v_i \neq v_{5m-3}$. Hence $v_i = v_{5m-5}$. Let $v_j \in V_2$ be the vertex dominating $v_{5m-3}$, then $v_j \in \{v_{5m-6}, v_{5m-4}, v_{5m-3}, v_{5m-2}\}$, it follows that $rd(V(G[V_0 \cup V_2])) > 0$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 0$ (see Fig. 3.1(1)).

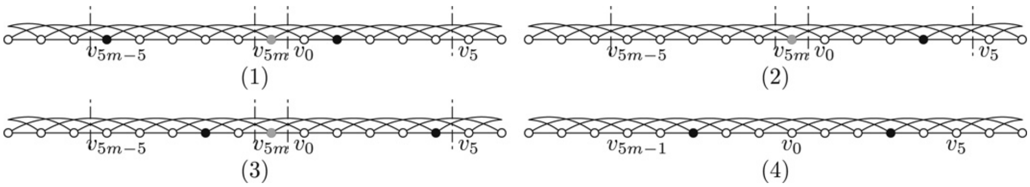


**Fig. 3.1.** The cases for $rd(v_0) = 1$ and $n_1 = 0$.

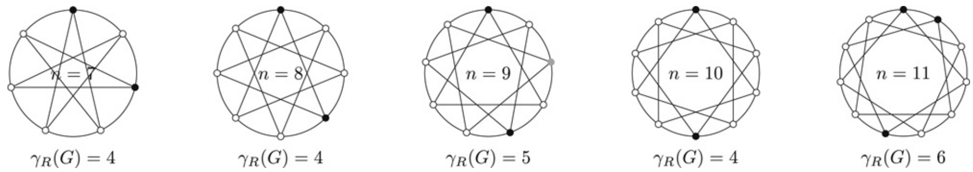


**Fig. 3.2.** A Roman dominating function on $G = C(n; \{1, 3\})$ for $7 \le n \le 11$.

*Case* 3.2. Suppose $v_3 \in V_2$. Let $v_i \in V_2$ be the vertex dominating $v_1$. By Proposition 1.5(b), we have $v_i \neq v_{5m-1}$. So $v_i \in \{v_0, v_1, v_2, v_4\}$, it follows that $rd(V(G[V_0 \cup V_2])) > 0$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 0$ (see Fig. 3.1(2)).

*Case* 3.3. Suppose $v_{5m-2} \in V_2$. Let $v_i \in V_2$ be the vertex dominating $v_1$, then since $rd(V(G[V_0 \cup V_2])) = 0$, we have $v_i \notin \{v_{5m-1}, v_0, v_1\}$. By Proposition 1.5(b), we have $v_i \neq v_2$. Hence $v_i = v_4$. Let $v_j \in V_2$ be the vertex dominating $v_2$, then $v_j \in \{v_2, v_3, v_4, v_5\}$, it follows that $rd(V(G[V_0 \cup V_2])) > 0$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 0$ (see Fig. 3.1(3)).

From cases 3.1–3.3, we have $\gamma_R(C(n; \{1, 3\})) \neq 2m + 1$ for $(t, n_1) = (1, 1)$, i.e. $\gamma_R(C(n; \{1, 3\})) \ge 2m + 2$.

*Case* 4. $t = 4$. By Lemma 2.2(2), $\gamma_R(C(n; \{1, 3\})) \ge \frac{2n+(r-1)n_1}{r+1} = \frac{2\times(5m+4)+(4-1)n_1}{4+1} = 2m + 1 + \frac{3n_1+3}{5}$. Hence $\gamma_R(C(n; \{1, 3\})) \ge 2m + 1 + \lceil \frac{3n_1+3}{5} \rceil$.

*Case* 4.1. Suppose $n_1 \neq 0$. Then $\gamma_R(C(n; \{1, 3\})) \ge 2m + 1 + \lceil \frac{3n_1+3}{5} \rceil \ge 2m + 3$.

*Case* 4.2. Suppose $n_1 = 0$. By Lemma 2.2(1), $n_2 \ge \lceil \frac{n-n_1}{5} \rceil = \lceil \frac{5m+4}{5} \rceil = m + 1$. Assume that $n_2 = m + 1$. Then by Lemma 2.1, $rd(V(G[V_0 \cup V_2])) = (r+1)n_2 - (n-n_1) = 5(m+1) - (5m+4) = 1$. Without loss of generality, we may assume that $rd(v_0) = 1$. Then we have $N[v_0] \cap V_2 = \{v_3, v_{5m+1}\}$. Let $v_i \in V_2$ be the vertex dominating $v_1$, we have $v_i \in \{v_{5m+2}, v_0, v_1, v_2, v_4\}$, it follows that $rd(V(G[V_0 \cup V_2])) > 1$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 1$ (see Fig. 3.1(4)). Hence $n_2 \neq m + 1$. i.e. $n_2 \ge m + 2$, $\gamma_R(C(n; \{1, 3\})) = 2n_2 + n_1 \ge 2m + 4$.

