NOTE

REPRESENTING A PLANAR GRAPH BY VERTICAL LINES JOINING DIFFERENT LEVELS

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Answering a problem of H. de Fraysseix and P. Rosenstiehl we prove that every planar graph can be represented by horizontal segments corresponding to vertices and vertical segments corresponding to edges in such a way that no crossing appears. For 2-connected planar graphs, the boundary of the representation can be prescribed.

In 1981, de Fraysseix and Rosenstiehl [2] proposed the following problem: Is every planar graph (finite and loopless) representable in the plane in such a way that the following conditions hold:

(1) To vertices of $G$ correspond pairwise disjoint horizontal line-segments (called the $S$-vertices).
(2) To edges of $G$ correspond pairwise disjoint vertical line-segments (called the $S$-edges).
(3) The segment that represents an edge $xy$ joins the segments that represent $x$ and $y$ but has an empty intersection with other horizontal segments.

Such a representation will be named an $S$-representation. Two examples of an $S$-representation for the complete bipartite graph $K_{2,4}$ are shown in Fig. 1.

Theorem 1. Every planar graph has an $S$-representation.

Proof. We use the well-known Fary's theorem [1]: Every planar graph has a straight-line embedding. An extended Fary representation is a collection of pairwise disjoint horizontal line-segments called $F$-vertices and pairwise disjoint non-horizontal line segments called $F$-edges such that every $F$-edge joins two $F$-vertices and has an empty intersection with the others. Theorem 1 is equivalent to the following: every planar graph has an extended Fary representation such that all edges are vertical.

Any Fary representation of a planar graph $G$ yields an extended Fary representation by choosing a horizontal direction different from directions of edges.
Consider any extended Fary representation of $G$ and let $e$ be an $F$-edge which is not vertical. We observe that there exists a pseudo-line $L$ of $\mathbb{R}^2$ such that:

1. $L$ meets every horizontal line in exactly one point.
2. $L$ contains the ends of $e$.
3. $L$ meets the representation only in $F$-vertices.

The details of the existence of $L$ are easy and left to the reader.

The pseudo-line $L$ separates $\mathbb{R}^2 \setminus L$ into two parts. We shift one of them horizontally on a distance $s$; in this operation the $F$-vertices cut by $L$ are also extended by $s$. Then clearly for $s$ large enough the edge $e$ can be replaced by a vertical edge joining the same $F$-edges, resulting in a new extended Fary representation of $G$. The theorem follows by induction on the number of edges. □

In the case of 2-connected planar graphs, one may prescribe the 'boundary' of the $S$-representation.

More precisely, by a left $S$-vertex of an $S$-representation $R$, we mean an $S$-vertex that can be extended to infinity to the left (the plane is supposed to be oriented, so we know what up, down, left and right mean) without meeting any other segment ($S$-vertex or $S$-edge). The left-side of $R$ then consists in the ordered sequence of the left $S$-vertices starting at the lowest element and ending at the highest. Right $S$-vertices and right-side are defined similarly. For short, we identify the left-side and the right-side with the corresponding sequences of vertices of the graph represented by $R$.

**Theorem 2.** Let $b$ and $t$ be two different vertices of a face $F$ in an embedded 2-connected planar graph $G$. Let $L$ and $R$ denote the two paths in $F$ starting at $b$ and ending at $t$. Then $G$ admits an $S$-representation with $L$ as its left-side and $R$ as its right-side.

**Proof.** The proof goes by induction on the number of vertices of the graph $G$, using a gluing procedure—the theorem is trivial for $K_3$, the smallest 2-connected graph. We may suppose $G$ is embedded in the plane in such a way that $F$ is its infinite face. Clearly it is no restriction to suppose that all other faces are triangles. (We may destroy multiple edges and add new edges to triangulate the interior of $F$.) By symmetry, we may also choose the embedding so that $L$ is the
left path from \( b \) to \( t \) and \( R \) the other path. So, the choice of vertices \( b \) and \( t \) in the face completely determines \( L \) and \( R \). Therefore, an \( S \)-representation of \( G \) satisfying the conditions of Theorem 2 will be called a \((b, t)\)-representation.

**Remark.** A \((b, t)\)-representation can obviously be transformed in another by extending some left-vertices to the left and some right-vertices to the right, as far as we need. Moreover, we can arbitrarily change the levels as long as we keep their order.

If \( F \) is reduced to vertices \( b \) and \( t \), having only two edges \( l \) and \( r \), then clearly a \((b, t)\)-representation of \( G \) can be deduced from a \((b, t)\)-representation of \( G \setminus r \), by a sufficiently large extension of right-\( S \)-vertices representing \( b \) and \( t \). So, we may suppose \(|F| \geq 3\), and interverting if necessary the role of \( R \) and \( L \) in the remainder of the proof, we may also suppose that \( L \) contains some vertex \( v \) different from \( b \) and \( t \). Two cases have to be considered:

(i) If \( v \) is adjacent to some vertex \( w \) in \( R \) with \( w \notin \{b, t\} \), then \( vw \) separates \( G \); the two corresponding pieces \( G_1 \), containing \( t \) and \( G_2 \) containing \( b \), are still 2-connected and strictly smaller than \( G \). Then applying the induction hypothesis, \( G_1 \) admits a \((v, t)\)-representation and \( G_2 \) admits a \((b, w)\)-representation. By the remark above, these representations can be easily glued by means of suitable extensions of \( S \)-vertices \( v \) and \( w \) to produce a \((b, t)\)-representation of \( G \).

(ii) If \( v \) is not incident to \( R \setminus \{b, t\} \), then the neighbours of \( v \) constitute a separating set \( N \) of \( G \). Denote by \( x \) and \( y \) respectively the first and last vertices of \( L \) belonging to \( N \). Then \( x, y \) and the neighbours of \( v \) that are not in \( L \), constitute a minimal separating set which induces a path \( P \) of \( G \). The corresponding pieces, \( G_1 \) that contains \( v \), and \( G_2 \), that does not, are again 2-connected. \( G_2 \) must contain a vertex not in \( N \), otherwise there would be a face of \( G \) which is not a triangle. Thus the induction hypothesis may be applied: \( G_1 \) admits an \((x, y)\)-representation and \( G_2 \) a \((b, t)\)-representation. Again, by the remark above, these representations can be glued by means of suitable extensions of \( S \)-vertices of \( P \) into a \((b, t)\)-representation of \( G \).

**References**