From above discussion, we have

$$\gamma_R(C(n; \{1, 3\})) = \begin{cases} 2m, & \text{if } t = 0; \\ 2m + 2, & \text{if } t = 1, 2, 3; \\ 2m + 3, & \text{if } t = 4. \end{cases} \quad \square$$

**Theorem 3.2.** *The circulant graphs $C(n; \{1, 3\})$ are Roman for $n \ge 7$ and $n \not\equiv 4 \pmod 5$.*

**Proof.** According to the proof of Lemma 3.1, we have $f = (V_0; V_1; V_2) = (S_{1,0}; S_{1,1}; S_{1,2})$ is a $\gamma_R$-function with $|V_1| = 0$. By Proposition 1.4, the circulant graphs $C(n; \{1, 3\})$ are Roman for $n \ge 7$ and $n \not\equiv 4 \bmod 5$. $\square$

In Fig. 3.2, we show a Roman dominating function on $C(n; \{1, 3\})$ for $7 \le n \le 11$.

Let $C_{n,k} = C(n; \{1, 2, \ldots, k\})$, then the graphs $C(n, k)$ are $2k$-regular.

**Lemma 3.3.** *For $n \ge 5$, $2 \le k \le \lfloor \frac{n}{2} \rfloor$, $n \neq 2k$.*

$$\gamma_R(C_{n,k}) = \begin{cases} 2m, & \text{if } t = 0; \\ 2m + 1, & \text{if } t = 1; \\ 2m + 2, & \text{if } t = 2, 3, \ldots, 2k. \end{cases}$$

**Proof.** Let

$$S_{2,2} = \begin{cases} \{v_{(2k+1)i+k} : 0 \le i \le m - 1\}, & \text{if } t = 0, 1; \\ \{v_{(2k+1)i+k} : 0 \le i \le m\}, & \text{if } t = 2, 3, \ldots, 2k. \end{cases}$$

$$S_{2,1} = \begin{cases} \{v_{5m}\}, & \text{if } t = 1; \\ \emptyset, & \text{if } t \neq 1. \end{cases}$$

$$S_{2,0} = N(S_{2,2}).$$

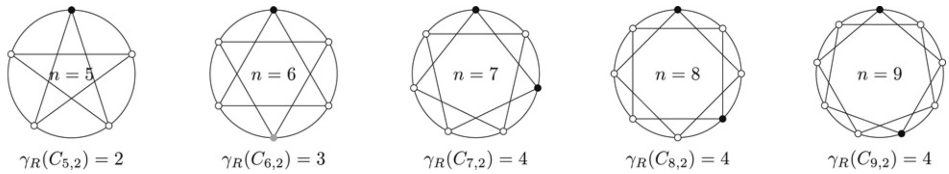


**Fig. 3.3.** A Roman dominating function on $C_{n,2} = C(n; \{1, 2\})$ for $5 \le n \le 9$.

Then $N[S_{2,2}] \cup S_{2,1} = V(C_{n,k})$, and $f = (V_0; V_1; V_2) = (S_{2,0}; S_{2,1}; S_{2,2})$ is a Roman dominating function of $C_{n,k}$ with

$$f(V(C_{n,k})) = \begin{cases} 2m, & \text{if } t = 0; \\ 2m+1, & \text{if } t = 1; \\ 2m+2, & \text{if } t = 2, 3, \dots, 2k. \end{cases}$$

Hence

$$\gamma_R(C_{n,k}) \le \begin{cases} 2m, & \text{if } t = 0; \\ 2m+1, & \text{if } t = 1; \\ 2m+2, & \text{if } t = 2, 3, \dots, 2k. \end{cases}$$

By Lemma 2.2(2), we have $\gamma_R(C_{n,k}) \ge 2m + \lceil \frac{2t+(r-1)n_1}{2k+1} \rceil = 2m + \lceil \frac{(2k-1)n_1+2t}{2k+1} \rceil$.

Hence

$$\gamma_R(C_{n,k}) \ge \begin{cases} 2m, & \text{if } t = 0; \\ 2m+1, & \text{if } t = 1. \end{cases}$$

If $t \ge 2$ and $n_1 \ne 0$, then we have $\gamma_R(C_{n,k}) \ge 2m + \lceil \frac{(2k-1)n_1+2t}{2k+1} \rceil \ge 2m + \lceil \frac{(2k-1)+2\times 2}{2k+1} \rceil = 2m+2$. If $t \ge 2$ and $n_1 = 0$, by Lemma 2.2(1), we have $n_2 \ge \lceil \frac{n-n_1}{r+1} \rceil = \lceil \frac{(2k+1)m+t}{2k+1} \rceil = m+1$, $\gamma_R(C_{n,k}) = 2n_2 \ge 2m+2$. From the above discussion, we have

$$\gamma_R(C_{n,k}) = \begin{cases} 2m, & \text{if } t = 0; \\ 2m+1, & \text{if } t = 1; \\ 2m+2, & \text{if } t = 2, 3, \dots, 2k. \quad \square \end{cases}$$

In Fig. 3.3, we show a Roman dominating function on $C(n; \{1, 2\})$ for $5 \le n \le 9$.

**Theorem 3.4.** *The circulant graphs $C(n; \{1, 2, \dots, k\})$ are Roman for $n \ge 4$ $(n \ne 2k)$, $2 \le k \le \lfloor \frac{n}{2} \rfloor$ and $n \not\equiv 1 \pmod{2k+1}$.*

**Proof.** According to the proof of Lemma 3.3, we have $f = (V_0; V_1; V_2) = (S_{20}; S_{21}; S_{22})$ is a $\gamma_R$-function with $|V_1| = 0$. By Proposition 1.4, the circulant graphs $C(n; \{1, 2, \dots, k\})$ are Roman for $n \ge 4$ $(n \ne 2k)$ and $n \not\equiv 1 \pmod{2k+1}$. $\square$

## 4. Roman domination in generalized Petersen graphs

In this section, we let $m^* = \lfloor \frac{n}{4} \rfloor$, $t^* = n \bmod 4$, then $n = 4m^* + t^*$, $0 \le t^* \le 3$. The graphs of this section are 3-regular, and the subscripts should be taken modulo $n$.

**Theorem 4.1.** *For $n \equiv 0 \pmod 4$, $0 \le k \le \frac{\lfloor \frac{n-1}{2} \rfloor - 1}{2}$, the generalized Petersen graphs $P(n, 2k+1)$ are Roman.*

**Proof.** Suppose $n \equiv 0 \pmod 4$, let

$$S_{3,2} = \{v_{4i}, u_{4i+2} : 0 \le i \le m^* - 1\},$$
$$S_{3,1} = \emptyset,$$
$$S_{3,0} = N(S_{3,2}),$$

then $N[S_{3,2}] \cup S_{3,1} = V(P(n, 2k+1))$, and $f = (V_0; V_1; V_2) = (S_{3,0}; S_{3,1}; S_{3,2})$ is a Roman dominating function of $P(n, 2k+1)$ with $f(V(P(n, 2k+1))) = 2 \times (2m^*) = 4m^*$. So we have $\gamma_R(P(n, 2k+1)) \le 4m^*$. By Lemma 2.2, $\gamma_R(P(n, 2k+1)) \ge \frac{2\times 2n+(3-1)n_1}{3+1} = \frac{4\times 4m^*+2n_1}{4} = 4m^* + \frac{n_1}{2} \ge 4m^*$. Hence $\gamma_R(P(n, 2k+1)) = 4m^*$ for $n \equiv 0 \pmod 4$.

Thus, $f = (V_0; V_1; V_2) = (S_{3,0}; S_{3,1}; S_{3,2})$ is a $\gamma_R$-function with $|V_1| = 0$. By Proposition 1.4, the generalized Petersen graphs $P(n, 2k+1)$ are Roman for $n \equiv 0 \pmod 4$ and $0 \le k \le \frac{\lfloor \frac{n-1}{2} \rfloor - 1}{2}$. $\square$

**Lemma 4.2.** *For $n \ge 3$, $\gamma_R(P(n, 1)) = 4m^* + t^* + 1$ for $t^* = 1, 2, 3$.*

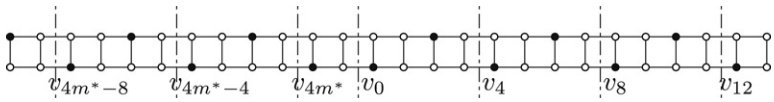


**Fig. 4.1.** The case for $n_2 = 2m + 1$ and $t^* = 2$.

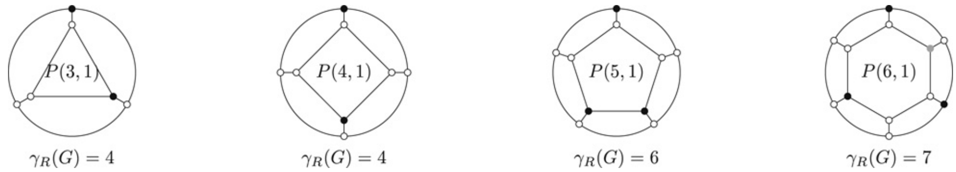


**Fig. 4.2.** A Roman dominating function on $G = P(n, 1)$ for $3 \le n \le 7$.

**Proof.** Let

$$S_{4,2} = \begin{cases} \{v_{4i}, u_{4i+2} : 0 \le i \le m^* - 1\} \cup \{v_{4m^*}\}, & \text{if } t^* = 1, 2; \\ \{v_{4i}, u_{4i+2} : 0 \le i \le m^*\}, & \text{if } t^* = 3. \end{cases}$$

$$S_{4,1} = \begin{cases} \{u_{4m+1}\}, & \text{if } t^* = 2; \\ \emptyset, & \text{if } t^* = 1, 3. \end{cases}$$

$$S_{4,0} = N(S_{4,2}).$$

Then $N[S_{4,2}] \cup S_{4,1} = V(P(n, 1))$, and $f = (V_0; V_1; V_2) = (S_{4,0}; S_{4,1}; S_{4,2})$ is a Roman dominating function of $P(n, 1)$ with $f(V(P(n, 1))) = 4m^* + t^* + 1$ for $t^* = 1, 2, 3$. Hence we have $\gamma_R(P(n, 1)) \le 4m^* + t^* + 1$ for $t^* = 1, 2, 3$. By Lemma 2.2(2), $f(V(P(n, 1))) \ge \frac{2\times 2n+(r-1)n_1}{r+1} = \frac{2\times 2\times(4m^*+t^*)+(3-1)n_1}{3+1} = 4m^* + t^* + \frac{n_1}{2}$. Hence $\gamma_R(P(n, 1)) \ge 4m^* + t^* + \lceil \frac{n_1}{2} \rceil$. If $n_1 \ne 0$, then $\gamma_R(P(n, 1)) \ge 4m^* + t^* + 1$.

If $n_1 = 0$, then by Lemma 2.2(1), $n_2 \ge \lceil \frac{2n}{4} \rceil = \lceil \frac{2(4m^*+t^*)}{4} \rceil = 2m^* + \lceil \frac{t^*}{2} \rceil$. Hence $\gamma_R(P(n, 1)) = 2n_2 + n_1 \ge 4m^* + 2\lceil \frac{t^*}{2} \rceil$. There are two cases :

*Case* 1. $t^* = 1, 3$. Then $\gamma_R(P(n, 1)) \ge 4m^* + 2\lceil \frac{t^*}{2} \rceil = 4m^* + t^* + 1$.

*Case* 2. $t^* = 2$. Then $n_2 \ge 2m^* + \lceil \frac{t^*}{2} \rceil = 2m^* + 1$. Assume that $n_2 = 2m^* + 1$. Then by Lemma 2.1, $rd(V(P(n, 1))) = rd(V(G[V_0 \cup V_2])) = (r + 1)n_2 - (n - n_1) = 4(2m^* + 1) - (8m^* + 4) = 0$. If $v_i \in V_0$ for every $0 \le i \le n - 1$, then $V_2 = \{u_0, u_1, \ldots, u_{n-1}\}$, $n_2 = 4m^* + 2$, a contradiction with $n_2 = 2m^* + 1$. Without loss of generality, we may assume that $v_0 \in V_2$. Let $x_i \in V_2$ be the vertex dominating $u_1$, then since $rd(V(P(n, 1))) = 0$, we have $x_i = u_2$. Let $x_j \in V_2$ be the vertex dominating $v_3$, then since $rd(V(P(n, 1))) = 0$, we have $x_j = v_4$. Continuing in this way, we have $\{v_{4i}, u_{4i+2}\} \subset V_2 (0 \le i \le m^*)$, i.e. $v_{4m^*} \in V_2$, $rd(v_{4m^*+1}) \ge 1$, a contradiction with $rd(V(P(n, 1))) = 0$. Hence $n_2 \ne 2m^* + 1$, i.e. $n_2 \ge 2m^* + 2$, $\gamma_R(P(n, 1)) = 2n_2 \ge 4m^* + 4 > 4m^* + t^* + 1$ (see Fig. 4.1).

From the above discussion, we have $\gamma_R(P(n, 1)) = 4m^* + t^* + 1$ for $t^* = 1, 2, 3$. □

**Theorem 4.3.** *The generalized Petersen graphs $P(n, 1)$ are Roman for $n \ge 3$ and $n \not\equiv 2 \pmod 4$.*

**Proof.** According to the proof of Lemma 4.2, we have $f = (V_0; V_1; V_2) = (S_{4,0}; S_{4,1}; S_{4,2})$ is a $\gamma_R$-function with $|V_1| = 0$ for $t = 1, 3$. By Proposition 1.4 and Theorem 4.1, we have that the generalized Petersen graphs $P(n, 1)$ are Roman for $n \ge 3$ and $n \not\equiv 2 \pmod 4$. □

In Fig. 4.2, we show a Roman dominating function on $P(n, 1)$ for $3 \le n \le 7$.

**Lemma 4.4.** *For the generalized Petersen graph $P(n, 3) (n \ge 7)$, if $n \equiv 2 \pmod 4$ and $n_1 = 1$, then $n_2 \ge 2m^* + 2$.*

**Proof.** By Lemma 2.2(1), $n_2 \ge \lceil \frac{2n-n_1}{4} \rceil = \lceil \frac{2\times(4m^*+2)-1}{4} \rceil = 2m^* + 1$. Assume that $n_2 = 2m^* + 1$. Then by Lemma 2.1, $rd(V(G[V_0 \cup V_2])) = (r + 1)n_2 - (n - n_1) = 4(2m^* + 1) - (8m^* + 4 - 1) = 1$, hence there exists an unique vertex $x$ with $rd(x) = 1$. If $x \in V_2$, then since $rd(x) = 1$, $x$ has to be dominated by another vertex, say $y \in V_2$. Thus, $rd(\{x, y\}) \ge 2$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 1$. Without loss of generality, we may assume that $x \in \{v_6, u_6\}$.

*Case* 1. $rd(v_6) = 1$. Then $N(v_6) \cap V_2 = \{\{v_5, v_7\}, \{v_5, u_6\}, \{u_6, v_7\}\}$. By symmetry, we only need to consider the cases for $N(v_6) \cap V_2 = \{\{v_5, v_7\}, \{v_5, u_6\}\}$.

*Case* 1.1. $N(v_6) \cap V_2 = \{v_5, v_7\}$. Consider the vertex $u_8$, we have:

*Case* 1.1.1. Suppose $u_8 \in V_1$. Let $x_i \in V_2$ be the vertex dominating $u_4$, since $rd(V(G[V_0 \cup V_2])) = 1$, we have $x_i = u_1$. Let $x_j \in V_2$ be the vertex dominating $v_2$, then $x_j \in \{v_1, v_2, u_2, v_3\}$, it follows that $rd(V(G[V_0 \cup V_2])) \ge 2$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 1$ (see Fig. 4.3(1)).

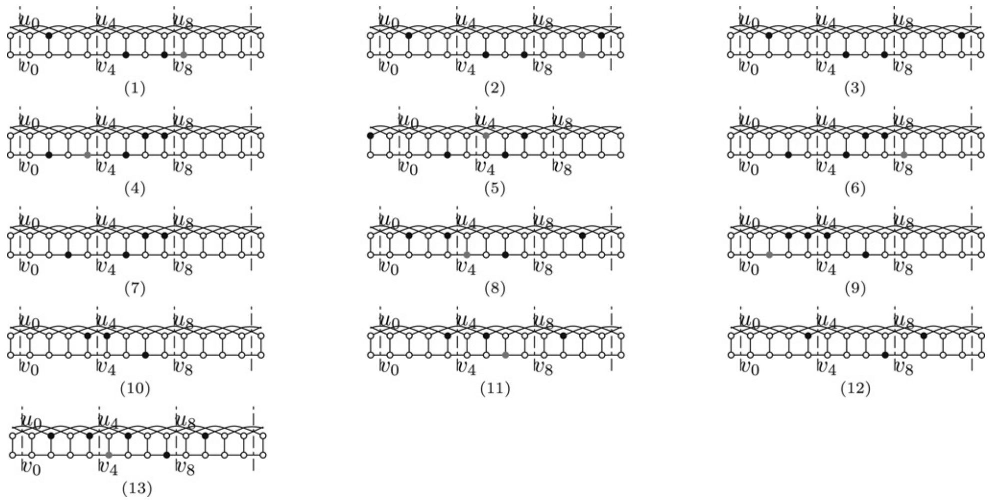

**Fig. 4.3.** The cases of $rd(x) = 1$ for $x \in \{v_6, u_6\}$.

*Case* 1.1.2. Suppose $u_8 \notin V_1$. Let $x_i \in V_2$ be the vertex dominating $u_8$, since $rd(V(G[V_0 \cup V_2])) = 1$, we have $x_i = u_{11}$. Consider the vertex $v_{10}$, we have:

*Case* 1.1.2.1. Suppose $v_{10} \in V_1$. Let $x_j \in V_2$ be the vertex dominating $u_4$, since $rd(V(G[V_0 \cup V_2])) = 1$, we have $x_j = u_1$. Let $x_h \in V_2$ be the vertex dominating $v_2$, then $x_h \in \{v_1, v_2, u_2, v_3\}$, it follows that $rd(V(G[V_0 \cup V_2])) \geq 2$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 1$ (see Fig. 4.3(2)).

*Case* 1.1.2.2. Suppose $v_{10} \notin V_1$. Let $x_j \in V_2$ be the vertex dominating $v_{10}$, then $x_j \in \{v_9, v_{10}, u_{10}, v_{11}\}$, it follows that $rd(V(G[V_0 \cup V_2])) \geq 2$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 1$ (see Fig. 4.3(3)).

*Case* 1.2. $N(v_6) \cap V_2 = \{\{v_5, u_6\}\}$. Consider the vertex $v_3$, we have:

*Case* 1.2.1 Suppose $v_3 \in V_1$. Let $x_i \in V_2$ be the vertex dominating $v_2$, then $x_i \in \{v_1, v_2, u_2\}$. By Proposition 1.5(b), $x_i \neq v_2$. Since $rd(V(G[V_0 \cup V_2])) = 1$, we have $x_i \neq u_2$, it follows that $x_i = v_1$. Let $x_j \in V_2$ be the vertex dominating $u_4$, since $rd(V(G[V_0 \cup V_2])) = 1$, $x_j = u_7$. Let $x_h \in V_2$ be the vertex dominating $v_8$, then $x_h \in \{v_7, v_8, u_8, v_9\}$, it follows that $rd(V(G[V_0 \cup V_2])) \geq 2$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 1$ (see Fig. 4.3(4)).

*Case* 1.2.2 Suppose $v_3 \notin V_1$. Since $rd(V(G[V_0 \cup V_2])) = 1$, we have $v_2 \in V_2$. Consider the vertex $u_4$, we have:

*Case* 1.2.2.1 Suppose $u_4 \in V_1$. Let $x_i \in V_2$ be the vertex dominating $u_1$, then $x_i \in \{u_{4m^*}, v_1, u_1, \}$. By Proposition 1.5(b), $x_i \neq u_1$. Since $rd(V(G[V_0 \cup V_2])) = 1$, we have $x_i \neq v_1$, it follows that $x_i = u_{4m^*}$. Let $x_j \in V_2$ be the vertex dominating $v_{4m^*+1}$, then $x_j \in \{v_{4m^*}, v_{4m^*+1}, u_{4m^*+1}, v_0\}$, it follows that $rd(V(G[V_0 \cup V_2])) \geq 2$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 1$ (see Fig. 4.3(5)).

*Case* 1.2.2.2 Suppose $u_4 \notin V_1$. Since $rd(V(G[V_0 \cup V_2])) = 1$, we have $u_7 \in V_2$. Consider the vertex $v_8$, Suppose $v_8 \in V_1$, Let $x_i \in V_2$ be the vertex dominating $v_9$, then $x_i \in \{v_9, u_9, v_{10}\}$. By Proposition 1.5(b), we have $x_i \neq v_9$, hence $x_i \in \{u_9, v_{10}\}$, it follows that $rd(V(G[V_0 \cup V_2])) \geq 2$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 1$ (see Fig. 4.3(6)). Suppose $v_8 \notin V_1$, Let $x_j \in V_2$ be the vertex dominating $v_8$, then $x_j \in \{v_7, v_8, u_8, v_9\}$, it follows that $rd(V(G[V_0 \cup V_2])) \geq 2$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 1$ (see Fig. 4.3(7)).

*Case* 2. $rd(u_6) = 1$. Then $N(u_6) \cap V_2 \in \{\{u_3, v_6\}, \{u_3, u_7\}, \{v_6, u_7\}\}$. By symmetry, we only need to consider the cases for $N(v_6) \cap V_2 \in \{\{u_3, v_6\}, \{u_3, u_7\}\}$.

*Case* 2.1. $N(u_6) \cap V_2 = \{u_3, v_6\}$. Consider the vertex $v_4$, we have:

*Case* 2.1.1. $v_4 \in V_1$. Let $x_i \in V_2$ be the vertex dominating $u_4$, then $x_i \in \{u_1, u_4, u_7\}$. By Proposition 1.5(b), $x_i \neq u_4$. Since $rd(V(G[V_0 \cup V_2])) = 1$, we have $x_i \neq u_7$, it follows that $x_i = u_1$. Let $x_j \in V_2$ be the vertex dominating $u_7$, since $rd(V(G[V_0 \cup V_2])) = 1$, we have $x_j = u_{10}$. Let $x_h \in V_2$ be the vertex dominating $v_9$, then $x_h \in \{v_8, v_9, u_9, v_{10}\}$, it follows that $rd(V(G[V_0 \cup V_2])) \geq 2$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 1$ (see Fig. 4.3(8)).

*Case* 2.1.2. $v_4 \notin V_1$. Since $rd(V(G[V_0 \cup V_2])) = 1$, we have $u_4 \in V_2$. Consider the vertex $v_1$, we have:

*Case* 2.1.2.1. Suppose $v_1 \in V_1$. Let $x_i \in V_2$ be the vertex dominating $v_2$, then $x_i \in \{v_2, v_3, u_2\}$. Since $rd(V(G[V_0 \cup V_2])) = 1$, we have $x_i = u_2$. Let $x_j \in V_2$ be the vertex dominating $v_0$, then $x_j \in \{v_{4m^*+1}, v_0, u_0\}$. By Proposition 1.5(b), we have $x_j \neq v_0$, hence $x_j \in \{v_{4m^*+1}, u_0\}$, it follows that $rd(V(G[V_0 \cup V_2])) \geq 2$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 1$ (see Fig. 4.3(9)).

*Case* 2.1.2.2. Suppose $v_1 \notin V_1$. Let $x_i \in V_2$ be the vertex dominating $v_1$, we have $x_i \in \{v_0, v_1, u_1, v_2\}$, it follows that $rd(V(G[V_0 \cup V_2])) \geq 2$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 1$ (see Fig. 4.3(10)).

*Case* 2.2. $N(u_6) \cap V_2 = \{u_3, u_9\}$. Consider the vertex $v_6$, we have:

*Case* 2.2.1. $v_6 \in V_1$. Let $x_i \in V_2$ be the vertex dominating $v_5$, then $x_i = \{v_4, v_5, u_5\}$. By Proposition 1.5(b), $x_i \neq v_5$. Since $rd(V(G[V_0 \cup V_2])) = 1$, we have $x_i \neq v_4$, it follows that $x_i = u_5$. Let $x_j \in V_2$ be the vertex dominating $v_8$, then

$x_j \in \{v_7, v_8, u_8, v_9\}$. By Proposition 1.5(b), $x_j \neq v_7$, hence $x_j \in \{v_8, u_8, v_9\}$, it follows that $rd(V(G[V_0 \cup V_2])) \geq 2$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 1$ (see Fig. 4.3(11)).

*Case* 2.2.2. $v_6 \notin V_1$. Since $rd(V(G[V_0 \cup V_2])) = 1$, we have $v_6 \in V_0$, $V_2 \cap \{v_5, v_7\} \neq \emptyset$. By symmetry, we only need to consider $v_7 \in V_2$. Consider the vertex $v_4$ and observe the following cases:

*Case* 2.2.2.1. Suppose $v_4 \notin V_1$. Let $x_i \in V_2$ be the vertex dominating $v_4$, then $x_j \in \{v_3, v_4, u_4, v_5\}$, it follows that $rd(V(G[V_0 \cup V_2])) \geq 2$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 1$ (see Fig. 4.3(12)).

*Case* 2.2.2.2. Suppose $v_4 \in V_1$. Let $x_i \in V_2$ be the vertex dominating $v_5$, then $x_i = \{v_5, u_5, v_6\}$. By Proposition 1.5(b), $x_i \neq v_5$. Since $rd(V(G[V_0 \cup V_2])) = 1$, we have $x_i \neq v_6$, it follows that $x_i = u_5$. Let $x_j \in V_2$ be the vertex dominating $u_4$, then $x_j \in \{u_3, u_4, u_7\}$. By Proposition 1.5(b), we have $x_j \neq u_4$. Since $rd(V(G[V_0 \cup V_2])) = 1$, we have $x_j \neq u_7$, it follows that $x_j = u_1$. Let $x_h \in V_2$ be the vertex dominating $v_2$, then $x_h \in \{v_1, v_2, u_2, v_3\}$, it follows that $rd(V(G[V_0 \cup V_2])) \geq 2$, a contradiction with $rd(V(G[V_0 \cup V_2])) = 1$ (see Fig. 4.3(13)).

From cases 1–2, we have $n_2 \neq 2m^* + 1$, i.e. $n_2 \geq 2m^* + 2$. □

**Lemma 4.5.** *For $n \geq 7$,*

$$\gamma_R(P(n,3)) = \begin{cases} 4m^*+2, & \text{if } t^*=1; \\ 4m^*+4, & \text{if } t^*=2; \\ 4m^*+4, & \text{if } t^*=3. \end{cases}$$

**Proof.** Let

$$S_{5,2} = \begin{cases} \{v_{4i}, u_{4i+2} : 0 \leq i \leq m^*-1\} \cup \{u_{4m^*-1}\}, & \text{if } t^*=1; \\ \{v_{4i}, u_{4i+2} : 0 \leq i \leq m^*-1\} \cup \{v_{4m^*-1}, u_{4m^*}\}, & \text{if } t^*=2; \\ \{v_{4i}, u_{4i+2} : 0 \leq i \leq m^*-1\} \cup \{v_{4m^*}\}, & \text{if } t^*=3. \end{cases}$$

$$S_{5,1} = \begin{cases} \emptyset, & \text{if } t^*=1,2; \\ \{u_1, u_{4m-1}\}, & \text{if } t^*=3. \end{cases}$$

$$S_{5,0} = N(S_{5,2}).$$

Then $N[S_{5,2}] \cup S_{5,1} = V(P(n,3))$, and $f = (V_0; V_1; V_2) = (S_{5,0}; S_{5,1}; S_{5,2})$ is a Roman dominating function of $P(n,3)$ with

$$f(P(n,3)) = \begin{cases} 4m^*+2, & \text{if } t^*=1; \\ 4m^*+4, & \text{if } t^*=2; \\ 4m^*+4, & \text{if } t^*=3. \end{cases}$$

Hence we have

$$\gamma_R(P(n,3)) \leq \begin{cases} 4m^*+2, & \text{if } t^*=1; \\ 4m^*+4, & \text{if } t^*=2; \\ 4m^*+4, & \text{if } t^*=3. \end{cases}$$

In the following part of this proof, we will prove that

$$\gamma_R(P(n;\{1,3\})) \geq \begin{cases} 4m^*+2, & \text{if } t^*=1; \\ 4m^*+4, & \text{if } t^*=2; \\ 4m^*+4, & \text{if } t^*=3. \end{cases}$$

By Lemma 2.2(2), $\gamma_R(P(n,3)) \geq \frac{2\times 2n+(r-1)n_1}{r+1} = \frac{2\times 2\times(4m^*+t^*)+(3-1)n_1}{3+1} = 4m^* + t^* + \frac{n_1}{2}$.

*Case* 1. $t^* = 1$. If $n_1 \neq 0$, then $f(V(P(n,2))) \geq 4m^* + t^* + \lceil \frac{n_1}{2} \rceil \geq 4m^* + 2$. If $n_1 = 0$, then by Lemma 2.2(1), $n_2 \geq \lceil \frac{2n-n_1}{3+1} \rceil = \lceil \frac{4m^*+1}{2} \rceil = 2m^* + 1$, $f(V(P(n,2))) = 2n_2 + n_1 \geq 2 \times (2m^*+1) = 4m^* + 2$. Hence $\gamma_R(P(n,2)) \geq 4m^* + 2$ for $t^* = 1$.

*Case* 2. $t^* = 2$. If $n_1 \geq 3$, then $f(V(P(n,2))) \geq 4m^* + t^* + \lceil \frac{n_1}{2} \rceil \geq 4m^* + 4$. So we only need to consider the cases for $n_1 = 0, 1, 2$.

*Case* 2.1. $n_1 = 0$, then $G[V_0 \cup V_2] = P(n,3)$. By Proposition 1.5(d), $V_2$ is a dominating set of $P(n,3)$. So $n_2 \geq \gamma(G)$. By Theorem 1.1, $n_2 \geq n - 2\lfloor \frac{n}{4} \rfloor = 4m^* + 2 - 2\lfloor \frac{4m^*+2}{4} \rfloor = 2m^* + 2$. So $f(V(P(n,3))) = 2n_2 \geq 4m^* + 4$.

*Case* 2.2. $n_1 = 1$. By Lemma 4.4, $n_2 \geq 2m^* + 2$. So $f(V(P(n,3))) = 2n_2 + n_1 \geq 2(2m^*+2) + 1 = 2m + 5$.

*Case* 2.3. $n_1 = 2$. Then by Lemma 2.2(1), $n_2 \geq \lceil \frac{2n-2}{4} \rceil = \lceil \frac{4m^*+2-1}{2} \rceil = 2m^* + 1$, $f(V(P(n,3))) = 2n_2 + n_1 \geq 2 \times (2m^*+1) + 2 = 4m^* + 4$.

Hence $\gamma_R(P(n,2)) \geq 4m^* + 4$ for $t^* = 2$.

*Case* 3. $t^* = 3$. If $n_1 \neq 0$, then $f(V(P(n,3))) \geq 4m^* + t^* + \lceil \frac{n_1}{2} \rceil \geq 4m^* + 4$. If $n_1 = 0$, then by Lemma 2.2(1), $n_2 \geq \lceil \frac{2n-n_1}{3+1} \rceil = \lceil \frac{4m^*+3}{2} \rceil = 2m^* + 2$, $f(V(P(n,2))) = 2n_2 + n_1 \geq 2(2m^*+2) = 4m^* + 4$. Hence $\gamma_R(P(n,2)) \geq 4m^* + 4$ for $t^* = 3$.

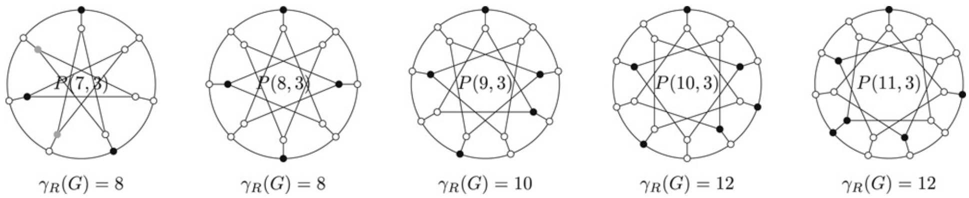


**Fig. 4.4.** A Roman dominating function on $G = P(n, 3)$ for $7 \le n \le 11$.

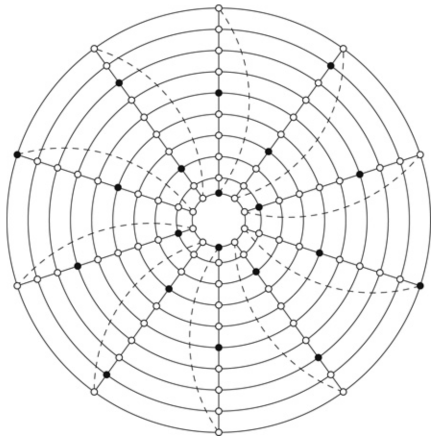

**Fig. 5.1.** A Roman dominating function on $C_{10}\Box C_{10}$.

Hence,

$$\gamma_R(P(n,3)) = \begin{cases} 4m^* + 2, & \text{if } t^* = 1; \\ 4m^* + 4, & \text{if } t^* = 2; \\ 4m^* + 4, & \text{if } t^* = 3. \quad \Box \end{cases}$$

**Theorem 4.6.** *The generalized Petersen graphs $P(n, 3)$ are Roman for $n = 11$ or $n \ge 7$ and $n \not\equiv 3 \pmod 4$.*

**Proof.** For $n = 11$, let $S_{6,2} = \{v_0, u_2, v_4, u_4, u_6, v_8\}$, $S_{6,1} = \emptyset$, $S_{6,0} = N(S_{6,2})$, by Lemma 4.5, $f = (V_0; V_1; V_2) = (S_{6,0}; S_{6,1}; S_{6,2})$ is a $\gamma_R$-function with $|V_1| = 0$. By Proposition 1.4, we have that the generalized Petersen graph $P(11, 3)$ is Roman.

For $n \ge 7$ and $n \not\equiv 3 \pmod 4$, by the proof of Lemma 4.5, we have $f = (V_0; V_1; V_2) = (S_{5,0}; S_{5,1}; S_{5,2})$ is a $\gamma_R$-function with $|V_1| = 0$ for $t = 1, 2$. By Proposition 1.4 and Theorem 4.1, we have that the Petersen graphs $P(n, 3)$ are Roman for $n \ge 7$ and $n \not\equiv 3 \pmod 4$. $\Box$

In Fig. 4.4, we show a Roman dominating function on $P(n, 3)$ for $7 \le n \le 11$.

## 5. Roman domination in Cartesian product graphs $C_{5m}\Box C_{5n}$

**Theorem 5.1.** *For $n \ge 1$, $m \ge 1$, the Cartesian product graphs $C_{5m}\Box C_{5n}$ are Roman.*

**Proof.** Let $V(C_{5m}\Box C_{5n}) = \{v_{ij} : \ 0 \le i \le 5m-1,\ 0 \le j \le 5n-1\}$,

$$S_{7,2} = \{v_{(5i)(5j)}, v_{(5i+1)(5j+3)}, v_{(5i+2)(5j+1)}, v_{(5i+3)(5j+4)}, v_{(5i+4)(5j+2)} : 0 \le i \le m-1, 0 \le j \le n-1\},$$
$$S_{7,1} = \emptyset,$$
$$S_{7,0} = N(S_{7,2}),$$

then $N[S_{7,2}] = V(C_{5m}\Box C_{5n})$, $f = (V_0; V_1; V_2) = (S_{7,0}; S_{7,1}; S_{7,2})$ is a Roman dominating function of $C_{5m}\Box C_{5n}$ with $f(V(C_{5m}\Box C_{5n})) = 2\times(5mn) = 10mn$. So we have $\gamma_R(C_{5m}\Box C_{5n}) \le 10mn$. By Proposition 1.1, we have $\gamma_R(C_{5m}\Box C_{5n}) \ge \frac{2\times(5m\times 5n)}{4+1} = 10mn$, hence $\gamma_R(C_{5m}\Box C_{5n}) = 10mn$. Thus, $f = (V_0; V_1; V_2) = (S_{7,0}; S_{7,1}; S_{7,2})$ is a $\gamma_R$-function with $|V_1| = 0$. By Proposition 1.4, the Cartesian product graphs $C_{5m}\Box C_{5n}$ are Roman. $\Box$

In Fig. 5.1, we show a Roman dominating function on $C_{10}\Box C_{10}$.

## Acknowledgements

We are very grateful to the referees for their careful reading with corrections and useful comments.


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